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Ulam-Hyers stability of Volterra integral equation in a $b_v(s)$-metric space

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Abstract

This work investigates investigate the Ulam-Hyers (UH) stability of the Volterra integral equation (*VIEs*) within a $b_v(s)$-metric space. We derive the desired result by employing a fixed-point approach.

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1. Introduction

Banach contraction principle or its generalization in defferent spaces has been investigated in several researches [1]. In 2017, Mitrovic et al. [17],

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traditional notion of MS was extended by introducing $b_{\mathfrak{v}}(\mathfrak{s})$-MSs . In this framework, authors proved many fixed point results [5, 19, 18, 17, 15, 3]. Moreover, in [17] a new method was used to prove the validity of Banach's fixed-point theorem within a $b_{\mathfrak{v}}(\mathfrak{s})-$ metric setting. The demonstration technique applied in [17] is different from the proof formulated in the early twentieth century and closely related to Banach's mapping theorem in the study of invariant points[10, 11]. The fixed point theory with its vast applications has allowed researchers to deal with several subjects in mathematics, namely those addressing the stability properties of equations in analysis introduced by Ulam [12] in 1940 and the collaboration with Hyers [2] led to the development of what is referred to as $\mathcal{UH}$-type stability. The establishment of these types of stabilities has attracted significant researchers attention who have contributed significantly to Investigation of the robustness properties of specific problems using a fixed-point approach [8]. Recently, Volterra integral equation has emerged as a highly promising fieled of research since it can be applied to model many different types of time-dependent phenomena. Given this importance, some Scientists have focused on the theoretical side to prove new results[14], while others have concentrated on developing new methods for solving certain problems involving the Volterra integral equation [7, 20, 6, 16, 9]. In a study that integrates the VIE, $\mathcal{UHS}$, and $\mathcal{FPT}$, the authors have explored the $\mathcal{UH}$ stability of a weakly singular $VIEs$ where they have used the Bielecki metric. [7]. Within this framework, investigating the $\mathcal{UH}$-type stability of certain classes of equations, we employ Banach's principle inside a $b_{\mathfrak{v}}(\mathfrak{s})-$MS. The article is structured in the subsequent way: first, attention is given to fundamental concepts with preliminary results for proving our main result, which will be presented in the third section. We conclude by summarizing our collaboration.

2. Preliminaries

First, we begin by presenting the necessary preliminaries and theoretical foundations cited in [17].

Definition 2.1: On a set $\mathfrak{E} \neq \emptyset$, we define the relation $d_{\mathfrak{E}}: \mathfrak{E} \times \mathfrak{E} \to [0, \infty)$, and consider a positive integer $\mathfrak{v}$. The paire $(\mathfrak{E}, d_{\mathfrak{E}})$ is termed a $b_{\mathfrak{v}}(\mathfrak{s})-$ metric space if, for all elements $\mathfrak{u}', \mathfrak{w}'$ in $\mathfrak{E}$ and for any different elements $\mathfrak{u}'_1, \mathfrak{u}'_2, \dots, \mathfrak{u}'_{\mathfrak{v}} \in \mathfrak{E}$, where each one is distinct from both $\mathfrak{u}'$ and $\mathfrak{w}'$ the following properties are met:

$i)$ $d_{\mathfrak{E}}(\mathfrak{u}', \mathfrak{w}') = 0 \quad iff \quad \mathfrak{u}' = \mathfrak{w}'$

$ii)$ $d_{\mathfrak{E}}(\mathfrak{u}', \mathfrak{w}') = d_{\mathfrak{E}}(\mathfrak{w}', \mathfrak{u}')$

$iii)$ $\exists \ \mathfrak{s} \in [1, +\infty[:$

$$d_{\mathfrak{E}}(\mathfrak{u}', \mathfrak{w}') \leq \mathfrak{s}[d_{\mathfrak{E}}(\mathfrak{u}, \mathfrak{u}'_1) + d_{\mathfrak{E}}(\mathfrak{u}'_1, \mathfrak{u}'_2) + d_{\mathfrak{E}}(\mathfrak{u}'_2, \mathfrak{u}'_3) + \dots + d_{\mathfrak{E}}(\mathfrak{u}'_{\mathfrak{v}}, \mathfrak{w}')].$$

The famous Banach contraction principle is cited in many references

Theorem 2.2: ([10, 11, 9]) *Consider the MS* $(\mathfrak{E}, d_{\mathfrak{E}})$; *assumed to be complete; with* Υ: $\mathfrak{E} \to \mathfrak{E}$ *satisfying a contractive condition:* $\exists 0 \leq \eta < 1$ *for which*

$$d_{\mathfrak{E}}(\Upsilon\mathfrak{u}', \Upsilon\mathfrak{w}') \leq \eta d_{\mathfrak{E}}(\mathfrak{u}', \mathfrak{w}'), \quad \forall \mathfrak{u}', \mathfrak{w}' \in \mathfrak{E}.$$

Thus, there is exactly one $\mathfrak{u}'^{*} \in \mathfrak{E}$ such that $\Upsilon\mathfrak{u}'^{*} = \mathfrak{u}'^{*}$.

The following theorem concerns the Banach contraction principle in $b_{\mathfrak{v}}(\mathfrak{s})$ −MS. It focuses on Mitrovic's and al key findings. [17]. This theorem is analogue of the original one in metric space but its proof (see, [17]) is short and different from Banach's one.

Theorem 2.3: *Consider the* $b_{\mathfrak{v}}(\mathfrak{s}) - MS$ $(\mathfrak{E}, d_{\mathfrak{E}})$; *assumed to be complete; and an application* $\Upsilon: \mathfrak{E} \to \mathfrak{E}$ *fulfilling:*

$$d_{\mathfrak{E}}(\Upsilon\mathfrak{u}, \Upsilon\mathfrak{w}) \leq \eta d_{\mathfrak{E}}(\mathfrak{u}, \mathfrak{w}), \quad \forall \mathfrak{u}, \mathfrak{w} \in \mathfrak{E}.$$

where $0 \leq \eta < 1$. *Hence, there exists a unique* $\mathfrak{u}^{*} \in \mathfrak{E}$ *satisfying* $\Upsilon\mathfrak{u}^{*} = \mathfrak{u}^{*}$.

Definition 2.4: ([13, 4]) Consider a MS $(\mathfrak{E}, d_{\mathfrak{E}})$ and an application $\Upsilon: \mathfrak{E} \to \mathfrak{E}$. Assuming the existence of a no-decreasing map $\mathfrak{f}: \mathfrak{R}^{+} \to \mathfrak{R}^{+}$, with ensured continuity at zero behind the property $\mathfrak{f}(0) = 0$, for which for every $\varepsilon > 0$ and for every $\mathfrak{u}$ satisfying

$$d_{\mathfrak{u}}(\mathfrak{u}, \Upsilon\mathfrak{u}) \leq \varepsilon$$

$\exists$ $\mathfrak{u}^{*} \in \mathfrak{E}$ ensuring that

$$\Upsilon\mathfrak{u}^{*} = \mathfrak{u}^{*} \tag{2.1}$$

and

$$d_{\mathfrak{u}}(\mathfrak{u}, \mathfrak{u}^{*}) \leq \mathfrak{f}(\varepsilon).$$

Then the Equation (2.1) has the Ulam stability which is called $\mathcal{UHS}$ when $\mathfrak{f}(\mathfrak{x}) = \mathfrak{t}\mathfrak{x}$, $\forall$ $\mathfrak{x} \in \mathfrak{R}^{+}$.

Definition 2.5: ([13, 4]) Consider the the MS $(\mathfrak{E}, d_{\mathfrak{E}})$ and $\mathfrak{X}' = C(\mathfrak{E}, \mathfrak{E})$. Plus $\Upsilon: \mathfrak{X}' \to \mathfrak{X}'$ be an operator. The Equation

$$\Upsilon\mathfrak{u} = \mathfrak{u} \tag{2.2}$$

has the generaized Ulam-Hyers-Rssias stability provided that $\exists$ $\sigma > 0$ so that, for every $\mathfrak{u}'$ satisfying

$$d_{\mathfrak{E}}(\mathfrak{u}'(\mathfrak{x}), \Upsilon\mathfrak{u}'(\mathfrak{x})) \leq \mathfrak{f}(\mathfrak{x}), \quad \forall \mathfrak{x} \in \mathfrak{E},$$

(2.2) admits a solution u^{*} fulfilling

$$d_{\mathfrak{E}}(\mathfrak{u}^{*}(\mathfrak{x}), \mathfrak{u}'(\mathfrak{x})) \leq \sigma\mathfrak{f}(\mathfrak{x}), \quad \forall \mathfrak{x} \in \mathfrak{E}$$

3. Main Result

In this section, the $\mathcal{UHS}$ has been investigated considering the equation:

$$\mathfrak{u}(\mathfrak{x}') = \mathfrak{g}^{*}(\mathfrak{x}) + \int_{x_0}^{\mathfrak{x}'} \Psi(\mathfrak{x}', z', \mathfrak{u}(z')dz'; \quad \forall \mathfrak{x}' \in [x_0, x_1], \tag{3.1}$$

where $\mathfrak{u} \in C([x_0, x_1])$ and Ψ a function defined by:

$$\Psi: [x_0, x_1] \times [x_0, x_1] \times \mathfrak{R} \to \mathfrak{R}$$
$$(\mathfrak{x}', z', y') \mapsto \Psi(\mathfrak{x}', z', y')$$

Postulate that assumptions below hold:

$a)$ $\Psi \in C([x_0, x_1] \times [x_0, x_1] \times \Re)$,

$b)$ there exists $\kappa > 0$ $such that$ $\forall \mathfrak{x}', z' \in [x_0, x_1]$, $\forall \mathfrak{u}', \mathfrak{w}' \in \Re$

$$|\Psi(\mathfrak{x}', z', u') - \Psi(\mathfrak{x}', z', \mathfrak{w}')| \leq \kappa |\mathfrak{u}' - w|$$

$c)$ $\kappa |x_1 - x_0| < 1$

$d)$ $\mathfrak{g}^* \in C([x_0, x_1])$

Let us consider the operator

$$A: C([x_0, x_1]) \to C([x_0, x_1]); \quad (A\mathfrak{u})(\mathfrak{x}') = \mathfrak{g}^*(\mathfrak{x}') + \int_{x_0}^{\mathfrak{x}'} \Psi(\mathfrak{x}', z', u(z'))\, dz', \quad \forall \mathfrak{x}' \in [x_0, x_1].$$

It is easily seen that A is well defined. The $\mathcal{UHS}$ of the problem $A\mathfrak{u} = \mathfrak{u}$ is ensuresd by the theorem below

Theorem 3.1: *Suppose* $\mathfrak{u} \in C([x_0, x_1])$, *and* $\varepsilon > 0$ *such that*

$$d_{\mathfrak{s}}(u, A\mathfrak{u}) \leq \varepsilon, \tag{3.2}$$

then, there exists $\rho > 0$ *and a unique* $\mathfrak{u}^* \in C([x_0, x_1])$ *satisfying*

$$d_{\mathfrak{s}}(\mathfrak{u}, \mathfrak{u}^*) \leq \rho\varepsilon,$$

Moreover, the Ulam-Heyers stability of the equation $A\mathfrak{u} = \mathfrak{u}$ *is is ensured.*

Proof: Set $\mathfrak{X}' = C([x_0, x_1])$ and consider the MS $(\mathfrak{X}',\ d_{\mathfrak{s}})$ with

$$d_{\mathfrak{s}}(\mathfrak{u}', \mathfrak{w}') = \mathfrak{s} \sup_{\mathfrak{x}' \in [x_0, x_1]} |\mathfrak{u}'(\mathfrak{x}') - \mathfrak{w}'(\mathfrak{x}')|, \quad \forall \mathfrak{u}', \mathfrak{w}' \in \mathfrak{X}',$$ and $\mathfrak{s}$ a real number satisfying $\mathfrak{s} \geq 1$.

First, we claim that A is a contracting operator in the $b_{\mathfrak{v}}(\mathfrak{s}) -$ MS $(\mathfrak{X}', d_{\mathfrak{s}})$. Indeed,

$$\begin{aligned}
|A\mathfrak{u}'(\mathfrak{x}') - A\mathfrak{w}'(\mathfrak{x}')| &= |\mathfrak{g}^*(\mathfrak{x}') + \int_{x_0}^{\mathfrak{x}'} \Psi(\mathfrak{u}', z', \mathfrak{u}'(z'))dz - \mathfrak{g}^*(\mathfrak{x}') - \int_{x_0}^{\mathfrak{x}'} \Psi(\mathfrak{x}', z', \mathfrak{w}'(z'))dz'| \\
&\leq \int_{x_0}^{\mathfrak{x}'} |\Psi(\mathfrak{x}', z', \mathfrak{u}'(z')) - \Psi(\mathfrak{x}', z', \mathfrak{w}'(z'))|dz' \\
&\leq \kappa \int_{x_0}^{\mathfrak{x}'} |\mathfrak{u}'(z') - \mathfrak{w}'(z')|dz' \\
&\leq \kappa \sup_{z' \in [x_0, x_1]} |\mathfrak{u}'(z') - \mathfrak{w}'(z')| \int_{x_0}^{\mathfrak{x}'} dz \\
&\leq \frac{1}{\mathfrak{s}} \kappa |x_1 - x_0| d_{\mathfrak{s}}(\mathfrak{u}', \mathfrak{w}')
\end{aligned}$$

we deduce that, $d_{\mathfrak{s}}(A\mathfrak{u}', A\mathfrak{w}') \leq \kappa |x_1 - x_0| d_{\mathfrak{s}}(\mathfrak{u}', \mathfrak{w}')$

Since $\eta = \kappa |x_1 - x_0| < 1$, by Theorem 2.3, there exits a unique $\mathfrak{u}^* \in \mathfrak{X}'$ so that $A\mathfrak{u}'^* = \mathfrak{u}^*$.

Now, let $\mathfrak{w}' \in \mathfrak{X}'$ and $\varepsilon > 0$ such that $d_{\mathfrak{s}}(\mathfrak{w}', A\mathfrak{w}') \leq \varepsilon$. As $(\mathfrak{X}', d_{\mathfrak{s}})$ is a $b_{\mathfrak{v}}(\mathfrak{s}) -$ MS, therefore :

$d_{\mathfrak{s}}(\mathfrak{w}', \mathfrak{u}'^{*}) \leq \mathfrak{s}[d(\mathfrak{w}', \mathfrak{u}'_1) + d(\mathfrak{u}'_1, \mathfrak{u}'_2) + d(\mathfrak{u}'_2, \mathfrak{u}'_3) + \cdots + d(\mathfrak{u}'_{\mathfrak{v}}, \mathfrak{u}'^{*})]$,

where $\mathfrak{u}'_1, \mathfrak{u}'_2, \ldots, \mathfrak{u}'_{\mathfrak{v}} \in \mathfrak{X}'$, are distinct from each other and all different from $\mathfrak{u}'^{*}$ and $\mathfrak{w}'$.

We put $\alpha_n = d_{\mathfrak{s}}(\mathfrak{u}'_n, \mathfrak{u}'_{n+1})$ and let us consider the sequence defined by $\mathfrak{u}'_1 = A\mathfrak{w}' \textit{ and } \mathfrak{u}'_{n+1} = A\mathfrak{u}'_n$.

$$\alpha_1 = d_{\mathfrak{s}}(\mathfrak{u}'_1, \mathfrak{u}'_2)$$

$$\alpha_2 = d_{\mathfrak{s}}(\mathfrak{u}'_2, \mathfrak{u}'_3) = d_{\mathfrak{s}}(A\mathfrak{u}'_1, A\mathfrak{u}'_2) \leq \eta d_{\mathfrak{s}}(\mathfrak{u}'_1, \mathfrak{u}'_2),$$

$$\alpha_3 = d_{\mathfrak{s}}(\mathfrak{u}'_3, \mathfrak{u}'_4) = d_{\mathfrak{s}}(A\mathfrak{u}'_2, A\mathfrak{u}'_3) \leq \eta^2 d_{\mathfrak{s}}(\mathfrak{u}'_1, \mathfrak{u}'_2),$$

by recurence we deduce that $\alpha_{v-1} = \eta^{v-2} d_{\mathfrak{s}}(\mathfrak{u}'_1, \mathfrak{u}'_2)$,

and $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{v-1} = (\eta^1 + \eta^2 + \eta^3 + \cdots + \eta^{v-2}) d_{\mathfrak{s}}(\mathfrak{u}'_1, \mathfrak{u}'_2)$,

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{v-1} = \eta \frac{1-\eta^{v-2}}{1-\eta} d_{\mathfrak{s}}(\mathfrak{u}'_1, \mathfrak{u}'_2).$$

On the other hand, we have

$$\begin{aligned} d_{\mathfrak{s}}(\mathfrak{u}'_1, \mathfrak{u}'_2) &= d_{\mathfrak{s}}(A\mathfrak{w}', A\mathfrak{w}'_1) \\ &= d_{\mathfrak{s}}(A(A\mathfrak{w}', A\mathfrak{w}') \\ &\leq \eta d_{\mathfrak{s}}(A\mathfrak{w}', \mathfrak{w}') \\ &\leq \eta\varepsilon. \end{aligned}$$

Hence,

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{v-1} \leq \frac{\eta^2}{1-\eta}(1 - \eta^{v-2})\varepsilon.$$

The last step is to calculate $d_{\mathfrak{s}}(\mathfrak{u}'_{\mathfrak{v}}, \mathfrak{u}'^{*})$.

$$\begin{aligned} d_{\mathfrak{s}}(\mathfrak{u}''_{\mathfrak{b}}, \mathfrak{u}'^{*}) &= d_{\mathfrak{s}}(A\mathfrak{u}'_{\mathfrak{b}-1}, A\mathfrak{u}'^{*}) \\ &= \eta d_{\mathfrak{s}}(\mathfrak{u}'_{\mathfrak{b}-1}, \mathfrak{u}'^{*}) \\ &\leq \eta^2 d_{\mathfrak{s}}(\mathfrak{u}'_{\mathfrak{b}-2}, \mathfrak{u}'^{*}) \\ &\leq \eta^3 d_{\mathfrak{s}}(\mathfrak{u}'_{\mathfrak{b}-3}, \mathfrak{u}'^{*}) \end{aligned}$$

by proceeding in the same way we get

$$d_{\mathfrak{s}}(\mathfrak{u}'_{\mathfrak{v}}, \mathfrak{u}'^{*}) \leq \eta^{v} d_{\mathfrak{s}}(\mathfrak{w}', \mathfrak{u}'^{*}).$$

Finaly, we have:

$$d_{\mathfrak{s}}(\mathfrak{w}', \mathfrak{u}'^{*}) \leq \mathfrak{s}(\varepsilon + \frac{\eta^2}{1-\eta}(1 - \eta^{\mathfrak{v}-2})\varepsilon + \eta^{\mathfrak{v}} d_{\mathfrak{s}}(\mathfrak{w}', \mathfrak{u}'^{*}))$$

then,

$$(1 - \mathfrak{s}\eta^{\mathfrak{v}}) d_{\mathfrak{s}}(\mathfrak{w}', \mathfrak{u}'^{*}) \leq \mathfrak{s}(\varepsilon + \frac{\eta^2}{1-\eta}(1 - \eta^{\mathfrak{v}-2})\varepsilon),$$

because $0 < \eta^{\mathfrak{v}}\mathfrak{s} < 1$, we have

$$d_{\mathfrak{s}}(\mathfrak{w}', \mathfrak{u}'^{*}) \leq \frac{\mathfrak{s}}{1-\mathfrak{s}\eta^{\mathfrak{v}}}(1 + \frac{\eta^2}{1-\eta}(1 - \eta^{\mathfrak{v}-2}))\varepsilon.$$

That is $d_{\mathfrak{s}}(\mathfrak{w}', \mathfrak{u}'^{*}) \leq \mathfrak{f}(\varepsilon)$ with

$$\mathfrak{f}(\varepsilon) = \frac{\mathfrak{s}}{1-\mathfrak{s}\eta^{\mathfrak{v}}}\left(1 + \frac{\eta^2}{1-\eta}\left(1 - \eta^{\mathfrak{v}-2}\right)\right)\varepsilon, \quad \text{and } \mathfrak{f}(0) = 0.$$

By Definition 2.4, the Ulam-Heyers stability of the equation $A\mathfrak{u} = \mathfrak{u}$ is guaranteed.

4. Summary

In this study, we analyze the Ulam-Hyers stability of a nonlinear Volterra integral equation. Our approach utilizes the $b_{\mathfrak{v}}(\mathfrak{s})$ metric to demonstrate the requisite stability. We establish a connection between the function bounding the distance between the exact solution of the fixed-point problem, the parameter $\mathfrak{s}$ and the integer $\mathfrak{v}$. By comparing the assumptions imposed in this study with those suggested in [7], we observe that we have relaxed them.

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