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Solving the fuzzy 1-center problem with an innovative ranking method for fuzzy numbers

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Abstract

For modeling real-world problems in mathematical form, we encounter uncertain and ambiguous quantities. In such situations, the efficient fuzzy set (FS) theory is employed to model these uncertain and ambiguous quantities in the form of fuzzy numbers (FNs). Since comparing FNs is essential in most optimization problems, we are compelled to utilize a suitable ranking method (RM). In this paper, the limitations of the RMs discussed in the literature are addressed, and a new revised approach for ranking various types of FNs is introduced. Additionally, it has been demonstrated that this innovative RM consistently ensures alignment among the rankings of fuzzy quantities and their associated representations. To illustrate the usages, advantages, and efficiency of the proposed method for ranking fuzzy numbers (RFN), we will present several examples, and several comparative examples are provided. Finally, in order to demonstrate the usage and applicability of our recommended RM, a practical application called the fuzzy 1-center problem is solved.

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1. Introduction

Ordering of FNs plays a crucial and determinative step in many fields of operations research, such as fuzzy decision-making problems [1], fuzzy optimization problems [2-5] and so forth. For this reason, many different ranking approaches were presented by researchers (see [6-10]). It should be noted that some of the RMs can only be applied in cases where the membership functions exhibit specific characteristics, including triangular, trapezoidal, normal, and similar types. A novel RM has been introduced by Nasseri et al. [11], which relies on the position of the incircle's center in triangles. This method leverages the unique characteristics of each triangle's incircle to enhance the ranking process. Additionally, the proposed method demonstrates alignment among the rankings of fuzzy quantities and their associated representations. Furthermore, Hussein and Abood addressed the challenges associated with fully fuzzy games, where the data involved is uncertain [12]. They implemented a fuzzy payoff framework that employs three distinct ranking functions to effectively tackle this issue.

Ingle and Ghadle [13] introduced a novel ranking function aimed at addressing fuzzy programming problems (FFPP) involving hexagonal fuzzy numbers (HFN). Dutta [14] proposed an uncomplicated general approach grounded in the idea of the exponential area of input FNs; however, this method has certain limitations. For instance, it fails to differentiate between crisp-valued FNs and regular FNs. In 2020, Saneifard and Saneifard [15] proposed a method to demonstrate that addressing fuzzy linear problems leads to precise results. Additionally, Dombi and Jónás [16] considered a fuzzy relation functioning as an index of preference intensity based on probability, presenting a closed formula to compute the integral necessary for determining this measure applicable to FSs characterized by trapezoidal membership functions.

In Pourabdollah [17], a cross-domain statistical approach is employed to evaluate defuzzification algorithms by utilizing the concepts of pairwise similarities and distance operators within FSs. This method assesses the correlation between variations in the defuzzified values of two FSs, allowing for a comparison based on their similarity or distance. Additionally, this approach facilitates the comparison of two support-based defuzzification techniques typically used in fuzzy controllers, as well as two level-based methods that are commonly applied for ranking FNs. Sam'an and Dasril [18] introduce a new method for evaluating and ordering generalized trapezoidal FNs using integral values to improve accuracy and consistency in decision-making processes. Salas-Molina et al. investigate [19] the necessary conditions for achieving total orderings of FNs using lexicographic methods, providing theoretical insights and practical examples to demonstrate their findings. Palash Dutta et al. [20] present an advanced technique for ordering FNs, which is designed to aid in decision-making under uncertain conditions.

1.1. *Literature review and problem statement*

Chu and Tsao [21] presented an innovative method for ordering FNs based on the measurement of distance from the centroid to the initial point. Abbasbandy and Asady [22] proposed a modified version of the distance-based method, which they termed the "sign distance method" for RFNs. A novel approach utilizing distance minimization to RFNs was introduced by Asady and Zendehnam [23]. In their method, all triangular fuzzy numbers (TFNs) $u = (x_0, \sigma, \beta)$ with $x_0 = \frac{\sigma-\beta}{4}$ and also all trapezoidal fuzzy numbers $u = (x_0, y_0, \sigma, \beta)$ with $x_0 + y_0 = \frac{\sigma-\beta}{2}$ have the same results. Nonetheless, it is evident that these FNs belong to different categories. The weakness of their method is that it was modified by Abbasbandy and Hajjari [24], and they present another ranking approach. Their method also has some disadvantages. For example, for all trapezoidal fuzzy numbers $u = (x_0, y_0, \sigma, \sigma)$ with different values of σ and all TFNs $u = (x_0, \sigma, \sigma)$ with different values of σ the same order is obtained based on their proposed RM. To address this limitation, Ezzati et al.[25] proposed an alternative approach. Nonetheless, Ezzati et al.'s approach fails to facilitate precise ranking of FNs due to the fact that it yields identical orders for all TFNs with the same average value of $\frac{x_0+y_0}{2}$, regardless of their differing σ values. To overcome this shortcoming, a modified approach is proposed by Ezzati et al.[26]. This method has other weaknesses. For example for any two TFNs $u_1 = (x_1, \sigma_1, \beta_1)$ and $u_2 = (x_2, \sigma_2, \beta_2)$ with $x_1 - x_2 = \frac{\sigma_1-\sigma_2}{6} = \frac{\beta_2-\beta_1}{6}$ it is yielded that $u_1 \sim u_2$. Nonetheless, it is evident that these FNs are from different classes. Also, according to the Ezzati el al.'s method [26], the same result is obtained for two trapezoidal fuzzy numbers $u_1 = (x_1, y_1, \sigma_1, \beta_1)$ and $u_2 = (x_2, y_2, \sigma_2, \beta_2)$ that satisfy the following relations:

$$\sigma_1 - \sigma_2 = 5(x_1 - x_2) + (y_1 - y_2)$$
$$\beta_1 - \beta_2 = -(x_1 - x_2) - 5(y_1 - y_2),$$

Clearly, these FNs belong to different classes. Therefore, this RM results in an unreasonable order. This paper introduces an innovative approach designed to address the limitations of the previously discussed methods. Moreover, it can be shown that the proposed method reliably maintains consistency between the rankings of TFNs and their corresponding representations. The efficiency and applicability of our presented ranking method are shown by solving a practical example called the 1-center problem.

The structure of the paper is organized as follows. Section 2 presents FNs along with some fundamental definitions. In Section 3, our new method is proposed, and certain characteristics of this new method are examined. Afterward, in Section 4, a comparative study is conducted using several numerical examples and various methods from the existing literature. In Section 5, the applicability and efficiency of our proposed method for solving a particular facility location problem are demonstrated. In conclusion, the findings of this study are presented in Section 6.

2. Preliminaries

This section outlines basic definitions and properties of FNs. For additional information, please see [27-29]).

Definition 2.1: ([29]). Let the function u, which maps real numbers to the interval $[0,1]$ be a FS that satisfies the following statements:

(i) The function u satisfies the property of upper semi-continuity,

(ii) Outside the interval $[0,1]$, $u(x) = 0$,

(iii) There exist four values a, b, c and d, ordered as $a \leq b \leq c \leq d$, all of which are real numbers and

(a) $u(x)$ increases monotonically over the interval [a,b],

(b) $u(x)$ decreases monotonically over the interval [c,d],

(c) $u(x) = 1, b \leq x \leq c$.

Thus, the function u can be represented in the following way:

$$u(x) = \begin{cases} f_L^u(x), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ f_R^u(x), & c \leq x \leq d, \\ 0, & \text{otherwise}, \end{cases}$$

Where continuous functions $f_L^u: [a,b] \to [0,1]$ and $f_R^u: [c,d] \to [0,1]$ serve as the left and right membership functions of the FN u, respectively. The collection of all FNs is denoted by E.

Definition 2.2: ([30]). A FN represented in parametric form can be described as a pair of ordered functions, $\left(\underline{u}(r), \overline{u}(r)\right)$, where the value of r lies between 0 and 1, inclusive. His representation must meet the following criteria:

1. The function $\underline{u}(r)$ represents a bounded function that is left-continuous and non-decreasing, defined on the interval $[0,1]$.
2. The function $\overline{u}(r)$ represents a bounded function that is left-continuous and non-decreasing, defined on the interval $[0,1]$.
3. $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1$.

Conversely, while the conditions (1)–(3) behold for the pair of functions $\underline{u}(r)$ and $\overline{u}(r)$, then there exists a unique FN $u \in E$ in which $u(r) = \left(\underline{u}(r), \overline{u}(r)\right), r \in [0,1]$.

Remark 2.1: A crisp number k is a FN that $\underline{u}(r) = \overline{u}(r) = k, 0 \leq r \leq 1$.

Definition 2.3: Trapezoidal FN $u = (x_0, y_0, \sigma, \beta)$ is a popular kind of FN. Its membership function is represented as follows:

$$u(x) = \begin{cases} \frac{x-x_0+\sigma}{\sigma}, & x_0-\sigma \le x \le x_0, \\ 1, & x_0 \le x \le y_0, \\ \frac{y_0+\beta-x}{\beta}, & y_0 \le x \le y_0+\beta, \\ 0, & \text{otherwise.} \end{cases}$$

Where $\sigma > 0$ and $\beta > 0$ are the left and the right fuzziness, respectively. Additionally, it can be expressed in the following parametric representation:

$$\underline{u}(r) = x_0 - \sigma + \sigma r, \quad \overline{u}(r) = y_0 + \beta - \beta r,$$

Definition 2.4: A TFN u is a trapezoidal fuzzy number with $x_0 = y_0$ and is denoted by $u = (x_0, \sigma, \beta)$.

Definition 2.5: Using the extension principle, for any arbitrary FNs $u = \left(\underline{u}(r), \overline{u}(r)\right)$ and $v = \left(\underline{v}(r), \overline{v}(r)\right)$ the addition and scalar multiplication are defined as

$$\begin{aligned} &\left(\underline{u+v}\right)(r) = \underline{u}(r) + \underline{v}(r), && \left(\overline{u+v}\right)(r) = \overline{u}(r) + \overline{v}(r), && \\ &\left(\underline{ku}\right)(r) = k\underline{u}(r), && \left(\overline{ku}\right)(r) = k\overline{u}(r), && k \ge 0, \\ &\left(\underline{ku}\right)(r) = k\overline{u}(r), && \left(\overline{ku}\right)(r) = k\underline{u}(r), && k < 0. \end{aligned} \tag{2.1}$$

Remark 2.2: ([31]). The negation of the FN $u = (x_0, y_0, \sigma, \beta)$ is represented as $-u = (-y_0, -x_0, \beta, \sigma)$.

Remark 2.3: Assuming that u represents a FN, the expression $u(1) = \left(\underline{u}(1), \overline{u}(1)\right)$ defines its core.

3. A new approach for RFNs based on the parametric form

For any given FN $u \in E$ represented in parametric form $\left(\underline{u}(r), \overline{u}(r)\right)$, $0 \le r \le 1$, the following indices are defined:

$$Mag(u) = \frac{1}{2}\int_0^1 \left(\underline{u}(r) + \overline{u}(r) + \underline{u}(1) + \overline{u}(1)\right) r \, dr$$

and (3.1)

$$Di(u) = -\frac{1}{4}\int_0^1 \left(\underline{u}'(r)\underline{u}(r) + \overline{u}(r)\overline{u}'(r)\right) dr$$

Where $\underline{u}'$ and $\overline{u}'$ are derivatives of $\underline{u}$ and $\overline{u}$, respectively.

Now, for the FN u, the new index named MD is introduced as follows:

$$MD(u) = Mag(u) + \lambda Di(u) \tag{3.2}$$

Where

$$\forall u, v \in E: \lambda = \begin{cases} 0 & Mag(u) \neq Mag(v), \\ 1 & Mag(u) = Mag(v) \geq 0, \\ -1 & Mag(u) = Mag(v) < 0. \end{cases} \tag{3.3}$$

According to the MD index, any two arbitrary FNs $u, v \in E$ are compared as follows:

1) $MD(u) < MD(v)$ is equivalent to $u \prec v$.
2) $MD(u) > MD(v)$ is equivalent to $u \succ v$.
3) $MD(u) = MD(v)$ is equivalent to $u \approx v$.

Then, '$\succcurlyeq$' and '$\preccurlyeq$' are defined as follows:

$u \preccurlyeq v$ is equivalent to $u \prec v$ or $u \approx v$,

$u \succcurlyeq v$ is equivalent to $u \succ v$ or $u \approx v$.

To put it differently, the initial step in this method involves comparing $Mag(u)$ and $Mag(v)$. If $Mag(v)$ equals $Mag(u)$, then the two FNs u and v are ordered using $Di(u)$ and $Di(v)$.

Remark 3.4: Let $u = (x_0, y_0, \sigma, \beta)$ be any arbitrary trapezoidal FN. Then:

$$Mag(u) = \frac{6(x_0+y_0)+(\beta-\sigma)}{12}, \quad Di(u) = \frac{2(\beta y_0 - \sigma x_0)+(\sigma^2+\beta^2)}{8}$$

Also, suppose that $u = (x_0, \sigma, \beta)$ is any arbitrary triangular FN. Then:

$$Mag(u) = \frac{12x_0+(\beta-\sigma)}{12}, \quad Di(u) = \frac{2x_0(\beta-\sigma)+(\sigma^2+\beta^2)}{8}.$$

In the following proposition, it is proved that our new proposed method always guarantees the alignment among the rankings of fuzzy quantities and their associated representations

Proposition 3.1: Let $u, v \in E$ and $u \succ v$. Then, $-u \prec -v$.

Proof: Suppose that $u = \left(\underline{u}(r), \overline{u}(r)\right)$ and $= \left(\underline{v}(r), \overline{v}(r)\right)$. Hence, $-u = \left(-\overline{u}(r), -\underline{u}(r)\right)$ and $-v = \left(-\overline{v}(r), -\underline{v}(r)\right)$. Therefore, it is obtained that:

$$Mag(-u) = \frac{1}{2}\int_0^1 \left(-\overline{u}(r) - \underline{u}(r) - \overline{u}(1) - \underline{u}(1)\right) r\, dr = -Mag(u) \tag{3.5}$$

And

$$Di(-u) = -\frac{1}{4}\int_0^1 \left(\left(-\overline{u}(r)\right)\left(-\overline{u}'(r)\right) + \left(-\underline{u}'(r)\right)\left(-\underline{u}(r)\right)\right) dr = \tag{3.6}$$

$$= -\frac{1}{4}\int_0^1 \left(\overline{u}(r)\overline{u}'(r) + \underline{u}'(r)\underline{u}(r)\right) dr = Di(u)$$

As $u \succ v$, one of the following three cases happened:

Case 1: $Mag(u) > Mag(v)$. Hence, $-Mag(u) < -Mag(v)$. Using Eq. (3.5), it is obtained that $Mag(-u) < Mag(-v)$. Thus, $-u \prec -v$.

Case 2: $Mag(u) = Mag(v) \geq 0$. Then, $\lambda = 1$ **(**for comparing u and v**)**. Therefore, since $MD(u) > MD(v) \geq 0$ it follows that $Di(u) > Di(v)$.

Now, by applying Eq. (3.6), it is obtained that:

$$Di(-u) > Di(-v)$$

Also, using Eq. (3.5), the following relation is yielded:

$$Mag(-u) = Mag(-v) \leq 0$$

Hence, λ is obtained equal to -1 (for comparing $-u$ and $-v$).

Therefore, $MD(-u) = Mag(-u) - Di(-u) < Mag(-v) - Di(-v) = MD(-v)$ and it is concluded that $-u \prec -v$.

Case 3: $Mag(u) = Mag(v) < 0$. Then, $\lambda = -1$ (for comparing u and v) and $Di(u) < Di(v)$.

Using Eq. (3.6), the following inequality is concluded:

$$Di(-u) < Di(-v)$$

And according to Eq. (3.5), it is obtained that:

$$Mag(-u) = Mag(-v) > 0$$

Also, it is yielded that $\lambda = 1$ (for comparing $-u$ and $-v$). Therefore:

$$MD(-u) = Mag(-u) + Di(-u) < Mag(-v) + Di(-v) = MD(-v).$$

Thus, $-u \prec -v$.

4. Comparative examples

In this section, some quantitative illustrations are presented in order to confirm the benefits of our newly proposed method.

Example 4.1: Consider two TFNs $u_1 = (5.5, 3, 0)$ and $u_2 = (5, 0, 3)$, which are represented in Fig. 1.

By applying our newly presented method, it is obtained that:

$$MD(u_1) = 5.25 - 3\lambda = 2.25$$

$$MD(u_2) = 5.25 + 4.87\lambda = 10.12$$

Thus, it can be easily concluded that $u_1 \prec u_2$.

According to Abbasbandy and Hajjari's method [24] $Mag(u_1) = Mag(u_2) = 5.25$. Thus, $u_1 \sim u_2$.

The ranking order based on Ezzati et al.'s method [25] is $u_1 \sim u_2$. Because

$$R(u_1, \delta = 1) = R(u_2, \delta = 1) = 6.75.$$

Also, using Ezzati et al.'s method [26], it is obtained that $R(u_1, \gamma = 1) = R(u_2, \gamma = 1) = 9.5$. Therefore, $u_1 \sim u_2$.

Hence, it can be easily concluded that all of these methods are unable to order these FNs and the corresponding obtained results are misaligned with human intuition. In contrast, our proposed method yields a more logical outcome.

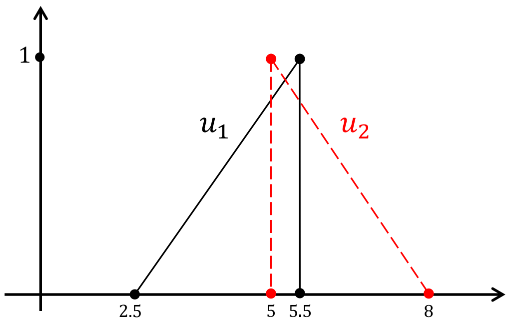

Figure 1
Visualization of the FNs presented in Ex 4.1.

Example 4.2. For analysis, consider the following sets:

Set 1: $A = (6, 8, 9, 9)$ and $B = (4, 10, 1, 1)$, (see Fig. 2).
Set 2: $A = (1, 2, 0, 6)$ and $B = (2, 3, 6, 0)$, (see Fig. 3).
Set 3: $A = (1, 1, 6)$ and $B = (1, 2, 2, 1)$, (see Fig. 4).

In Table 1, the comparative results are given, which conclude the following:

(1) The two FNs A and B in Set 1 cannot be ranked according to Abbasbandy and Hajjari's [24] and Ezzati et al.'s methods [26]. However, based on the Ezzati et al. [25] method, Palash Dutta et al. [20], and our proposed method, the same reasonable order is achieved.

(2) The FNs identified as A and B in Set 2 cannot be ranked using Abbasbandy and Hajjari [24], Ezzati et al. [25] and Ezzati et al. [26] methods. Nevertheless, the outcome produced by our proposed method is logical. And clearly, the RM of Palash Dutta et al.[20] provides a completely incorrect ranking for the two numbers.

Table 1
Comparison of Example 4.2. results.

Fuzzy numbers	**New method**	**Abbasbandy and Hajjari [24]**	**Ezzati et al. [25]**	**Ezzati et al. [26]**	**Palash Dutta et al. [20]**
A	31.75	7	17	0.5	8
B	8.75	7	11	0.5	4
In Set 1	$A \succ B$	$A \sim B$	$A \succ B$	$A \sim B$	$A \succ B$
A	9.5	2	5.5	-0.5	2.25
B	3.5	2	5.5	-0.5	2.75
In Set 2	$A \succ B$	$A \sim B$	$A \sim B$	$A \sim B$	$B \succ A$
A	7.29	1.42	4.92	-0.33	2.67
B	2.04	1.42	3.42	-0.33	1.5
In Set 3	$A \succ B$	$A \sim B$	$A \succ B$	$A \sim B$	$A \succ B$

(3) For FNs A and B in Set 3, corrected order is not obtained based on Abbasbandy and Hajjari's method [24] and also Ezzati et al.'s method [26]. However, using our proposed method, Ezzati et al.'s method [25], and Palash Dutta et al.'s method [20], the same and reasonable result is concluded.

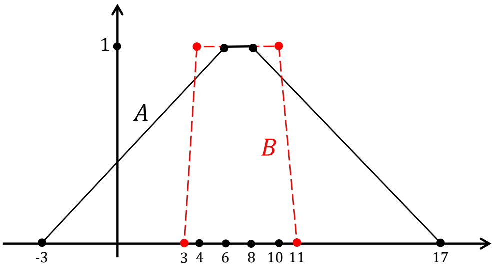

Figure 2
Set 1 of Example 4.2.

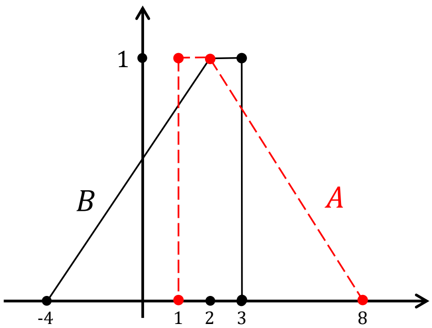

Figure 3
Set 2 of Example 4.2.

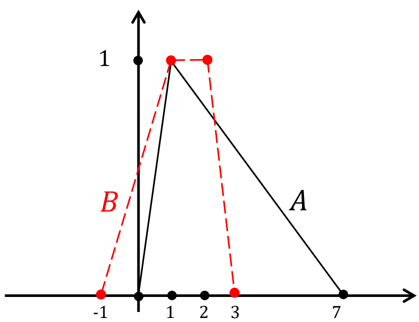

Figure 4
Set 3 of Example 4.2.

Example 4.3: ([25]). Consider the following FNs A, B, C, and D, represented in Fig. 5.

$$A = \left(6 - 4(1 - r)^{\frac{1}{3}}, 8 + 2(1 - r)^3\right),$$
$$B = (1.5 + 3r, 10.5 - 3r),$$
$$C = \left(6 - 4(1 - r^2)^{\frac{1}{2}}, 6 + 4(1 - r^2)^{\frac{1}{2}}\right),$$
$$D = (1 + 5r, 11 - 5r).$$

Using the proposed method, the following results are obtained:

$$MD(A) = 6.4 + 0.5\lambda$$
$$MD(B) = 6 + 4.5\lambda$$
$$MD(C) = 6 + 4\lambda$$
$$MD(D) = 6 + 6.25\lambda$$

Clearly, it follows that $A \succ D \succ B \succ C$. The order obtained using our proposed method and the method proposed by Ezzati et al. [25] is the same. However, based on Abbasbandy and Hajjari's method [24] and Ezzati et al.'s method [26], these FNs are ordered as $A \succ D \sim B \sim C$ and $D \succ C \succ A \succ B$, respectively, which are illogical and misaligned with human intuition.

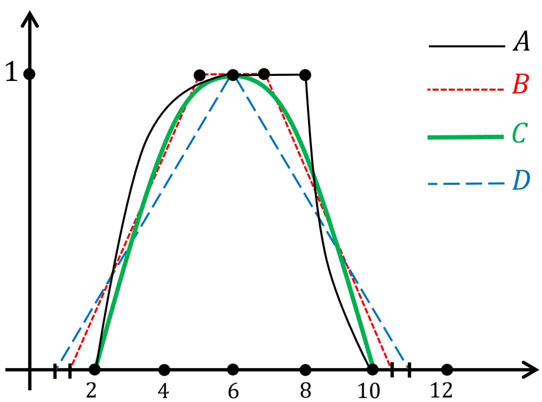

Figure 5
FNs of Example 4.3.

5. An application of our presented method in facility location problems

In this section, our newly proposed ranking approach is utilized to address the 1-center problem in facility location. Consider the problem of finding the best place among existing points $A_j, j = 1, \dots, 5$ (where represent the customers' locations) for establishing a facility center in a way that minimizes the maximum weighted distance. These existing points are presented in Figure 6, their coordinates are provided in Table 2, and the distances between these points are calculated using the Euclidean norm. Also, it is considered that the customers' demands are all fuzzy numbers, as shown in Table 2.

Table 2
The coordinates of the points and fuzzy demands

j	a_{j1}	a_{j2}	$\widetilde{w}_j$
1	1	1	(5,2,1.5)
2	3	7	(8,1,1)
3	5	5	(7,1.5,2.5)
4	7	10	(12,1,2.5)
5	10	3	(15,2,3)

Moreover, for each method, the properties satisfied by the proposed method are listed.

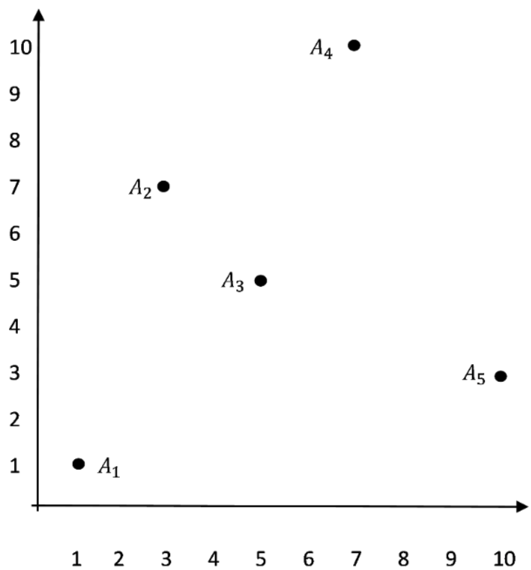

Figure 6
Existing points $A_j, j = 1, \dots, 5$.

The distance matrix, which is obtained using the Euclidean norm, is as follows:

$$D = [d_{ij}]_{5\times 5} = \begin{bmatrix} 0 & 6.325 & 5.657 & 10.817 & 9.22 \\ 6.325 & 0 & 2.828 & 5 & 8.062 \\ 5.657 & 2.828 & 0 & 5.385 & 5.385 \\ 10.817 & 5 & 5.385 & 0 & 7.616 \\ 9.22 & 8.062 & 5.385 & 7.616 & 0 \end{bmatrix}$$

Generally, a fuzzy 1- center problem is formulated as follows ([32]):

$$Z^* = \min_i \max_j (\tilde{w}_j d_{ij}).$$

Let $\tilde{M}_i = max_{j=1,\dots,5}(\tilde{w}_j d_{ij})$. Hence it concluded that:

$$Z^* = \min_i \quad \tilde{M}_i.$$

For $i = 1, \dots, 5$, the following maximum amounts, are obtained using our presented RM:

$$\tilde{M}_1 = \widetilde{max}\begin{pmatrix} (50.6,6.325,6.325), (39.599, 8.486,11.314) \\ , (129.804,10.817,27.042), (138.3,18.44,27.66) \end{pmatrix}$$
$$= (138.3,18.44,27.66)$$

$$\tilde{M}_2 = \widetilde{max}\begin{pmatrix} (31.625,12.65,9.487), (19.796,4.242,5.656) \\ , (60,5,12.5), (120.93,16.124,24.186) \end{pmatrix}$$
$$= (120.93,16.124,24.186)$$

$$\tilde{M}_3 = \widetilde{max}\begin{pmatrix} (28.285,11.314,8.485), (22.616,2.888,2.827) \\ , (64.62,5.385,13.462), (80.775,10.77,16.155) \end{pmatrix}$$
$$= (80.775,10.77,16.155)$$

$$\tilde{M}_4 = \widetilde{max}\begin{pmatrix} (54.0850,21.634,16.2250), (40,5,5), \\ (37.695, 8.078,10.77), (114.24,15.232, 22.848) \end{pmatrix}$$
$$= (114.24,15.232,22.848)$$

$$\tilde{M}_5 = \widetilde{max}\begin{pmatrix} (46.1000 \quad 18.4400 \quad 13.8300), (64.496,8.062,8.0620), \\ (37.695,8.078,10.7700), (91.392, 7.616,19.04) \end{pmatrix}$$
$$= (91.392, 7.616,19.04)$$

Now, based on our presented RM, the following results are obtained:

$$MD(\tilde{M}_1) = MD((138.3,18.44,27.66)) = 139.068 + \lambda\, 920.$$
$$MD(\tilde{M}_2) = MD((120.93,16.124,24.186)) = 212.601 + \lambda\, 349.352.$$
$$MD(\tilde{M}_3) = MD((80.775,10.77,16.155)) = 81.223 + \lambda\, 155.865.$$
$$MD(\tilde{M}_4) = MD((114.24,15.232,22.848)) = 114.874 + \lambda\, 311.768.$$
$$MD(\tilde{M}_5) = MD((91.392, 7.616,19.04)) = 92.344 + \lambda\, 313.58.$$

Therefore, it results that:

$$\tilde{M}_3 < \tilde{M}_5 < \tilde{M}_4 < \tilde{M}_2 < \tilde{M}_1$$

Hence, the optimal fuzzy value is obtained as follows:

$$Z^* = \widetilde{\min}(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{M}_4, \tilde{M}_5) = \tilde{M}_3 = (80.775,10.77,16.155).$$

Therefore, the best place for locating facility center is A_3. Thus, placing the facility center at A_3 minimizes the maximum weighted distance

6. Necessary properties of analysis

In a 2015 study, N. A. Taghi-Nezhad [33] conducted a comprehensive analysis of over a hundred widely used full citation ranking indexes, meticulously examining their strengths and weaknesses. Based on this extensive investigation, Taghi-Nezhad proposed a set of novel properties that serve as a framework for evaluating and accepting new ranking indexes. These properties, outlined in the following sections, provide a rigorous set of criteria to ensure the robustness and reliability of any proposed ranking methodology.

1) Capability to Rank Non-Normal FMs.
2) Evaluating Symmetric FMs with Identical Cores and Different Spreads.
3) Responsiveness to Every Point in the Support of FMs.
4) Simplicity and Computational Efficiency.
5) Linearity of the Ranking Index.
6) Adherence to Common Properties of Fuzzy Quantity Ranking [34, 35].
7) Assessing FMs and Their Symmetries.
8) Unified Ranking for Real and FMs.

The findings from the evaluation of these properties are summarized in Table 3.

Table 3
Evaluation Results of the Proposed Method Based on Taghi-Nezhad's Properties

Essential properties	1	2	3	4	5	6	7	8
Proposed Method	No	Yes	Yes	Medium	No	A_1, A_2, A_3, A_4, A_5	Yes	Yes

Explanation for Table 3

Table 3 presents the results obtained from evaluating each method against the eight specified properties. The findings are organized across columns one to eight, respectively. In the first column, A 'Yes' signifies that the method is capable of differentiating between normal and non-normal FMs, while a 'No' signifies that it cannot.

The second column evaluates whether the method is capable of identifying symmetric FMs that share the same core, yet have different spreads. If it can, the answer is 'Yes'; if not, it is 'No.' Some methods may yield both 'Yes' and 'No' responses, in which case the phrase "dependent on the decision-maker" is noted in the corresponding column to provide a broader perspective.

The third column evaluates whether a ranking approach accounts for every data point within the entire range of FMs. A "Yes" signifies full responsiveness to these points, whereas a "No" suggests the method overlooks portions of the FM's domain.

In the fourth column, each method is evaluated using linguistic classifications including 'Low,' 'Medium,' 'High,' and 'Very High' to reflect time consumption and implementation complexity. As evident from the second to fourth rows, the method lacks simplicity, primarily because it incorporates the Nipper number 'e' alongside intricate mathematical computations.

The fifth column evaluates the linearity of ranking methods, marked as "Yes" (linear) or "No" (non-linear). Rows 2–4 reveal that methods incorporating multiplicative or divisional operations inherently violate linearity principles, disqualifying them from being classified as linear.

The sixth column scrutinizes adherence to seven axiomatic criteria (A_1–A_7) proposed by Wang and Kerre [34, 35]. The proposed ranking method fully satisfies A_1–A_5, but A_6 exhibits inconsistencies in specific scenarios. Under A_7, if $u \preccurlyeq v$ and $w \succcurlyeq 0$ (or $w \preccurlyeq 0$), then $uw \preccurlyeq vw$ (or $uw \succcurlyeq vw$). Notably, no existing RM universally fulfills all seven properties, highlighting inherent limitations in current frameworks.

In the seventh column, a 'Yes' is recorded if a method possesses the symmetry property; otherwise, a 'No' is indicated.

Ultimately, the eighth column classifies each method according to its proficiency in ranking real numbers, employing 'Yes' or 'No' to signify this capability.

7. Conclusion

A novel technique for arranging FMs in order is presented in this research. Our proposed method is a modified approach that is able to overcome all the disadvantages and weaknesses of earlier approaches mentioned in the literature. It is important to highlight that our new approach possesses several notable mathematical properties, which are demonstrated in this paper. It has been demonstrated that FMs with identical modes and symmetrical spreads can be perfectly ranked using our newly proposed method. The proposed methodology demonstrates exceptional capability in accurately ranking diverse FMs, their corresponding images, and even crisp numbers. Through the presentation of numerical examples, we establish that this novel approach exhibits enhanced reasonableness and alignment with human intuition. To conclude, we showcase the practical applicability and effectiveness of this new method by employing it to resolve a significant challenge in facility location, specifically the 1-center problem.

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