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Second kind Chebyshev polynomials for solving logarithmic Volterra-Fredholm integral equations

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Abstract

It is known that the logarithmic Volterra integral equations are so hard to find its exact analytic solution, so we are obliged to find a new numerical method strong and reliable.. In this work, we develop a technical approximation to obtain a numerical solution of the logarithmic Volterra and Fredholm integral equations, This new approximation is based on the Shifted second Chebyshev polynomials the unknown function will be approximated by a truncated sum of shifted Chebyshev Polynomials. our spectral method sends the given equation into an algebraic system of equations with the unknown expansion coefficients. The convegence, the efficiency and the high accuracy of this method are realized by many examples.

Subject Classification: *45B05, 45D05, 45L05, 45E10.*

Keywords: *Shifted second Chebyshev polynomials, Logarithmic kernel Volterra and Fredholm integral equations, Collocation and projection methods.*

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1. Introduction

Many problems arising in mathematical physics, applied mathematics or biological system, can be transformed into integral equation with logarithmic kernel as the form

$$\phi(x)-\int_a^x \ln(x-\tau)\phi(\tau)d\tau-\int_a^b k(x,\tau)\phi(\tau)d\tau=f(x), \tag{1}$$

Where the second member $f(x)$ is a given continuous function defined on $[a,b]$, $k(x,t)$ is a given continuous function defined for every pair (x,t) in $[a,b]\times[a,b]$ and $\phi(x)$ is an unknown density function in $[a,b]$ to looked for. Denoting the equation (1) by the operator equation

$$\phi(x)-A_1\phi(x)-A_2\phi(x)=f(x), \tag{2}$$

where the operators $A_1\phi(x)$ and $A_2\phi(x)$ are defined by

$$A_1\phi(x)=\int_a^x \ln(x-\tau)\phi(\tau)d\tau,\ A_2\phi(x)=\int_a^b k(x,\tau)\phi(\tau)d\tau, \tag{3}$$

or still

$$A_1\phi(x)=\int_a^x \ln(x-\tau)(\phi(\tau)-\phi(x))d\tau+\phi(x)\int_a^x \ln(x-\tau)d\tau. \tag{4}$$

Specific methods are proposed to approximate a logarithmic Volterra integral equations. The author in [1,2] observes that with the analysis is feasible, Abel equations of the first kind may be reduced to Volterra equations of the second kind having a definite and significant role. In [3] the authors introduce in [3] the authors introduce in the integral equation a change of variable for to obtain an equation with a perturbed kernel and give a smooth solution. The work in [4] treats the conventional boundary element methods where the sparse matrix is replaced by the dense one. The product integration for nonlinear weakly singular integral leads to the convergence analysis in [5]. A Finite Difference Method for the Smooth Solution of weakly Volterra integral equations. This method is discussed for example, in [6]. An efficient discrete collocation method for solving a logarithmic Volterra integral equations is investigated by the author in [7]. The authors in [8] suggest a numerical method based on the hat functions for solving a stochastic integral equation. A a modified linear and quadratic spline approximation are discussed in [9, 10], the authors in [11] give a numerical approach for approximating the solution of second kind Volterra integral equation with Logarithmic kernel using Block Pulse Functions (BPFs) and Taylor series expansion.

2. Chebyshev polynomials

i. *The Second Chebyshev polynomials U_n*

The second Chebyshev polynomial $U_n(x)$ is a polynomial of degree n, defined as

$$U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with} \quad x=\cos\theta,$$

where the element $x\in[-1,1]$, this leads the corresponding element $\theta\in[0,\pi]$. Note that, we can see $U_0(x)=1$, $U_1(x)=2x$ and by the recurrence we get

$$\sin(n+1)\theta+\sin(n-1)\theta=2\cos\theta\sin n\theta, \tag{5}$$

or still the fundamental relation

$$U_{n+1}(x)=2xU_n(x)-U_{n-1}(x), \quad n=1,2,3,.... \tag{6}$$

We note that the functions $\{U_n(x), n\geq 0.\}$ is a complete system and orthogonal on the interval $[-1,1]$ with its knowen weight function $w(x)$ = $(1-x^2)^{\frac{1}{2}}$ this implies that, we define a new system $S_n(x)$ given by

$$\left\{S_0(x)=\sqrt{\frac{2}{\pi}}U_0(x),...S_n(x)=\sqrt{\frac{2}{\pi}}U_n(x)...\right\}, \tag{7}$$

form an orthonormal closed system on $[-1,1]$ with respect its weight function $w(x)$ = $\sqrt{1-x^2}$. That is to say

$$\left\langle S_k(x),S_l(x)\right\rangle=\int_{-1}^{1}S_k(x)S_l(x)\sqrt{1-x^2}\,dx=\begin{cases}0 & \text{if} \quad k\neq l\\ 1 & \text{if} \quad k=l\end{cases} \tag{8}$$

ii. *Shifted Chebyshev Polynomial*

For the construction of the shifted Chebyshev polynomials we use the change of variable $x=\frac{2}{b-a}t-\frac{b+a}{b-a}$. So, the shifted Chebyshev polynomials $U_n^s(t)$ with $t\in[a,b]$ and $a,b\in R$, $\infty\prec a\prec b\prec\infty$ is given as $U_n^s(t)$ = $U_n\left(\frac{2t-b-a}{b-a}\right)$ and so $U_0^s(x)$ = 1, $U_1^s(x)$ = $2\left(\frac{2x-b-a}{b-a}\right)$.

Therefore, we get the fundamental relation for the shifted polynomial

$$U_n^s(x)=2\left(\frac{2x-b-a}{b-a}\right)U_{n-1}^s(x)-U_{n-2}^s(x), \quad n=2,3,.. \tag{9}$$

We note that the functions $\{U_n^s(x), n \geq 0\}$, is a complete and orthogonal on the interval $[a,b]$ with its new weight function

$$w^s(x) = \sqrt{1 - \left(\frac{2x-b-a}{b-a}\right)^2} \tag{10}$$

and so the elements $S_n^s(x)$ given by

$$\left\{ S_n^s(x) = \sqrt{\frac{2}{b-a}} S_n\left(\frac{2x-b-a}{b-a}\right), .., \ n \geq 0 \right\}, \tag{11}$$

Forms on the interval $[a,b]$ an orthonormal system with its weight function (10). In other words

$$\begin{aligned} \left\langle S_k^s(x), S_l^s(x) \right\rangle &= \int S_k^s(x) S_l^s(x) \sqrt{1 - \left(\frac{2x-b-a}{b-a}\right)^2} dx \\ &= \int S_k\left(\frac{2x-b-a}{b-a}\right) S_l\left(\frac{2x-b-a}{b-a}\right) w^s(x) \frac{2}{b-a} dx \end{aligned} \tag{12}$$

with the change of variables $t = \left(\frac{2x-b-a}{b-a}\right)$, we obtain (8).

3. Discretization of integral equation

Applying the method of collocation to our equation (1) this later will be transformed into an algebraic system system on the interval $[a,b]$. For this replacing the unknown density function $\phi(x)$ by its approximation $\phi_N(x)$ given as

$$\phi_N(x) \approx \sum_{k=0}^{N} \alpha_k S_k^s(x), \tag{13}$$

where $S_n^s(x)$ represents the nth shifted second Chebyshev polynomials. substituting the expression (13) into the given equation (1) we obtain the approximate equation of (1) as

$$\sum_{k=0}^{N} \alpha_k S_k^s(x) - \int_a^x \ln(x-\tau) \sum_{k=0}^{N} \alpha_k S_k^s(\tau) - \int_a^b k(x,\tau) \sum_{k=0}^{N} \alpha_k S_k^s(\tau) = f(x). \tag{14}$$

Seeking for the coefficients of Fourier α_k such that (14) is verified on the interval $[a,b]$. For this, choosing the equidistant point of collocation reparting as follows

$$\tau_\sigma = a + \sigma h, \quad h = \frac{b-a}{N}, \ \sigma = 0,1,...N. \tag{15}$$

Define the residual of the equation (14)

$$R_N(x) = \sum_{k=0}^{N} \alpha_k S_k^s(x) - A_1\left(\sum_{k=0}^{N} \alpha_k S_k^s(\tau)\right) - A_2\left(\sum_{k=0}^{N} \alpha_k S_k^s(\tau)\right) - f(x). \tag{16}$$

Where at the points of collocation we have

$$R_N(x_\sigma) = 0, \quad \sigma = 0,1,....N, \tag{17}$$

the expression (14) is converted into an algebraic system of equations.

Theorem : *Let* $G = A_1 + A_2 : X \to X$ *be a sum compact operators* A_1 *and* A_2 *and supposing that*

$$(I-G)\phi = f, \tag{18}$$

admits a unique solution. For the projections $P_N \ X \to X_N$ *with a condition*

$$\left\| P_N G - G \right\| \to 0, \quad N \to \infty.$$

The approximate of the equation (1) given by

$$\phi_N - P_N G \phi_N = P_N f, \tag{19}$$

has always a unique solution for any $f \in X$ *provided that N,* is *sufficiently large, besides*

$$\left\| \phi - \phi_N \right\| \le M \left\| \phi - P_N \phi \right\|, \tag{20}$$

where M is some positive constant depending on G.

Proof: The operator $(I-G)$ is invertible and so the operator $(I - P_N G)$ is also invertible for sufficiently large *N* applying the production of the projection P_N to the equation (18) the equation comes

$$P_N \phi - P_N G \phi = P_N f, \tag{21}$$

or still

$$\phi - P_N G \phi = P_N f + \phi - P_{Nn} \phi. \tag{22}$$

Subtracting (22) from (19) we get

$$(I - P_N G)(\phi - \phi_N) = (I - P_N)\phi.$$

Therefore the result (20) is achieved.

4. Illustrating examples

Example 1: Let the logarithmic integral equation

$$\phi(x)-\int_0^x \ln(x-\tau)\phi(\tau)d\tau-\int_0^1 e^{(x+\tau)}\phi(\tau)d\tau = f(x), \tag{23}$$

where the function

$$f(x)=\frac{3}{4}x^2+x-e^x-\frac{1}{2}x^2\ln x,$$

is chosen in the way that

$$\phi(x)=x.$$

Applying the second shifted polynomial of Chebyshev $S_8^s(x)$ to the function $\phi_N(x)$ and find the solution of the algebraic system in order to get the approech solution of $\phi(x)$, and errors.

Table 1

The points of collocaton, the exact solution, the approximate one and the error of the equation (23) using the second shifted Chebyshev polynomial S_8 (x), n=8

Points x	Exact sol	Approx sol	Error
0.0000	0.0000	0.0000	1.4821e-08
0.25000	0.25000	0.25000	3.6202e-08
0.50000	0.50000	0.50000	7.0938e-08
0.75000	0.75000	0.75000	1.0005e-07
1.00000	1.00000	1.00000	1.4795e-07

Example 2: Let the logarithmic integral equation

$$\phi(x)-\int_0^x \ln(x-\tau)\phi(\tau)d\tau-\int_0^1\left(\sqrt{x-\tau}\right)\phi(\tau)d\tau = f(x), \tag{24}$$

such that the function

$$f(x)=x^2+\frac{11}{18}x^3+\frac{2}{105}(x-1)^{\frac{3}{2}}(8x^2+12x+15)-\frac{16x^2}{105}(x)^{\frac{3}{2}}-\frac{1}{3}x^3\ln(x),$$

is chosen in the way that

$$\phi(x)=x^2$$

Applying the second shifted polynomial of Chebyshev $S_8^s(x)$ to the function $\phi_N(x)$ and find the solution of the algebraic system in order to get the approech solution of $\phi(x)$, and errors.

Table 2

The points of collocaton, the exact solution, the approximate one and the error of the equation (24) using the second shifted Chebyshev polynomial S_8 (x), n=8

Points x	Exact sol	Approx sol	Error
0.0000	0.0000	0.0000	4.9453e-07
0.25000	0.06250	0.06250	2.5544e-07
0.50000	0.25000	0.25000	4.8371e-07
0.75000	0.56250	0.56250	9.7846e-07
1.00000	1.00000	1.00000	1.6378e-06

Example 3: Let the logarithmic integral equation

$$\phi(x) - \int_0^x \ln(x-\tau)\phi(\tau)d\tau - \int_0^1 (x\tau)\phi(\tau)d\tau = f(x), \tag{25}$$

such that the function

$$f(x) = \frac{1}{(2-x)^2} + x(\ln 2 - 1) - \frac{\ln|x-2|}{(2-x)} + \frac{\ln 2}{(2-x)} - \frac{x\ln|x|}{2(2-x)},$$

is chosen in the way that

$$\phi(x) = \frac{1}{(2-x)^2}$$

Applying the second shifted polynomial of Chebyshev $S_8^s(x)$ to the function $\phi_N(x)$ and find the solution of the algebraic system in order to get the approech solution of $\phi(x)$, and errors.

Table 3

The points of collocaton, the exact solution, the approximate one and the error of the equation (25) using the second shifted Chebyshev polynomial S_8 (x), n=8

Points x	Exact sol	Approx sol	Error
0.0000	0.25000	0.25000	2.9152e-06
0.25000	0.32653	0.32652	8.1942e-07
0.50000	0.44444	0.44444	6.6476e-08
0.75000	0.64000	0.64000	8.5619e-07
1.00000	1.00000	0.99999	4.6824e-06

Example 4: Let the logarithmic integral equation

$$\phi(x)-\int_0^x \ln(x-\tau)\phi(\tau)d\tau-\int_0^1 (x+\tau^2)\phi(\tau)d\tau = f(x), \tag{26}$$

such that the function

$$f(x)=\frac{1}{(1+x)^3}+\frac{x+\ln(x+1)}{2(1+x)^3}-\frac{3}{8}x+\frac{5}{8}-\ln 2-\frac{(x^2+2x)\ln|x|}{2(1+x)^3},$$

is chosen in the way that

$$\phi(x)=\frac{1}{(1+x)^3}$$

Applying the second shifted polynomial of Chebyshev $S_8^s(x)$ to the function $\phi_N(x)$ and find the solution of the algebraic system in order to get the approech solution of $\phi(x)$, and errors.

Table 4

The points of collocaton, the exact solution, the approximate one and the error of the equation (26) using the second shifted Chebyshev polynomial S_8 (x), n=8

Points x	Exact sol	Approx sol	Error
0.0000	1.00000	0.99998	1.7358e-05
0.25000	0.51200	0.51200	4.0756e-06
0.50000	0.29629	0.29629	1.6911e-06
0.75000	0.18658e	0.18658	1.6911e-06
1.0000	0.12500	0.12501	1.1534e-05

Example 5: Let the logarithmic integral equation

$$\phi(x)-\int_0^x \ln(x-\tau)\phi(\tau)d\tau-\int_0^1 \sqrt{(x\tau)}\phi(\tau)d\tau = f(x), \tag{27}$$

such that the function

$$f(x)=x^{\frac{9}{2}}-\frac{2x^{\frac{11}{2}}(-13016+6930\ln 2)}{38115}-\frac{1}{6}\sqrt{x}-\frac{2x^{\frac{11}{2}}(3465\ln x)}{38115},$$

is chosen in the way that

$$\phi(x)=x^{\frac{9}{2}}$$

Applying the second shifted polynomial of Chebyshev $S_8^s(x)$ to the function $\phi_N(x)$ and find the solution of the algebraic system in order to get the approech solution of $\phi(x)$, and errors.

Table 5

The points of collocaton, the exact solution, the approximate one and the error of the equation (27) using the second shifted Chebyshev polynomial S_8 (x), n=8

Points x	Exact sol	Approx sol	Error
0.0000	0.00000	9.0007e-07	9.0007e-07
0.25000	1.9531e-03	1.9529e-03	2.1356e-07
0.50000	4.4194e-02	4.4194e-02	1.2109e-07
0.75000	2.7401e-01	2.7401e-01	8.0923e-08
1.00000	1.0000	0.99999	8.8088e-07

5. Conclusion

This new approximation is based on the Shifted second polynomials of Chebyshev where the unknown density will be approximted by a truncated sum from shifted second polynomials of Chebyshev and substituting this expression of the density into the integral equation (1) so that to get an algebraic system of equations contains $N+1$ unknowns expansion coefficients. The stability of the equation (1) may be examined in terms of the inverse, formally $\phi=(I-A)^{-1}f$ [1,2], the convergence, the efficiency and the high accuracy of this method are realized by many examples being found in table 1-5.

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