

Optimal mixed control problem dominating by triple system of parabolic equations

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Abstract

This paper is devoted to study one of the important subject in the branch of applied mathematics, namely the optimal mixed control vector problem (OMCVP) dominating by triple system of parabolic PDEs (for brief TSPE). The Galerkin approach is employed to demonstrate the existence and uniqueness theorem of a triple state vector solution (TSVS) of the weak formulation system (SWF) of the TSPE when the mixed control vector (MCV) is specified. The continuity of the operator between the MCV and its conforming TSVS is established. The demonstration of the existence of an optimal mixed control vector (OMCV) controlling by TSVS is done. The Fréchet derivative (FD) of the objective function (OF) is secured. The derivation of the adjoint triple system (ATS) associated to the TSVS is found. The theorem for necessary optimality conditions for the problem is demonstrated.

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1. Introduction

The subject of optimal control problem (OCPs) it is considered as a branch of mathematical science which interests about detecting the policy for a control of a dynamical system over time for which the criterion of a certain optimality is attained. The aim is usually to maximize or minimize the objective function which reflects the system performance. In OCPs the variables are classified for two classes; the control and the state variables. To formulate the OCPs, there is a need for a mathematical model, a determination for the index of performance, a determination for the constraints which they are satisfied by control and state, and determination for all the boundary conditions. OCPs in general take a main place in various implementations for real life problems, as in the fields of medicine [1], economic sciences [2], the shuttle of space [4], and the robots [3], besides to many other fields of implementations. In particular, and in the mathematical field, the subject of OCP are usually dominating by both PDEs [5] and ODEs [6]. Mathematical OCPs may be of one of the types; classical or relaxed, and each one of these two type may be of continuous mathematical form or discrete mathematical form.

Through the previous years, a lot of researchers took their attentions about the OCPs in general, and in particular about the classical and continuous type of these problems (CCOCPs). Later these attentions extended to studding CCOCPs dominating by the common three kinds for PDEs, the hyperbolic, elliptic and the parabolic [7]- [9]. These studies encouraged other researchers about studding mixed CCOCPs dominating by more general forms of these three types like couple PDEs [10]-[12], and triple PDEs [13]-[15]. All these studies and their results motivated us to think about studding mixed OCPs dominating by triple system of parabolic PDEs

These researches motivate us to search about the OCP controlling by parabolic TPDEs. This paper seeks about studying the optimal mixed control vector problem (OMCVP) dominating by TSPE with boundary and initial

conditions. The existence theorem of a unique TSVS for the system of weak formulation (SWF) of the TSPE is demonstrated by utilizing the Galerkin approach (GAP) when the mixed control vector (MCV) is specified. The operator between the MCV and its conforming TSVS is established and is demonstrated be continuous. The existence theorem for an optimal mixed control vector (OMCV) governing by the TSVS is demonstrated. The FD of the OF is secured. The mathematical formulation of the adjoint triple system (ATS) that is associated to the TSVS is found. The necessity optimality conditions for the problem is demonstrated.

2. Problem Description

Consider $Q=[0,T] \times \Omega$, $\Omega \subset R^2$, $I=[0,T]$, $\Gamma=\partial\Omega$, $\Sigma = \Gamma \times I$, the OMCVP is proposed by the TSPEs :

$$\gamma_{1\mathfrak{b}} - \Delta\gamma_1 + \gamma_1 - \gamma_2 - \gamma_3 = f_1(x, \mathfrak{b}) \text{ in } Q \quad (1)$$

$$\gamma_{2\mathfrak{b}} - \Delta\gamma_2 + \gamma_2 + \gamma_3 + \gamma_1 = f_2(x, \mathfrak{b}) + r_2 \text{ in } Q \quad (2)$$

$$\gamma_{3\mathfrak{b}} - \Delta\gamma_3 + \gamma_3 + \gamma_1 - \gamma_2 = f_3(x, \mathfrak{b}) \text{ in } Q \quad (3)$$

with the initial and boundary conditions (ICs) and (BCs))

$$\frac{\partial \gamma_l}{\partial n_l} = \sum_{i=1}^2 \frac{\partial \gamma_l}{\partial x_j} \cos(n_l, x_j) = r_l(x, \mathfrak{b}), \text{ on } \Sigma, \text{ for } l = 1, 3 \quad (4)$$

$$\gamma_2(x, \mathfrak{b}) = 0, \text{ on } \Sigma \quad (5)$$

$$\gamma_l(x, 0) = \gamma_l^0(x), \text{ on } \Omega \quad \text{for } l = 1, 2, 3 \quad (6)$$

where $x=(x_1, x_2)$, n_l ($\forall l = 1, 3$) is an outer normal vector on the boundary Σ , and (n_l, x_j) represents the angle between n_l and the x_j - axis, (f_1, f_2, f_3) is given on Ω , $\vec{r} = (r_1, r_2, r_3) \in L^2(\Sigma) \times L^2(Q) \times L^2(\Sigma) = L^2(\Sigma, Q)$ is the MCV and $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in H^1(Q) \times H_0^1(Q) \times H^1(Q) = H_{0,2}^1(Q)$ is their corresponding TSVS. The set of admissible MCV is expressed by $\vec{V}_A = \{\vec{r} = (r_1, r_2, r_3) \in L^2(\Sigma, Q) | (r_1, r_2, r_3) \in \vec{V} \subset R^3 \text{ a.e. in } \Sigma\}$. The OF for $\vec{r} \in \vec{V}_A$, and $\beta > 0$ is

$$G_0(\vec{r}) = \frac{1}{2} \sum_{l=1}^3 \|\gamma_1 - \gamma_{1d}\|_Q^2 + \frac{\beta}{2} \|r_1\|_\Sigma^2 + \frac{\beta}{2} \|r_2\|_Q^2 + \frac{\beta}{2} \|r_3\|_\Sigma^2, \quad (7)$$

Let $\vec{W} = W_1 \times W_2 \times W_3 = \{\vec{\omega}: \vec{\omega} = (\omega_1, \omega_2, \omega_3) \in H_{0,2}^1(Q)\}$.

The system of weak formulation (SWF)of problem (1)-(6), is $(\forall \omega_l \in W_l)$, by

$$\langle \gamma_{1\mathfrak{b}}, \omega_1 \rangle + (\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2, \omega_1) - (\gamma_3, \omega_1) = (f_1, \omega_1) + (r_1, \omega_1)_\Gamma \quad (8)$$

$$\langle \gamma_{2\mathfrak{b}}, \omega_2 \rangle + (\nabla \gamma_2, \nabla \omega_2) + (\gamma_2, \omega_2) + (\gamma_3, \omega_2) + (\gamma_1, \omega_2) = (f_2 + r_2, \omega_2) \quad (9)$$

$$\langle \gamma_{3\mathfrak{b}}, \omega_3 \rangle + (\nabla \gamma_3, \nabla \omega_3) + (\gamma_3 + \gamma_1, \omega_3) - (\gamma_2, \omega_3) = (f_3, \omega_3) + (r_3, \omega_3)_\Gamma \quad (10)$$

$$(\gamma_l^0, \omega_l) = (\gamma_l(0), \omega_l), \quad \text{for } l = 1, 2, 3 \quad (11)$$

3. The Solution for the SWF

Theorem 3.1: For any given $MCV \vec{u} \in L^2(\Sigma, Q)$, the SWF ((8)-(11)) has a unique TSVS $\vec{\gamma} \in \vec{W}$.

Proof: Let, for any $n, \vec{W}_n = W_n \times W_n \times W_n \subset \vec{W}$ be the set of continuous functions in Ω , has the basis ω_{lj} ($j = 1, 2, \dots, n$ & $l = 1, 2, 3$), then by the GM, the TSVS $\vec{\gamma}$ is approximated by $\vec{\gamma}_n$ with: $\gamma_{ln} = \sum_{j=1}^n c_{lj}(\mathfrak{b}) \omega_{lj}(x)$ (12)

The SWF (8-11), ($\forall \omega_l \in W_n (l = 1, 2, 3)$).

$$\langle \gamma_{1n\mathfrak{b}}, \omega_1 \rangle + (\nabla \gamma_{1n}, \nabla \omega_1) + (\gamma_{1n} - \gamma_{2n} - \gamma_{3n}, \omega_1) = (f_1, \omega_1) + (\mathfrak{r}_1, \omega_1)_\Gamma, \quad (13)$$

$$\langle \gamma_{2n\mathfrak{b}}, \omega_2 \rangle + (\nabla \gamma_{2n}, \nabla \omega_2) + (\gamma_{2n} + \gamma_{3n} + \gamma_{1n}, \omega_2) = (f_2 + \mathfrak{r}_2, \omega_2), \quad (14)$$

$$\langle \gamma_{3n\mathfrak{b}}, \omega_3 \rangle + (\nabla \gamma_{3n}, \nabla \omega_3) + (\gamma_{3n} + \gamma_{1n} - \gamma_{2n}, \omega_3) = (f_3, \omega_3) + (\mathfrak{r}_3, \omega_3)_\Gamma, \quad (15)$$

$(\gamma_{ln}^0, \omega_l) = (\gamma_l^0, \omega_l)$, for $l = 1, 2, 3$ (16). Through the projection of γ_l^0 , the following holds $(\gamma_{ln}^0, \omega_l) = (\gamma_l^0, \omega_l), \forall \omega_l \in W_n \Leftrightarrow \|\gamma_{ln}^0 - \gamma_l^0\|_0 \leq \|\gamma_l^0 - \omega_l\|_0, \forall \omega_l \in W_n$. Using (12) in the SWF ((13)-(16)), putting $\omega_l = \omega_{li}$ ($j = 1, 2, \dots, n$ & $l = 1, 2, 3$), the SWF, can be expressed as a linear system of 1st order of ODEs and the ICs (the uniqueness of a TSVS is satisfied), i.e.

$$AC_1'(\mathfrak{b}) + BC_1(\mathfrak{b}) - DC_2(\mathfrak{b}) - EC_3(\mathfrak{b}) = b_1 \quad (17)$$

$$AC_1(0) = b_1^0 \quad (18)$$

$$FC_2'(\mathfrak{b}) + GC_2(\mathfrak{b}) + HC_3(\mathfrak{b}) + KC_1(\mathfrak{b}) = b_2 \quad (19)$$

$$FC_2(0) = b_2^0 \quad (20)$$

$$MC_3'(\mathfrak{b}) + NC_3(\mathfrak{b}) + RC_1(\mathfrak{b}) - WC_2(\mathfrak{b}) = b_3 \quad (21)$$

$$MC_l(0) = b_l^0, \quad (22)$$

where $A = (a_{ij})_{n \times n}$, $a_{ij} = (\omega_{1j}, \omega_{1i})$, $B = (b_{lj})_{n \times n}$, $b_{lj} = (\nabla \omega_{1j}, \nabla \omega_{1i}) + (\omega_{1j}, \omega_{1i})$, $D = (d_{ij})_{n \times n}$, $d_{ij} = (\omega_{2j}, \omega_{1i})$, $E = (e_{ij})_{n \times n}$, $e_{ij} = (\omega_{3j}, \omega_{1i})$, $F = (f_{ij})_{n \times n}$, $f_{ij} = (\omega_{2j}, \omega_{2i})$, $G = (g_{ij})_{n \times n}$, $g_{ij} = (\nabla \omega_{2j}, \nabla \omega_{2i}) + (\omega_{2j}, \omega_{2i})$, $H = (h_{ij})_{n \times n}$, $h_{ij} = (\omega_{3j}, \omega_{2i})$, $K = (k_{ij})_{n \times n}$, $k_{ij} = (\omega_{1j}, \omega_{2i})$, $M = (m_{ij})_{n \times n}$, $m_{ij} = (\omega_{3j}, \omega_{3i})$, $N = (n_{ij})_{n \times n}$, $n_{ij} = (\nabla \omega_{3j}, \nabla \omega_{3i}) + (\omega_{3j}, \omega_{3i})$, $R = (r_{ij})_{n \times n}$, $r_{ij} = (\omega_{1j}, \omega_{3i})$, $V = (\mathfrak{r}_{ij})_{n \times n}$, $\mathfrak{r}_{ij} = (\omega_{2j}, \omega_{3i})$, $b_l^0 = (b_{li}^0)$, $b_{li}^0 = (\gamma_l^0, \omega_{li})$, $b_l = (b_{li})_{n \times 1}$, $b_{ki} = (f_k, \omega_k) + (\mathfrak{r}_k, \omega_{kl})_\Gamma$, $b_{2i} = (f_2 + \mathfrak{r}_2, \omega_{2i})$, $C_l'(t) = (c_{lj}'(t))_{n \times 1}$, $C_l(\mathfrak{b}) = (c_{lj}(\mathfrak{b}))_{n \times 1}$, $l = 1, 2, 3$, $k = 1, 3$, $\forall i = 1, 2, \dots, n$.

The boundedness of $\|\vec{\gamma}_n^0\|_0$: Since $\gamma_l^0 \in L^2(\Omega)$, for $l = 1, 2, 3$, then there is

$$L^2(\Omega)$$

a sequence (seq.) $\{\omega_{ln}^0\}$, $\omega_{ln}^0 \in W_n$, s.t. $\omega_{ln}^0 \rightarrow \gamma_l^0$, then from the projection

$$S$$

$$L^2(\Omega)$$

theorem [16] and (16), it obtains $\gamma_{ln}^0 \rightarrow \gamma_l^0$ with $\|\gamma_{ln}^0\|_0 \leq b_l$. Then $\|\vec{\gamma}_n^0\|_0$ is

$$S$$

bounded in $(L^2(\Omega))^3$.

The boundedness of the norms $\|\vec{\gamma}_n\|_{L^\infty(I, L^2(\Omega))}$ and $\|\vec{\gamma}_n\|_Q$:

Putting $\mathfrak{r}_l = \gamma_{ln}$ in ((17) –(19)) for $l = 1, 2, 3$, integrating on $[0, T]$, collecting the resulting equalities, to obtain $\int_0^T \langle \vec{\gamma}_{n\mathfrak{b}}, \vec{\gamma}_n \rangle d\mathfrak{b} + \int_0^T \|\vec{\gamma}_n\|_1^2 d\mathfrak{b} =$

$$\int_0^T [\sum_{l=1}^3 (f_l, \gamma_l) + (r_1, \gamma_1)_\Gamma + (r_2, \gamma_2) + (r_3, \gamma_3)_\Gamma] d\mathfrak{b} \quad (23)$$

Utilizing Lemma (1.2) [17] to the 1^{st} integral in the LHS for (23), with the 2^{nd} integral is nonnegative, then utilizing the property that is assumed f_l , it gives

$$\begin{aligned} \frac{1}{2} \int_0^{\mathfrak{b}} \frac{d}{d\mathfrak{b}} \|\vec{\gamma}_n(\mathfrak{b})\|_0^2 d\mathfrak{b} &\leq \sum_{l=1}^3 \int_0^{\mathfrak{b}} \int_\Omega \eta_l |\gamma_{ln}| dx d\mathfrak{b} + \int_0^{\mathfrak{b}} \int_\Omega |r_2| |\gamma_{2n}| dx d\mathfrak{b} + \\ &\int_0^{\mathfrak{b}} \int_\Gamma (|r_1|^2 + |\gamma_{1n}|^2) d\gamma d\mathfrak{b} + \int_0^{\mathfrak{b}} \int_\Gamma (|r_3|^2 + |\gamma_{3n}|^2) d\gamma d\mathfrak{b} \Rightarrow \int_0^{\mathfrak{b}} \frac{d}{d\mathfrak{b}} \|\vec{\gamma}_n(\mathfrak{b})\|_0^2 d\mathfrak{b} \\ &\leq \sum_{l=1}^3 \|\eta_l\|_Q^2 + \|r_1\|_\Sigma^2 + \|r_2\|_Q^2 + \|r_3\|_\Sigma^2 + 2 \sum_{l=1}^3 \int_0^{\mathfrak{b}} \|\gamma_{ln}\|_0^2 d\mathfrak{b} \end{aligned}$$

Since $\|\eta_l\|_Q^2 \leq b_l$, $\|r_l\|_\Sigma^2 \leq c_l$, $\forall l = 1, 3$, $\|r_2\|_Q^2 \leq c_2$, $\|\vec{\gamma}_n(0)\|_0^2 \leq b$, then

$$\|\vec{\gamma}_n(\mathfrak{b})\|_0^2 \leq c^* + 2 \int_0^{\mathfrak{b}} \|\vec{\gamma}_n\|_0^2 d\mathfrak{b}, \text{ with } c^* = b_1 + b_2 + b_3 + c_1 + c_2 + c_3 + b,$$

using the inequality of Gronwall, to obtain $(\forall \mathfrak{b} \in [0, T])$. $\|\vec{\gamma}_n(\mathfrak{b})\|_0^2 \leq c^* e^{2T} = b^2(c)$, or $\|\vec{\gamma}_n(\mathfrak{b})\|_{L^\infty(I, L^2(\Omega))} \leq b(c) \Rightarrow \|\vec{\gamma}_n(\mathfrak{b})\|_Q \leq b_1(c)$. The boundedness of $\|\vec{\gamma}_n(\mathfrak{b})\|_{L^2(I, W)}$. Also through utilizing Lemma (1.2)[17] for the LHS, taking advantage of the steps that resulted from above with $\|\vec{\gamma}_n(\mathfrak{b})\|_0^2$ is nonnegative, then (23), for $\mathfrak{b} = T$ gives $\|\vec{\gamma}_n(T)\|_0^2 - \|\vec{\gamma}_n(0)\|_0^2 + 2 \int_0^T \|\vec{\gamma}_n\|_1^2 d\mathfrak{b} \leq \sum_{l=1}^3 \|\eta_l\|_Q^2 + \|r_1\|_\Sigma^2 + \|r_2\|_Q^2 + \|r_3\|_\Sigma^2 + 2 \|\vec{\gamma}_n\|_Q^2$, which gives

$$\int_0^T \|\vec{\gamma}_n\|_1^2 d\mathfrak{b} \leq b_2^2(c), \text{ with } b_2^2(c) = \frac{(b+b_1'+b_2'+b_3'+c_1+c_2+c_3+2b_1(c))}{2},$$

thus $\|\vec{\gamma}_n\|_{L^2(I, W)} \leq b_2(c)$.

The solution strongly convergence:

Let the sequence for subspace of $\vec{W}$ be $\{\vec{W}_n\}_{n=1}^\infty$, s.t. $\forall \vec{\omega} \in \vec{W}$, there is $\vec{\omega}_n \in \vec{W}_n$, $\forall n$ for which $\vec{\omega}_n \xrightarrow[S]{\vec{W}} \vec{\omega} \Rightarrow \vec{\omega}_n \xrightarrow[S]{(L^2(\Omega))^3} \vec{\omega}$, but ((13)-16)) has a TSVS $\vec{\gamma}_n$,

$\vec{W}_n \subset \vec{W}$, $\forall n$ hence then conforming to the subspaces $\{\vec{W}_n\}_{n=1}^\infty$, there is a sequence of WFS like ((13)-16)), one has the following WFS for $n = 1, 2, \dots$

$$\begin{aligned} \langle \gamma_{1n}, \omega_{1n} \rangle + (\nabla \gamma_{1n}, \nabla \omega_{1n}) + (\gamma_{1n}, \omega_{1n}) - (\gamma_{2n}, \omega_{1n}) - (\gamma_{3n}, \omega_{1n}) = \\ (f_1, \omega_{1n}) + (r_1, \omega_{1n})_\Gamma \end{aligned} \quad (24)$$

$$\begin{aligned} \langle \gamma_{2n}, \omega_{2n} \rangle + (\nabla \gamma_{2n}, \nabla \omega_{2n}) + (\gamma_{2n}, \omega_{2n}) + (\gamma_{3n}, \omega_{2n}) + (\gamma_{1n}, \omega_{2n}) \\ (f_2 + r_2, \omega_{2n}) \end{aligned} \quad (25)$$

$$\begin{aligned} \langle \gamma_{3n}, \omega_{3n} \rangle + (\nabla \gamma_{3n}, \nabla \omega_{3n}) + (\gamma_{3n}, \omega_{3n}) + (\gamma_{1n}, \omega_{3n}) - (\gamma_{2n}, \omega_{3n}) = \\ (f_3, \omega_{3n}) + (r_3, \omega_{3n})_\Gamma \end{aligned} \quad (26)$$

$(\gamma_{ln}^0, \omega_{ln}) = (\gamma_l^0, \omega_{ln})$, for $l = 1, 2, 3$, (27) which has a sequence of TSVS $\{\vec{\gamma}_n\}_{n=1}^\infty$, (by the first part of proof for this theorem, with $\|\vec{\gamma}_n\|_{L^2(Q)}$ and $\|\vec{\gamma}_n\|_{L^2(I, W)}$ satisfy the bounded property), then by utilizing the Alaoglu's theorem (AT), the sequence $\{\vec{\gamma}_n\}_{n \in \mathbb{N}}$ has a subsequence

“let be the same” $\{\vec{\gamma}_n\}_{n \in \mathbb{N}}$ s.t. $\vec{\gamma}_n \xrightarrow[W]{(L^2(Q))^3} \vec{\gamma}$ and $\vec{\gamma}_n \xrightarrow[W]{(L^2(I, W))^3} \vec{\gamma}$, then integrate both sides of ((24)-(27)) on $[0, T]$ after multiply them by $\theta_l(\mathfrak{h}) \in C^1[0, T]$ respectively, s.t. $\theta_l(T) = 0$, ($l = 1, 2, 3$), then integrating by parts in the terms which have the first time derivative, they give

$$\begin{aligned} & - \int_0^T (\gamma_{1n}, \omega_{1n}) \theta'_1(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_{1n}, \nabla \omega_{1n}) + (\gamma_{1n} - \gamma_{2n} - \gamma_{3n}, \omega_{1n})] \theta_1(\mathfrak{h}) d\mathfrak{h} \\ & = \int_0^T [(f_1, \omega_{1n}) + (r_1, \omega_{1n})_\Gamma] \theta_1(\mathfrak{h}) + (\gamma_{1n}^0, \omega_{1n}) \theta_1(0) \end{aligned} \quad (28)$$

$$\begin{aligned} & - \int_0^T (\gamma_{2n}, \omega_{2n}) \theta'_2(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_{2n}, \nabla \omega_{2n}) + (\gamma_{2n} + \gamma_{3n} + \gamma_{1n}, \omega_{2n})] \theta_2(\mathfrak{h}) d\mathfrak{h} \\ & = \int_0^T (f_2 + r_2, \omega_{2n}) \theta_2(\mathfrak{h}) d\mathfrak{h} + (\gamma_{2n}^0, \omega_{2n}) \theta_2(0), \end{aligned} \quad (29)$$

$$\begin{aligned} & - \int_0^T (\gamma_{3n}, \omega_{3n}) \theta'_3(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_{3n}, \nabla \omega_{3n}) + (\gamma_{3n} + \gamma_{1n} - \gamma_{2n}, \omega_{3n})] \theta_3(\mathfrak{h}) d\mathfrak{h} \\ & = \int_0^T [(f_3, \omega_{3n}) + (r_3, \omega_{3n})_\Gamma] \theta_3(\mathfrak{h}) + (\gamma_{3n}^0, \omega_{3n}) \theta_3(0), \end{aligned} \quad (30)$$

$$\text{Since } \left. \begin{array}{c} L^2(\Omega) \\ \omega_{ln} \rightarrow \omega_l \\ S \\ W_1 \\ \omega_{ln} \rightarrow \omega_l \\ S \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} L^2(Q) \\ \omega_{ln} \theta'_l \rightarrow \omega_l \theta'_l \\ S \\ L^2(I, W) \\ \omega_{ln} \theta'_l \rightarrow \omega_l \phi \theta'_l \\ S \end{array} \right\}, \quad \text{and} \quad \text{since} \quad \left\{ \begin{array}{c} L^2(Q) \\ \gamma_{ln} \rightarrow \gamma_l \\ S \\ L^2(\Omega) \\ \gamma_{ln}^0 \rightarrow \gamma_l^0 \\ S \end{array} \right\},$$

$$\begin{aligned} & \int_0^T (\gamma_{1n}, \omega_{1n}) \theta'_1(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_{1n}, \nabla \omega_{1n}) + (\gamma_{1n} - \gamma_{2n} - \gamma_{3n}, \omega_{1n})] \theta_1(\mathfrak{h}) d\mathfrak{h} \\ & \rightarrow \int_0^T (\gamma_1, \omega_1) \theta'_1(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2 - \gamma_3, \omega_1)] \theta_1(\mathfrak{h}) d\mathfrak{h} \end{aligned} \quad (31)$$

$$\begin{aligned} & \int_0^T (\gamma_{2n}, \omega_{2n}) \theta'_2(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_{2n}, \nabla \omega_{2n}) + (\gamma_{2n} + \gamma_{3n} + \gamma_{1n}, \omega_{2n})] \theta_2(\mathfrak{h}) d\mathfrak{h} \\ & \rightarrow \int_0^T (\gamma_2, \omega_2) \theta'_2(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_2, \nabla \omega_2) + (\gamma_2 + \gamma_3 + \gamma_1, \omega_2)] \theta_2(\mathfrak{h}) d\mathfrak{h} \end{aligned} \quad (32)$$

$$\begin{aligned} & \int_0^T (\gamma_{3n}, \omega_{3n}) \theta'_3(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_{3n}, \nabla \omega_{3n}) + (\gamma_{3n} + \gamma_{1n} - \gamma_{2n}, \omega_{3n})] \theta_3(\mathfrak{h}) d\mathfrak{h} \\ & \rightarrow \int_0^T (\gamma_3, \omega_3) \theta'_3(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_3, \nabla \omega_3) + (\gamma_3 + \gamma_1 - \gamma_2, \omega_3)] \theta_3(\mathfrak{h}) d\mathfrak{h} \end{aligned} \quad (33)$$

$$(\gamma_{ln}^0, \omega_{ln}) \theta_l(0) \rightarrow (\gamma_l^0, \omega_l) \theta_l(0) \quad (34). \quad \text{Since } \omega_{ln} \xrightarrow[W]{L^2(\Omega)} \omega_l, \text{ then}$$

$$\int_0^T [(f_l, \omega_{ln}) + (r_l, \omega_{ln})_\Gamma] \theta_l(\mathfrak{h}) \rightarrow \int_0^T [(f_l, \omega_l) + (r_l, \omega_l)_\Gamma] \theta_l(\mathfrak{h}), \quad l = 1, 3 \quad (34)$$

$$\int_0^T (f_2 + r_2, \omega_{2n}) \theta_2(\mathfrak{h}) d\mathfrak{h} \rightarrow \int_0^T (f_2 + r_2, \omega_2) \theta_2(\mathfrak{h}) d\mathfrak{h} \quad (35)$$

Thus ((28)-(30)) converge to ((36)-(38)), i.e.

$$\begin{aligned} & - \int_0^T (\gamma_1, \omega_1) \theta'_1(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2 - \gamma_3, \omega_1)] \theta_1(\mathfrak{h}) d\mathfrak{h} \\ & = \int_0^T [(f_1, \omega_1) + (r_1, \omega_1)_\Gamma] \theta_1(\mathfrak{h}) d\mathfrak{h} + (\gamma_1^0, \omega_1) \theta_1(0) \end{aligned} \quad (36)$$

$$\begin{aligned}
& - \int_0^T \langle \gamma_2, \omega_2 \rangle \theta_2'(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_2, \nabla \omega_2) + (\gamma_2 + \gamma_3 + \gamma_1, \omega_2)] \theta_2(\mathfrak{h}) d\mathfrak{h} \\
& = \int_0^T (f_2 + r_2, \omega_2) \theta_2(\mathfrak{h}) d\mathfrak{h} + (\gamma_2^0, \omega_2) \theta_2(0)
\end{aligned} \tag{37}$$

$$\begin{aligned}
& - \int_0^T \langle \gamma_3, \omega_3 \rangle \theta_3'(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_3, \nabla \omega_3) + (\gamma_3 + \gamma_1 - \gamma_3, \omega_3)] \theta_3(\mathfrak{h}) d\mathfrak{h} \\
& = \int_0^T [(f_3, \omega_3) + (r_3, \omega_3)_\Gamma] \theta_3(\mathfrak{h}) d\mathfrak{h} + (\gamma_3^0, \omega_3) \theta_3(0)
\end{aligned} \tag{38}$$

Case 1: Pick $\theta_l(\mathfrak{h}) \in C^1[0, T]$, with $\theta_l(0) = \theta_l(T) = 0$, for $l = 1, 2, 3$. Substituting in ((36)-(38)), integrating by parts all the terms in LHS that have first time derivative, to acquire

$$\begin{aligned}
& \int_0^T \langle \gamma_{1\mathfrak{h}}, \omega_1 \rangle \theta_1(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2 - \gamma_3, \omega_1)] \theta_1(\mathfrak{h}) d\mathfrak{h} \\
& = \int_0^T [(f_1, \omega_1) + (r_1, \omega_1)_\Gamma] \theta_1(\mathfrak{h}) d\mathfrak{h}
\end{aligned} \tag{39}$$

$$\begin{aligned}
& \int_0^T \langle \gamma_{2\mathfrak{h}}, \omega_2 \rangle \theta_2(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_2, \nabla \omega_2) + (\gamma_2 + \gamma_3 + \gamma_1, \omega_2)] \theta_2(\mathfrak{h}) d\mathfrak{h} \\
& = \int_0^T (f_2 + r_2, \omega_2) \theta_2(\mathfrak{h}) d\mathfrak{h}
\end{aligned} \tag{40}$$

$$\begin{aligned}
& \int_0^T \langle \gamma_{3\mathfrak{h}}, \omega_3 \rangle \theta_3(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_3, \nabla \omega_3) + (\gamma_3 + \gamma_1 - \gamma_2, \omega_3)] \theta_3(\mathfrak{h}) d\mathfrak{h} \\
& = \int_0^T [(f_3, \omega_3) + (r_3, \omega_3)_\Gamma] \theta_3(\mathfrak{h}) d\mathfrak{h}
\end{aligned} \tag{41}$$

i.e. $\vec{\gamma}$ is solution of the SWF ((8)-(10)).

Case 2: Pick $\theta_l \in C^1[0, T]$ s.t. $\theta_l(T) = 0$ & $\theta_l(0) \neq 0$, ($l = 1, 2, 3$), integrating by parts the term in LHS that have first time derivative, to acquire

$$\begin{aligned}
& - \int_0^T \langle \gamma_1, \omega_1 \rangle \theta_1'(\mathfrak{h}) d\mathfrak{h} + \int_0^T [(\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2 - \gamma_3, \omega_1)] \theta_1(\mathfrak{h}) d\mathfrak{h} \\
& = \int_0^T [(f_1, \omega_1) + (r_1, \omega_1)_\Gamma] \theta_1(\mathfrak{h}) d\mathfrak{h} + (\gamma_1(0), \omega_1) \theta_1(0)
\end{aligned} \tag{42}$$

Subtract (39) from (42), to acquire

$$(\gamma_1^0, \omega_1) \theta_1(0) = (\gamma_1(0), \omega_1) \theta_1(0), \theta_1(0) \neq 0, \forall \theta_1 \in [0, T] \Rightarrow (\gamma_1^0, \omega_1) = (\gamma_1(0), \omega_1)$$

Same manner can be utilized to acquire (11) are held.

The strongly convergence for $\vec{\gamma}_n$:

Utilizing $\omega_l = \gamma_{ln}$ (for $l = 1, 2, 3$) in (13)-(18), gathering all the resulting equalities, then integrating on $[0, T]$, and then utilizing in LHS Lemma (1.2) [17] for all the terms that have first time derivative, to conclude

$$\begin{aligned}
& \frac{1}{2} \|\vec{\gamma}_n(T)\|_0^2 - \frac{1}{2} \|\vec{\gamma}_n(0)\|_0^2 + \int_0^T a(\vec{\gamma}_n, \vec{\gamma}_n) d\mathfrak{h} = \int_0^T [(f_1, \gamma_{1n}) + (r_1, \gamma_{1n})_\Gamma + \\
& (f_2 + r_2, \gamma_{2n})] d\mathfrak{h} + \int_0^T [(f_3, \gamma_{3n}) + (r_3, \gamma_{3n})_\Gamma] d\mathfrak{h}
\end{aligned} \tag{43}$$

Also, utilizing $\omega_l = \gamma_l$, ($l = 1, 2, 3$) in ((8)-(11)) gathering all the resulting equalities, then integrating on $[0, T]$, and then utilizing in LHS Lemma (1.2) [17] for all the terms that have first time derivative, to conclude

$$\begin{aligned}
& \frac{1}{2} \|\vec{\gamma}(T)\|_0^2 - \frac{1}{2} \|\vec{\gamma}(0)\|_0^2 + \int_0^T a(\vec{\gamma}, \vec{\gamma}) d\mathfrak{h} = \int_0^T [(f_1, \gamma_1) + (r_1, \gamma_1)_\Gamma] d\mathfrak{h} \\
& \int_0^T (f_2 + r_2, \gamma_2) d\mathfrak{h} + \int_0^T [(f_3, \gamma_3) + (r_3, \gamma_3)_\Gamma] d\mathfrak{h}
\end{aligned} \tag{44}$$

$$\begin{aligned}
& \text{But } \frac{1}{2} \|\vec{\gamma}_n(T) - \vec{\gamma}(T)\|_0^2 - \frac{1}{2} \|\vec{\gamma}_n(0) - \vec{\gamma}(0)\|_0^2 + \int_0^T a(\vec{\gamma}_n - \vec{\gamma}, \vec{\gamma}_n - \vec{\gamma}) d\mathfrak{t} = \\
& E_1 - E_2 - E_3 \quad (45). \quad E_1 = \frac{1}{2} \|\vec{\gamma}_n(T)\|_0^2 - \frac{1}{2} \|\vec{\gamma}_n(0)\|_0^2 + \int_0^T a(\vec{\gamma}_n(\mathfrak{t}), \vec{\gamma}_n(\mathfrak{t})) d\mathfrak{t} \\
& E_2 = \frac{1}{2} (\vec{\gamma}_n(T), \vec{\gamma}(T)) - \frac{1}{2} (\vec{\gamma}_n(0), \vec{\gamma}(0)) + \int_0^T a(\vec{\gamma}_n(\mathfrak{t}), \vec{\gamma}(\mathfrak{t})) d\mathfrak{t}, \\
& E_3 = \frac{1}{2} (\vec{\gamma}(T), \vec{\gamma}_n(T) - \vec{\gamma}(T)) - \frac{1}{2} (\vec{\gamma}(0), \vec{\gamma}_n(0) - \vec{\gamma}(0)) + \int_0^T a(\vec{\gamma}(\mathfrak{t}), \vec{\gamma}_n(\mathfrak{t}) - \\
& \vec{\gamma}(\mathfrak{t})) d\mathfrak{t}
\end{aligned}$$

$$\begin{aligned}
& \text{Since } \vec{\gamma}_n^0 = \vec{\gamma}_n(0) \xrightarrow[S]{} \vec{\gamma}^0 = \vec{\gamma}(0) \text{ and } \vec{\gamma}_n(T) \xrightarrow[S]{} \vec{\gamma}(T), \text{ then} \\
& (\vec{\gamma}(0), \vec{\gamma}_n(0) - \vec{\gamma}(0)) \rightarrow 0, (\vec{\gamma}(T), \vec{\gamma}_n(T) - \vec{\gamma}(T)) \rightarrow 0 \quad (46) \\
& \|\vec{\gamma}_n(0) - \vec{\gamma}(0)\|_0^2 \rightarrow 0 \text{ and } \|\vec{\gamma}_n(T) - \vec{\gamma}(T)\|_0^2 \rightarrow 0 \quad (47)
\end{aligned}$$

$$\begin{aligned}
& \text{Besides, since } \vec{\gamma}_n \xrightarrow[W]{} \vec{\gamma} \text{ \& } \vec{\gamma}_n \xrightarrow[W]{} \vec{\gamma}, \text{ then } \int_0^T a(\vec{\gamma}(\mathfrak{t}), \vec{\gamma}_n(\mathfrak{t}) - \\
& \vec{\gamma}(\mathfrak{t})) d\mathfrak{t} \rightarrow 0 \quad (48) \text{ and } \int_0^T [((f_1, \gamma_{1n}) + (r_1, \gamma_{1n})_\Gamma + (f_2 + r_2, \gamma_{2n}))] d\mathfrak{t} + \\
& \int_0^T [(f_3, \gamma_{3n}) + (r_3, \gamma_{3n})_\Gamma] d\mathfrak{t} \rightarrow \int_0^T [((f_1, \gamma_1) + (r_1, \gamma_1)_\Gamma + (f_2 + r_2, \gamma_2))] d\mathfrak{t} + \\
& \int_0^T [(f_3, \gamma_3) + (r_3, \gamma_3)_\Gamma] d\mathfrak{t} \quad (49). \text{ Now, as } n \rightarrow \infty \text{ in (45), the following points} \\
& \text{are concluded:}
\end{aligned}$$

- (1) The first two terms in the LHS of (45) are tending to zero (from (46)),
- (2) $E_1 \xrightarrow[\text{from (43)}]{} \int_0^T [((f_1, \gamma_{1n}) + (r_1, \gamma_{1n})_\Gamma + (f_2 + r_2, \gamma_{2n}))] d\mathfrak{t} + \int_0^T [(f_3, \gamma_{3n}) + (r_3, \gamma_{3n})_\Gamma] d\mathfrak{t}$
 $\xrightarrow[\text{from (49)}]{} \int_0^T [((f_1, \gamma_1) + (r_1, \gamma_1)_\Gamma + (f_2 + r_2, \gamma_2))] d\mathfrak{t} + \int_0^T [(f_3, \gamma_3) + (r_3, \gamma_3)_\Gamma] d\mathfrak{t}$
- (3) $E_2 \rightarrow \text{LHS of (44)} = \int_0^T [((f_1, \gamma_1) + (r_1, \gamma_1)_\Gamma + (f_2 + r_2, \gamma_2))] d\mathfrak{t} + \int_0^T [(f_3, \gamma_3) + (r_3, \gamma_3)_\Gamma] d\mathfrak{t}$ (4) In E3, the 1st two expressions converge to zero (by (46)), and the remaining one converges to zero (by (48)).

Through the above resulting four points, and as $n \rightarrow \infty$ in (45), it gives

$$\int_0^T \|\vec{\gamma}_n - \vec{\gamma}\|_1^2 d\mathfrak{t} = \int_0^T a(\vec{\gamma}_n - \vec{\gamma}, \vec{\gamma}_n - \vec{\gamma}) d\mathfrak{t} \rightarrow 0 \Rightarrow \vec{\gamma}_n \xrightarrow[S]{} \vec{\gamma} \text{ in } H_{0,2}^1(Q)$$

The Uniqueness of the TSVS: Let $\vec{\gamma}, \vec{\bar{\gamma}}$ be two TSVSs of ((8)-(10)), i.e. γ_1 & $\bar{\gamma}_1$ are satisfied (8), or
 $\langle \gamma_{1\mathfrak{t}}, \omega_1 \rangle + a_1(\gamma_1, \omega_1) - (\gamma_2, \omega_1) - (\gamma_3, \omega_1) = (f_1, \omega_1) + (r_1, \omega_1)_\Gamma, \forall \omega_1 \in W_1$
 $\langle \bar{\gamma}_{1\mathfrak{t}}, \omega_1 \rangle + a_1(\bar{\gamma}_1, \omega_1) - (\bar{\gamma}_2, \omega_1) - (\bar{\gamma}_3, \omega_1) = (f_1, \omega_1) + (r_1, \omega_1)_\Gamma,$
 $\forall \omega_1 \in W_1$

Subtracting each equation from the other, replacing $r_1 = \gamma_1 - \bar{\gamma}_1$, to acquire that

$$\langle (\bar{\gamma}_1 - \gamma_1)_{\mathfrak{h}}, \bar{\gamma}_1 - \gamma_1 \rangle + a_1(\bar{\gamma}_1 - \gamma_1, \bar{\gamma}_1 - \gamma_1) - (\bar{\gamma}_2 - \gamma_2, \bar{\gamma}_1 - \gamma_1) - (\bar{\gamma}_3 - \gamma_3, \bar{\gamma}_1 - \gamma_1) = 0. \quad (50)$$

By the similar manner, one gets

$$\langle (\bar{\gamma}_2 - \gamma_2)_{\mathfrak{h}}, \bar{\gamma}_2 - \gamma_2 \rangle + a_2(\bar{\gamma}_2 - \gamma_2, \bar{\gamma}_2 - \gamma_2) + (\bar{\gamma}_3 - \gamma_3, \bar{\gamma}_2 - \gamma_2) + (\bar{\gamma}_1 - \gamma_1, \bar{\gamma}_2 - \gamma_2) = 0, \quad (51)$$

$$\langle (\bar{\gamma}_3 - \gamma_3)_{\mathfrak{h}}, \bar{\gamma}_3 - \gamma_3 \rangle + a_3(\bar{\gamma}_3 - \gamma_3, \bar{\gamma}_3 - \gamma_3) + (\bar{\gamma}_1 - \gamma_1, \bar{\gamma}_3 - \gamma_3) - (\bar{\gamma}_2 - \gamma_2, \bar{\gamma}_3 - \gamma_3) = 0. \quad (52)$$

Adding ((50)-(52)), using Lemma (1.2)[17] in the terms that have first time derivative, to acquire

$$\frac{1}{2} \frac{d}{dt} \|\vec{\gamma} - \bar{\gamma}\|_0^2 + \|\vec{\gamma} - \bar{\gamma}\|_1^2 = 0 \quad (53)$$

From the positive property for 2nd term in (53), integrating the resulting on $[0, T]$, to obtain $\int_0^{\mathfrak{h}} \frac{d}{dt} \|\vec{\gamma} - \bar{\gamma}\|_0^2 d\mathfrak{h} \leq 0 \Rightarrow \|(\vec{\gamma} - \bar{\gamma})(\mathfrak{h})\|_0^2 \leq 0 \Rightarrow \|\vec{\gamma} - \bar{\gamma}\|_0^2 = 0$, for each $\mathfrak{h} \in \tilde{I}$. Again, integrating (53) on $[0, T]$, utilizing (11) and the above result, it concludes $\int_0^T \|\vec{\gamma} - \bar{\gamma}\|_1^2 d\mathfrak{h} = 0 \Rightarrow \|\vec{\gamma} - \bar{\gamma}\|_{L^2(I, V)} = 0 \Rightarrow \vec{\gamma} = \bar{\gamma}$.

4. Existence of an OMTC

Theorem 4.1: Assume that $\vec{\gamma}$, and $\vec{\gamma} + \delta\vec{\gamma}$ are the TSVS conforming for the MCV $\vec{r}, \vec{r} + \delta\vec{r} \in L^2(\Sigma, Q)$, then (with $M, \tilde{M}, \bar{M}_1 \in \mathbb{R}^+$)

$$\|\delta\vec{\gamma}\|_{L^\infty(I, L^2(\Omega))} \leq M \|\delta\vec{r}\|_{\Sigma, Q}, \|\delta\vec{\gamma}\|_{L^2(Q)} \leq \tilde{M} \|\delta\vec{r}\|_{\Sigma, Q},$$

$$\|\delta\vec{\gamma}\|_{L^2(I, V)} \leq \bar{M}_1 \|\delta\vec{r}\|_{\Sigma, Q}$$

Proof: Take $\vec{r}, \vec{\bar{r}} \in L^2(\Sigma, Q)$, then by Theorem 3.1, there are $\vec{\gamma}, \vec{\bar{\gamma}}$ which satisfy the WFS ((8)-(10)) (for $\vec{\gamma}$) and the WFS for $\vec{\bar{\gamma}}$, i.e.

$$\langle \bar{\gamma}_{1\mathfrak{h}}, \omega_1 \rangle + (\nabla \bar{\gamma}_1, \nabla \omega_1) + (\bar{\gamma}_1 - \bar{\gamma}_2 - \bar{\gamma}_3, \omega_1) = (f_1, \omega_1) + (\bar{r}_1, \omega_1)_\Gamma \quad (54)$$

$$\langle \bar{\gamma}_{2\mathfrak{h}}, \omega_2 \rangle + (\nabla \bar{\gamma}_2, \nabla \omega_2) + (\bar{\gamma}_2 + \bar{\gamma}_3 + \bar{\gamma}_1, \omega_2) = (f_2 + \bar{r}_2, \omega_2) \quad (55)$$

$$\langle \bar{\gamma}_{3\mathfrak{h}}, \omega_3 \rangle + (\nabla \bar{\gamma}_3, \nabla \omega_3) + (\bar{\gamma}_3 + \bar{\gamma}_1 - \bar{\gamma}_2, \omega_3) = (f_3, \omega_3) + (\bar{r}_3, \omega_3)_\Gamma \quad (56)$$

$$(\bar{\gamma}_l(0), r_l) = (\gamma_l^0, r_l), \text{ for } l = 1, 2, 3 \quad (57)$$

Subtracting ((8)-(10)) from ((54)-(56)), and (11) from (57) (for $l = 1, 2, 3$) then letting $\delta\gamma_l = \bar{\gamma}_l - \gamma_l$, and $\delta r_l = \bar{r}_l - r_l$, they acquire

$$\langle \delta\gamma_{1\mathfrak{h}}, \omega_1 \rangle + (\nabla \delta\gamma_1, \nabla \omega_1) + (\delta\gamma_1 - \delta\gamma_2 - \delta\gamma_3, \omega_1) = (\delta r_1, \omega_1)_\Gamma \quad (58)$$

$$\langle \delta\gamma_{2\mathfrak{h}}, \omega_2 \rangle + (\nabla \delta\gamma_2, \nabla \omega_2) + (\delta\gamma_2 + \delta\gamma_3 + \delta\gamma_1, \omega_2) = (\delta r_2, \omega_2) \quad (59)$$

$$\langle \delta\gamma_{3\mathfrak{h}}, \omega_3 \rangle + (\nabla \delta\gamma_3, \nabla \omega_3) + (\delta\gamma_3 + \delta\gamma_1 - \delta\gamma_2, \omega_3) = (\delta r_3, \omega_3)_\Gamma \quad (60)$$

$$(\delta\gamma_l(0), \omega_l) = 0, \quad (61)$$

Replacing $\omega_l = \delta\gamma_l$, in (58)-(61), collecting the results equations together, utilizing Lemma (1.2) [17], they acquire

$$\frac{1}{2} \frac{d}{d\mathfrak{b}} \|\delta\vec{\gamma}\|_0^2 + \|\delta\vec{\gamma}\|_1^2 = (\delta\mathbf{r}_1, \delta\gamma_1)_\Gamma + (\delta\mathbf{r}_2, \delta\gamma_2) + (\delta\mathbf{r}_3, \delta\gamma_3)_\Gamma \quad (62)$$

But the 2^{nd} term is nonnegative, then taking the integral for $\mathfrak{b} \in [0, \mathfrak{b}]$, and then utilizing the inequality of the Cauchy Schwartz (CSI), it acquires

$$\begin{aligned} \int_0^{\mathfrak{b}} \frac{d}{d\mathfrak{b}} \|\delta\vec{\gamma}\|_0^2 d\mathfrak{b} &\leq 2 \int_0^{\mathfrak{b}} \int_\Gamma |\delta\mathbf{r}_1| |\delta\gamma_1| dx d\mathfrak{b} + 2 \int_0^{\mathfrak{b}} \int_\Omega |\delta\mathbf{r}_2| |\delta\gamma_2| dx d\mathfrak{b} + \\ &2 \int_0^{\mathfrak{b}} \int_\Gamma |\delta\mathbf{r}_3| |\delta\gamma_3| dx d\mathfrak{b} \leq \int_0^T [\|\delta\mathbf{r}_1\|_\Sigma^2 + \|\delta\mathbf{r}_2\|_Q^2 + \|\delta\mathbf{r}_3\|_\Sigma^2] d\mathfrak{b} \\ &+ \int_0^{\mathfrak{b}} [\|\delta\gamma_1\|_\Sigma^2 + \|\delta\gamma_2\|_Q^2 + \|\delta\gamma_3\|_\Sigma^2] d\mathfrak{b} \end{aligned}$$

$$\leq \int_0^T \|\delta\vec{r}\|_{\Sigma,Q}^2 d\mathfrak{b} + \int_0^{\mathfrak{b}} \|\delta\vec{\gamma}\|_0^2 d\mathfrak{b}, \quad \mathfrak{b} \in [0, T], \text{ (by the trace operator)}$$

Through the inequality of Gronwall, it is yield, for $\mathfrak{b} \in [0, T]$ that

$$\|\delta\vec{\gamma}(\mathfrak{b})\|_0^2 \leq M^2 \|\delta\vec{r}\|_{\Sigma,Q}^2 \Rightarrow \|\delta\vec{\gamma}(\mathfrak{b})\|_0 \leq M \|\delta\vec{r}\|_{\Sigma,Q}^2, \text{ where } M^2 = e^T, \quad M > 0.$$

$$\Rightarrow \|\delta\vec{\gamma}\|_{L^\infty(I, L^2(\Omega))} \leq M \|\delta\vec{r}\|_{\Sigma,Q}^2.$$

Since $\|\delta\vec{\gamma}\|_{L^2(Q)}^2 \leq TM^2 \|\delta\vec{r}\|_{\Sigma,Q}^2$, then $\|\delta\vec{\gamma}\|_{L^2(Q)} \leq \tilde{M} \|\delta\vec{r}\|_Q$, $\tilde{M}^2 = TM^2$. Similar steps and utilizing the above results, to acquire

$$\begin{aligned} \int_0^T \frac{d}{d\mathfrak{b}} \|\delta\vec{\gamma}\|_0^2 + 2 \int_0^T \|\delta\vec{\gamma}\|_1^2 d\mathfrak{b} &\leq \|\delta\vec{r}\|_{\Sigma,Q}^2 + \int_0^T \|\delta\vec{\gamma}\|_0^2 d\mathfrak{b} \leq \tilde{M}^2 \|\delta\vec{r}\|_{\Sigma,Q}^2 \Rightarrow \\ \|\delta\vec{\gamma}\|_{L^2(I,V)}^2 &\leq \bar{M}_1^2 \|\delta\vec{r}\|_{\Sigma,Q}^2, \text{ with } \bar{M}_1^2 = (1 + \tilde{M}^2)/2 \Rightarrow \|\delta\vec{\gamma}\|_{L^2(I,V)} \leq \bar{M}_1 \|\delta\vec{r}\|_{\Sigma,Q}^2. \end{aligned}$$

Theorem 4.2: The operator $\vec{u} \mapsto \vec{\gamma}_{\vec{u}}$ has the continuity property from $L^2(\Sigma, Q)$ in to $(L^\infty(I, L^2(\Omega)))^3$ or in to $(L^2(I, V))^3$ or in to $(L^2(Q))^3$.

Proof: $\delta\vec{r} = \vec{r} - \vec{r}^\epsilon$ & $\delta\vec{\gamma} = \vec{\gamma} - \vec{\gamma}^\epsilon$, then from the result of Theorem 4.1,

$$\|\vec{\gamma} - \vec{\gamma}^\epsilon\|_{L^\infty(I, L^2(\Omega))} \leq M \|\vec{r} - \vec{r}^\epsilon\|_{\Sigma,Q}, \text{ If } \vec{r} \xrightarrow{L^2(\Sigma,Q)} \vec{r}^\epsilon \text{ then } \vec{\gamma} \xrightarrow{(L^\infty(I, L^2(\Omega)))^3} \vec{\gamma}^\epsilon.$$

Which prove the Lipchitz continuity (LC) of $\vec{r} \mapsto \vec{\gamma}_{\vec{r}}$ form $L^2(\Sigma, Q)$ in to $(L^\infty(I, L^2(\Omega)))^3$. The other two results are obtained similarly.

Lemma 4.1 [15]: $\|\cdot\|_0$ is weakly lower semi continuous (WLSC).

Lemma 4.1: The OF $G_0(\vec{r})$ is WLSC

Proof: By Lemma (4.1) $\|\vec{r}\|_{\Sigma,Q}$, is WLSC and as $\vec{r}_k \xrightarrow{L^2(\Sigma,Q)} \vec{r}$ gives $\vec{\gamma}_k \xrightarrow{(L^2(Q))^3} \vec{\gamma}$
 $\xrightarrow{W}$

by Th. 4.1, thus $\|\vec{\gamma} - \vec{\gamma}_d\|_Q \leq \lim_{k \rightarrow \infty} \inf_{\gamma_k \in V_k} \|\vec{\gamma}_k - \vec{\gamma}_d\|_Q$.

Hence $\|\vec{\gamma} - \vec{\gamma}_d\|_Q$ is WLSC, thus $G_0(\vec{r})$ is WLSC.

Lemma 4.3 [13]: $\|\cdot\|_0^2$ is convex strictly.

Theorem 4.3: If $G_0(\vec{r})$ in (7), is coercive, then there is MOCV for MOCVP.

Proof: Since $G_0(\vec{r}) \geq 0$ and $G_0(\vec{r})$ satisfy the coercively property, then $\vec{V}_A$ has the sequence $\{\vec{r}_k\} = \{(\vec{r}_{1k}, \vec{r}_{2k}, \vec{r}_{3k})\}$, which minimize $G_0(\vec{r})$, s.t.

$$\lim_{n \rightarrow \infty} G_0(\vec{r}_k) = \inf_{\vec{r}_k \in \vec{V}_A} G_0(\vec{r}) , \text{ and } \|\vec{r}_k\|_{\Sigma, Q}^2 \leq c, \text{ then by AT there is}$$

a subsequence of $\{\vec{r}_k\}$, say again $\{\vec{r}_k\}$ (for simplicity), s.t. $\vec{r}_k \xrightarrow{L^2(\Sigma, Q)} \vec{r}$. But

Theorem 3.1, ensures there is a unique TSVS $\vec{\gamma}_k = \vec{\gamma}_{\vec{r}_k}$ of each conforming MCV $\vec{r}_k$, then conforming to the sequence $\{\vec{r}_k\}$ there is a sequence $\{\vec{\gamma}_k\}$ and that $\|\vec{\gamma}_k\|_{L^\infty(I, L^2(\Omega))}, \|\vec{\gamma}_k\|_{L^2(Q)} \& \|\vec{\gamma}_k\|_{L^2(I, W)}$ have the bounded property, thus utilizing AT there the sequence $\{\vec{\gamma}_k\}$ has a subsequence let be for simplicity $\{\vec{\gamma}_k\}$ (again), for which

$$\vec{\gamma}_k \xrightarrow[W]{L^\infty(I, L^2(\Omega))} \vec{\gamma}, \vec{\gamma}_k \xrightarrow[W]{L^2(Q)} \vec{\gamma}, \text{ and } \vec{\gamma}_k \xrightarrow[W]{L^2(I, W)} \vec{\gamma}.$$

Now, to prove $\|\vec{\gamma}_{k\mathbb{b}}\|_{(L^2(I, W^*))}^3$ is bounded, system (13)-(16) can be rewritten as

$$\begin{aligned} \left| \int_0^T \langle \vec{\gamma}_{k\mathbb{b}}, \vec{\omega} \rangle d\mathbb{b} \right| &\leq \|\nabla \gamma_{1k}\|_Q \|\nabla \omega_1\|_Q + \|\gamma_{1k}\|_Q \|\omega_1\|_Q + \|\gamma_{2k}\|_Q \|\omega_1\|_Q + \\ &\|\gamma_{3k}\|_Q \|\omega_1\|_Q + \|\nabla \gamma_{2k}\|_Q \|\nabla \omega_2\|_Q + \|\gamma_{2k}\|_Q \|\omega_2\|_Q + \|\gamma_{3k}\|_Q \|\omega_2\|_Q + \\ &\|\gamma_{1k}\|_Q \|\omega_2\|_Q + \|\nabla \gamma_{3k}\|_Q \|\nabla \omega_3\|_Q + \|\gamma_{3k}\|_Q \|\omega_3\|_Q + \|\gamma_{1k}\|_Q \|\omega_3\|_Q + \\ &\|\gamma_{2k}\|_Q \|\omega_3\|_Q + \|\eta_1\|_Q \|\omega_1\|_Q + \|\eta_2\|_Q \|\omega_2\|_Q + \|\eta_3\|_Q \|\omega_3\|_Q + \\ &\|\vec{r}_{1k}\|_\Sigma \|\omega_1\|_\Sigma + \|\vec{r}_{2k}\|_Q \|\omega_2\|_Q + \|\vec{r}_{3k}\|_\Sigma \|\omega_3\|_\Sigma \end{aligned}$$

Now, for $l = 1, 2, 3$ the following relations for the norms are held

$$\begin{aligned} \|\nabla \gamma_{lk}\|_Q &\leq \|\nabla \vec{\gamma}_k\|_Q \leq \|\vec{\gamma}_k\|_{L^2(I, V)}, \|\omega_l\|_Q \leq \|\vec{\omega}\|_Q \leq \\ \|\vec{\omega}\|_{L^2(I, W)}, \|\nabla \omega_l\|_Q &\leq \|\nabla \vec{\omega}\|_Q \leq \|\vec{\omega}\|_{L^2(I, W)}, \|\gamma_{lk}\|_Q \leq \|\vec{\gamma}_k\|_Q \leq \\ \|\vec{\gamma}_k\|_{L^2(I, V)}, \|\omega_l\|_\Sigma &\leq \|\vec{\omega}\|_\Sigma \leq c_1 \|\vec{\omega}\|_{L^2(I, W)}, \|\eta_i\|_Q \leq d'_i, \|\vec{r}_{lk}\|_\Sigma \leq \|\vec{r}_k\|_\Sigma \leq b, \\ \text{for } l = 1, 3 \text{ and } \|\vec{r}_{2k}\|_Q &\leq \|\vec{r}_k\|_\Sigma \leq b, \text{ then } \left| \int_0^T \langle \vec{\gamma}_{k\mathbb{b}}, \vec{\omega} \rangle d\mathbb{b} \right| \leq [3 \|\vec{\gamma}_k\|_{L^2(I, V)} + \\ 9 \|\vec{\gamma}_k\|_{L^2(I, W)} + (d'_1 + d'_2 + d'_3 + 3b)] &\|\vec{\omega}\|_{L^2(I, W)}, \text{ Or} \end{aligned}$$

$$\left| \int_0^T \langle \vec{\gamma}_{k\mathbb{b}}, \vec{\omega} \rangle d\mathbb{b} \right| \leq (12d_2(b) + d'(b)) \|\vec{\omega}\|_{L^2(I, W)}, \text{ where } \|\vec{\gamma}_k\|_{L^2(I, V)} \leq b_2(b) \text{ and } d'(b) = d'_1 + d'_2 + d'_3 + 3b \Rightarrow \frac{\left| \int_0^T \langle \vec{\gamma}_{k\mathbb{b}}, \vec{\omega} \rangle d\mathbb{b} \right|}{\|\vec{r}\|_{L^2(I, W)}} \leq d_3(b), \text{ with } d_3(b) =$$

$12d_2(b) + d'(b) \Rightarrow \|\vec{\gamma}_k\|_{L^2(I, V^*)} \leq d_3(b)$ since $\vec{\gamma}_k$ is TSVS of ((1)-(6)), then

$$\langle \gamma_{1k\mathbb{b}}, \omega_1 \rangle + (\nabla \gamma_{1k}, \nabla \omega_1) + (\gamma_{1k} - \gamma_{2k} - \gamma_{3k}, \omega_1) = (f_1, \omega_1) + (\vec{r}_{1k}, \omega_1)_\Gamma \quad (63)$$

$$\langle \gamma_{2k\mathbb{b}}, \omega_2 \rangle + (\nabla \gamma_{2k}, \nabla \omega_2) + (\gamma_{2k} + \gamma_{3k} + \gamma_{1k}, \omega_2) = (f_2 + \vec{r}_{2k}, \omega_2) \quad (64)$$

$$\langle \gamma_{3k\mathbb{b}}, \omega_3 \rangle + (\nabla \gamma_{3k}, \nabla \omega_3) + (\gamma_{3k} + \gamma_{1k} - \gamma_{2k}, \omega_3) = (f_3, \omega_3) + (\vec{r}_{3k}, \omega_3)_\Gamma \quad (65)$$

Integrating ((63)-(65)) on $[0, T]$, after multiplying each equality by $\theta_l \in C^1[0, T]$, with $\theta_l(T) = 0$, for each $l = 1, 2, 3$, rewriting each 1^{st} term of LHS of each resulting equations, and then integrating by parts all the terms which have first time derivative, they acquire

$$\begin{aligned} & - \int_0^T (\gamma_{1k}, \omega_1) \theta_1'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_{1k}, \nabla \omega_1) + (\gamma_{1k} - \gamma_{2k} - \gamma_{3k}, \omega_1)] \theta_1(\mathfrak{B}) d\mathfrak{B} \\ & = \int_0^T [(f_1, \omega_1) + (r_{1k}, \omega_1)_\Gamma] \theta_1(\mathfrak{B}) d\mathfrak{B} + (\gamma_{1k}(0), \omega_1) \theta_1(0) \end{aligned} \quad (66)$$

$$\begin{aligned} & - \int_0^T (\gamma_{2k}, \omega_2) \theta_2'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_{2k}, \nabla \omega_2) + (\gamma_{2k} + \gamma_{3k} + \gamma_{1k}, \omega_2)] \theta_2(\mathfrak{B}) d\mathfrak{B} \\ & = \int_0^T (f_2 + r_{2k}, \omega_2) \theta_2(\mathfrak{B}) d\mathfrak{B} + (\gamma_{2k}(0), \omega_2) \theta_2(0) \end{aligned} \quad (67)$$

$$\begin{aligned} & - \int_0^T (\gamma_{3k}, \omega_3) \theta_3'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_{3k}, \nabla \omega_3) + (\gamma_{3k} + \gamma_{1k} - \gamma_{2k}, \omega_3)] \theta_3(\mathfrak{B}) d\mathfrak{B} \\ & = \int_0^T [(f_3, \omega_3) + (r_{3k}, \omega_3)_\Gamma] \theta_3(\mathfrak{B}) d\mathfrak{B} + (\gamma_{3k}(0), \omega_3) \theta_3(0) \end{aligned} \quad (68)$$

since $\vec{\gamma}_k \xrightarrow[W]{L^2(Q)^3} \vec{\gamma}$ and $\vec{\gamma}_k \xrightarrow[W]{L^2(I, W)^3} \vec{\gamma}$, then the following are hold

$$\begin{aligned} & - \int_0^T (\gamma_{1k}, \omega_1) \theta_1'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_{1k}, \nabla \omega_1) + (\gamma_{1k} - \gamma_{2k} - \gamma_{3k}, \omega_1)] \theta_1(\mathfrak{B}) d\mathfrak{B} \\ & \rightarrow - \int_0^T (\gamma_1, \omega_1) \theta_1'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2 - \gamma_3, \omega_1)] \theta_1(\mathfrak{B}) d\mathfrak{B} \end{aligned} \quad (69)$$

$$\begin{aligned} & - \int_0^T (\gamma_{2k}, \omega_2) \theta_2'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_{2k}, \nabla \omega_2) + (\gamma_{2k} + \gamma_{3k} + \gamma_{1k}, \omega_2)] \theta_2(\mathfrak{B}) d\mathfrak{B} \\ & \rightarrow - \int_0^T (\gamma_2, \omega_2) \theta_2'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_2, \nabla \omega_2) + (\gamma_2 + \gamma_3 + \gamma_1, \omega_2)] \theta_2(\mathfrak{B}) d\mathfrak{B} \end{aligned} \quad (70)$$

$$\begin{aligned} & - \int_0^T (\gamma_{3k}, \omega_3) \theta_3'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_{3k}, \nabla \omega_3) + (\gamma_{3k} + \gamma_{1k} - \gamma_{2k}, \omega_3)] \theta_3(\mathfrak{B}) d\mathfrak{B} \\ & \rightarrow - \int_0^T (\gamma_3, \omega_3) \theta_3'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_3, \nabla \omega_3) + (\gamma_3 + \gamma_1 - \gamma_2, \omega_3)] \theta_3(\mathfrak{B}) d\mathfrak{B} \end{aligned} \quad (71)$$

From the bounded $(\gamma_{1k}(0), \gamma_{2k}(0), \gamma_{3k}(0))$ in $(L^2(\Omega))^3$ and the projection theorem one has

$$(\gamma_{lk}^0, \omega_l) \theta_l(0) \rightarrow (\gamma_l^0, \omega_l) \theta_l(0), \text{ for } l = 1, 2, 3 \quad (72)$$

and since $\vec{r}_k \xrightarrow[W]{L^2(\Sigma, Q)} \vec{r}$, then

$$\int_0^T (f_l, \omega_l) + (r_{lk}, \omega_l)_\Gamma \theta_l(\mathfrak{B}) d\mathfrak{B} \rightarrow \int_0^T (f_l, \omega_l) + (r_l, \omega_l)_\Gamma \theta_l(\mathfrak{B}) d\mathfrak{B}, \forall l = 1, 2 \quad (73)$$

$$\int_0^T (f_2 + r_{2k}, \omega_2) \theta_2(\mathfrak{B}) d\mathfrak{B} \rightarrow \int_0^T (f_2 + r_2, \omega_2) \theta_2(\mathfrak{B}) d\mathfrak{B} \quad (74)$$

Utilizing (69) –(74) in ((66)–(68)), to acquire

$$\begin{aligned} & - \int_0^T (\gamma_1, \omega_1) \theta_1'(\mathfrak{B}) d\mathfrak{B} + \int_0^T [(\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2 - \gamma_3, \omega_1)] \theta_1(\mathfrak{B}) d\mathfrak{B} = \\ & \int_0^T [(f_1, \omega_1) + (r_1, \omega_1)_\Gamma] \theta_1(\mathfrak{B}) d\mathfrak{B} + (\gamma_1^0, \omega_1) \theta_1(0) \end{aligned} \quad (75)$$

$$\begin{aligned}
& - \int_0^T \langle \gamma_2, \omega_2 \rangle \theta_2'(\mathfrak{b}) d\mathfrak{b} + \int_0^T [(\nabla \gamma_2, \nabla \omega_2) + (\gamma_2 + \gamma_3 + \gamma_1, \omega_2)] \theta_2(\mathfrak{b}) d\mathfrak{b} \\
& = \int_0^T \langle \mathfrak{f}_2 + \mathfrak{r}_2, \omega_2 \rangle \theta_2(\mathfrak{b}) d\mathfrak{b} + (\gamma_2^0, \omega_2) \theta_2(0)
\end{aligned} \tag{76}$$

$$\begin{aligned}
& - \int_0^T \langle \gamma_3, \omega_3 \rangle \theta_3'(\mathfrak{b}) d\mathfrak{b} + \int_0^T [(\nabla \gamma_3, \nabla \omega_3) + (\gamma_3 + \gamma_1 - \gamma_2, \omega_3)] \theta_3(\mathfrak{b}) d\mathfrak{b} \\
& = \int_0^T [\langle \mathfrak{f}_3, \omega_3 \rangle + (\mathfrak{r}_3, \omega_3)_\Gamma] \theta_3(\mathfrak{b}) d\mathfrak{b} + (\gamma_3^0, \omega_3) \theta_3(0)
\end{aligned} \tag{77}$$

Case 1: Select $\theta_l \in C^1[0, T]$, i.e. $\theta_l(0) = \theta_l(T) = 0$, ($l = 1, 2, 3$), utilizing integrating “by parts” all terms that have first time derivative in LHS of ((77)–(79)), to acquire ($\forall \omega_l \in W_l$)

$$\begin{aligned}
& \int_0^T \langle \gamma_{1\mathfrak{b}}, \omega_1 \rangle \theta_1(\mathfrak{b}) d\mathfrak{b} + \int_0^T [(\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2 - \gamma_3, \omega_1)] \theta_1(\mathfrak{b}) d\mathfrak{b} = \\
& \int_0^T [\langle \mathfrak{f}_1, \omega_1 \rangle + (\mathfrak{r}_1, \omega_1)_\Gamma] \theta_1(\mathfrak{b}) d\mathfrak{b}
\end{aligned} \tag{78}$$

$$\begin{aligned}
& \int_0^T \langle \gamma_{2\mathfrak{b}}, \omega_2 \rangle \theta_2(\mathfrak{b}) d\mathfrak{b} + \int_0^T [(\nabla \gamma_2, \nabla \omega_2) + (\gamma_2 + \gamma_3 + \gamma_1, \omega_2)] \theta_2(\mathfrak{b}) d\mathfrak{b} \\
& = \int_0^T \langle \mathfrak{f}_2 + \mathfrak{r}_2, \omega_2 \rangle \theta_2(\mathfrak{b}) d\mathfrak{b}
\end{aligned} \tag{79}$$

$$\begin{aligned}
& \int_0^T \langle \gamma_{3\mathfrak{b}}, \omega_3 \rangle \theta_3(\mathfrak{b}) d\mathfrak{b} + \int_0^T [(\nabla \gamma_3, \nabla \omega_3) + (\gamma_3 + \gamma_1 - \gamma_2, \omega_3)] \theta_3(\mathfrak{b}) d\mathfrak{b} \\
& = \int_0^T [\langle \mathfrak{f}_3, \mathfrak{r}_3 \rangle + (\mathfrak{r}_3, \omega_3)_\Gamma] \theta_3(\mathfrak{b}) d\mathfrak{b}
\end{aligned} \tag{80}$$

Therefore (78)–(80) give a.e. on I that

$$\begin{aligned}
& \langle \gamma_{1\mathfrak{b}}, \omega_1 \rangle + (\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_3 - \gamma_2, \omega_1) = \langle \mathfrak{f}_1, \omega_1 \rangle + (\mathfrak{r}_1, \omega_1)_\Gamma, \forall \omega_1 \in W_1. \\
& \langle \gamma_{2\mathfrak{b}}, \omega_2 \rangle + (\nabla \gamma_2, \nabla \omega_2) + (\gamma_2 + \gamma_3 + \gamma_1, \omega_2) = \langle \mathfrak{f}_2 + \mathfrak{r}_2, \omega_2 \rangle, \forall \mathfrak{r}_2 \in W_2. \\
& \langle \gamma_{3\mathfrak{b}}, \omega_3 \rangle + (\nabla \gamma_3, \nabla \omega_3) + (\gamma_2 + \gamma_1 - \gamma_2, \omega_3) = \langle \mathfrak{f}_3, \omega_3 \rangle + (\mathfrak{r}_3, \omega_3)_\Gamma, \forall \omega_3 \in W_3.
\end{aligned}$$

i.e. $\vec{\gamma}$ satisfies the SWF .

Case 2: Select $\theta_l \in C^1[I]$, s.t. $\theta_l(T) = 0$, $\theta_l(0) \neq 0$, ($l = 1, 2, 3$). Utilizing integration by parts for all term that have first time derivative in LHS of ((78)–(80)), to acquire

$$\begin{aligned}
& - \int_0^T \langle \gamma_1, \omega_1 \rangle \theta_1'(\mathfrak{b}) d\mathfrak{b} + \int_0^T [(\nabla \gamma_1, \nabla \omega_1) + (\gamma_1 - \gamma_2 - \gamma_3, \omega_1)] \theta_1(\mathfrak{b}) d\mathfrak{b} = \\
& \int_0^T [\langle \mathfrak{f}_1, \omega_1 \rangle + (\mathfrak{r}_1, \omega_1)_\Gamma] \theta_1(\mathfrak{b}) d\mathfrak{b} + (\gamma_1(0), \omega_1) \theta_1(0)
\end{aligned} \tag{81}$$

$$\begin{aligned}
& - \int_0^T \langle \gamma_2, \omega_2 \rangle \theta_2'(\mathfrak{b}) d\mathfrak{b} + \int_0^T [(\nabla \gamma_2, \nabla \omega_2) + (\gamma_2 + \gamma_3 - \gamma_2, \omega_2)] \theta_2(\mathfrak{b}) d\mathfrak{b} \\
& = \int_0^T \langle \mathfrak{f}_2 + \mathfrak{r}_2, \omega_2 \rangle \theta_2(\mathfrak{b}) d\mathfrak{b} + (\gamma_2(0), \omega_2) \theta_2(0)
\end{aligned} \tag{82}$$

$$\begin{aligned}
& - \int_0^T \langle \gamma_3, \omega_3 \rangle \theta_3'(\mathfrak{b}) d\mathfrak{b} + \int_0^T [(\nabla \gamma_3, \nabla \omega_3) + (\gamma_3 + \gamma_1 - \gamma_2, \omega_3)] \theta_3(\mathfrak{b}) d\mathfrak{b} \\
& = \int_0^T [\langle \mathfrak{f}_3, \omega_3 \rangle + (\mathfrak{r}_3, \omega_3)_\Gamma] \theta_3(\mathfrak{b}) d\mathfrak{b} + (\gamma_3(0), \omega_3) \theta_3(0)
\end{aligned} \tag{83}$$

Subtracting ((81)–(83)) from ((75)–(77)), to get

$$(\gamma_l^0, \omega_l) \theta_l(0) = (\gamma_l(0), \omega_l) \theta_l(0), \forall l = 1, 2, 3 \Rightarrow \gamma_l^0 = \gamma_l(0) = \gamma_l^0(x),$$

Then $\vec{r}$ is a TSVS of the SWF. Since $G_0(\vec{r})$ is WLSC (by Lemma 4.1) and $L^2(\Sigma, Q)$ since $\vec{r}_k \rightarrow \vec{r}$, therefore

$$G_0(\vec{r}) \leq \lim_{k \rightarrow \infty} \inf_{\vec{r}_k \in \vec{V}_A} G_0(\vec{r}_k) = \lim_{k \rightarrow \infty} G_0(\vec{r}_k) = \inf_{\vec{r} \in \vec{V}_A} G_0(\vec{r}) \Rightarrow G_0(\vec{r}) = \inf_{\vec{r} \in \vec{V}_A} G_0(\vec{r}) = \min_{\vec{r} \in \vec{V}_A} G_0(\vec{r}).$$

Then $\vec{u}$ is a MOCV.

5. The Necessity optimality condition

To study the NCs for OMCVP, the TAEs associated to the TSPE must be derived beside to the FD for the OF (7).

Theorem 5.1: Consider $G_0(\vec{u})$ in (7) & the TAEs of TSPE (1-6) are given by

$$-z_{1b} - \Delta z_1 + z_1 + z_2 + z_3 = (\gamma_1 - \gamma_{1d}) \quad (84)$$

$$-z_{2b} - \Delta z_2 + z_2 - z_1 - z_3 = (\gamma_2 - \gamma_{2d}) \quad (85)$$

$$-z_{3b} - \Delta z_3 + z_3 - z_1 + z_2 = (\gamma_3 - \gamma_{3d}) \quad (86)$$

$$z_l(T) = 0, \forall l = 1, 2, 3 \quad (87)$$

$$\frac{\partial z_l}{\partial n_l} = \sum_{i=1}^2 \frac{\partial z_l}{\partial x_j} \cos(n_l, x_j) = 0, \text{ on } \Sigma, \text{ for } l = 1, 3 \quad (88)$$

$$z_2(x, b) = 0, \text{ on } \Sigma \quad (89)$$

Then $\vec{r} \in \vec{V}_A$ and the FD of G_0 is given by $(G'_0(\vec{r}), \delta \vec{r}) = (\vec{z} + \beta \vec{r}, \delta \vec{r})$

Proof: The SWF of ((84)–(89)) for $\omega_l \in W_l, \forall l = 1, 2, 3$ is given by

$$-\langle z_{1b}, \omega_1 \rangle + (\nabla z_1, \nabla \omega_1) + (z_1 + z_2 + z_3, \omega_1) = (\gamma_1 - \gamma_{1d}, \omega_1) \quad (90)$$

$$-\langle z_{2b}, \omega_2 \rangle + (\nabla z_2, \nabla \omega_2) + (z_2 - z_1 - z_3, \omega_2) = (\gamma_2 - \gamma_{2d}, \omega_2) \quad (91)$$

$$-\langle z_{3b}, \omega_3 \rangle + (\nabla z_3, \nabla \omega_3) + (z_3 - z_1 + z_2, \omega_3) = (\gamma_3 - \gamma_{3d}, \omega_3) \quad (92)$$

Now, substituting $\omega_l = z_l$, for $l = 1, 2, 3$ in ((58)–(60)) respectively, these equations become,

$$\langle \delta \gamma_{1b}, z_1 \rangle + (\nabla \delta \gamma_1, \nabla z_1) + (\delta \gamma_1 - \delta \gamma_2 - \delta \gamma_3, z_1) = (\delta r_1, z_1)_\Gamma \quad (93)$$

$$\langle \delta \gamma_{2b}, z_2 \rangle + (\nabla \delta \gamma_2, \nabla z_2) + (\delta \gamma_2 + \delta \gamma_3 + \delta \gamma_1, z_2) = (\delta r_2, z_2)_\Gamma \quad (94)$$

$$\langle \delta \gamma_{3b}, z_3 \rangle + (\nabla \delta \gamma_3, \nabla z_3) + (\delta \gamma_3 + \delta \gamma_1 - \delta \gamma_2, z_3) = (\delta r_3, z_3)_\Gamma \quad (95)$$

Also, substituting $\omega_l = \delta \gamma_l$ for $l = 1, 2, 3$ in (90–92) respectively, to get

$$-\langle z_{1b}, \delta \gamma_1 \rangle + (\nabla z_1, \nabla \delta \gamma_1) + (z_1 + z_2 + z_3, \delta \gamma_1) = (\gamma_1 - \gamma_{1d}, \delta \gamma_1) \quad (96)$$

$$-\langle z_{2b}, \delta \gamma_2 \rangle + (\nabla z_2, \nabla \delta \gamma_2) + (z_2 - z_1 - z_3, \delta \gamma_2) = (\gamma_2 - \gamma_{2d}, \delta \gamma_2) \quad (97)$$

$$-\langle z_{3b}, \delta \gamma_3 \rangle + (\nabla z_3, \nabla \delta \gamma_3) + (z_3 - z_1 + z_2, \delta \gamma_3) = (\gamma_3 - \gamma_{3d}, \delta \gamma_3) \quad (98)$$

Integrating ((93)–(95)) on $[0, T]$, utilizing integration by parts for all the 1st terms in LHS of each resulting equality of ((96)–(98)), and then subtracting each resulting equality from its conforming equality ((93)–(95)), add the resulting equality to gather, it yields that

$$\int_0^T [(\delta r_1, z_1)_\Gamma + (\delta r_2, z_2) + (\delta r_3, z_3)_\Gamma] d\mathfrak{t} = \int_0^T [(\gamma_1 - \gamma_{1d}, \delta \gamma_1) + (\gamma_2 - \gamma_{2d}, \delta \gamma_2) + (\gamma_3 - \gamma_{3d}, \delta \gamma_3)] d\mathfrak{t} \quad (99)$$

Now, adding each of the pair of equations ((8)&(58)), ((9)&(59)), ((10)&(60) and ((11)-(61)), one has

$$\begin{aligned} & \langle (\gamma_1 + \delta \gamma_1)_\mathfrak{h}, \omega_1 \rangle + (\nabla(\gamma_1 + \delta \gamma_1), \nabla \omega_1) + (\gamma_1 + \delta \gamma_1, \omega_1) - (\gamma_2 + \delta \gamma_2, \omega_1) - (\gamma_3 + \delta \gamma_3, \omega_1) = (f_1, \omega_1) + (\delta r_1, \omega_1)_\Gamma \\ & \langle (\gamma_2 + \delta \gamma_2)_\mathfrak{h}, \omega_2 \rangle + (\nabla(\gamma_2 + \delta \gamma_2), \nabla \omega_2) + (\gamma_2 + \delta \gamma_2, \omega_2) + (\gamma_3 + \delta \gamma_3, \omega_2) + (\gamma_1 + \delta \gamma_1, \omega_2) = (f_2 + u_2 + \delta r_2, r_2) \\ & \langle (\gamma_3 + \delta \gamma_3)_\mathfrak{h}, \omega_3 \rangle + (\nabla(\gamma_3 + \delta \gamma_3), \nabla \omega_3) + (\gamma_3 + \delta \gamma_3, \omega_3) + (\gamma_1 + \delta \gamma_1, \omega_3) - (\gamma_2 + \delta \gamma_2, \omega_3) = (f_3, \omega_3) + (\delta r_3, r_3)_\Gamma. \end{aligned}$$

Thus, the MCV $(r_1 + \delta r_1, r_2 + \delta r_2, r_3 + \delta r_3)$ gives TSVS $(\gamma_1 + \delta \gamma_1, \gamma_2 + \delta \gamma_2, \gamma_3 + \delta \gamma_3)$. Now, from of (7), so, $G_0(\vec{r} + \delta \vec{r}) - G_0(\vec{r}) = (\delta r_1, z_1)_\Sigma + (\delta r_2, z_2) + (\delta r_3, z_3)_\Sigma + (\beta r_1, \delta r_1) + (\beta r_2, \delta r_2) + (\beta r_3, \delta r_3) + \frac{1}{2} \|\delta \vec{r}\|_Q^2 + \frac{\beta}{2} \|\delta \vec{r}\|_{\Sigma, Q}^2$, Or $G_0(\vec{r} + \delta \vec{r}) - G_0(\vec{r}) = (\vec{z} + \beta \vec{r}, \delta \vec{r})_{\Sigma, Q} + \frac{1}{2} \|\delta \vec{r}\|_Q^2 + \frac{\beta}{2} \|\delta \vec{r}\|_{\Sigma, Q}^2$. From the first results of Theorem 4.1, it acquires $\frac{1}{2} \|\delta \vec{r}\|_Q^2 = \varepsilon_1(\delta \vec{r}) \|\delta \vec{r}\|_{\Sigma, Q}$ and $\frac{\beta}{2} \|\delta \vec{r}\|_{\Sigma, Q}^2 = \varepsilon_2(\delta \vec{r}) \|\delta \vec{r}\|_{\Sigma, Q}$ with $\varepsilon_1(\delta \vec{r}) = \frac{1}{2} M^2 \|\delta \vec{r}\|_{\Sigma, Q}$, where $\varepsilon_1(\delta \vec{r}), \varepsilon_2(\delta \vec{r}) \rightarrow 0$, as $\|\delta \vec{r}\|_{\Sigma, Q} \rightarrow 0$

Then $G_0(\vec{r} + \delta \vec{r}) - G_0(\vec{r}) = (\vec{z} + \beta \vec{r}, \delta \vec{r})_{\Sigma, Q} + \varepsilon(\delta \vec{r}) \|\delta \vec{r}\|_{\Sigma, Q}$ with $\varepsilon(\delta \vec{r}) = \varepsilon_1(\delta \vec{r}) + \varepsilon_2(\delta \vec{r}) \rightarrow 0$, as $\|\delta \vec{r}\|_{\Sigma, Q} \rightarrow 0$. Applying the definition of FD of G_0 , one has $(G'_0(\vec{r}), \delta \vec{r}) = (\vec{z} + \beta \vec{r}, \delta \vec{r})_{\Sigma, Q}$.

Theorem 5.2: If $\vec{r}$ is OMCV of the problem, then $(\vec{z} + \beta \vec{r}, \vec{r})_{\Sigma, Q} \geq (\vec{z} + \beta \vec{r}, \vec{\tilde{r}})_{\Sigma, Q}, \forall \vec{r} \in L^2(\Sigma, Q)$.

Proof: If $\vec{r}$ is OMCV of the problem, then $G_0(\vec{r}) = \min_{\vec{r} \in \bar{V}_A} G_0(\vec{r}), \forall \vec{r} \in L^2(\Sigma, Q)$. Hence the FD of $G_0(\vec{r}) = 0$, thus $G'_0(\vec{r}) = 0 \Rightarrow \vec{z} + \beta \vec{r} = 0 \Rightarrow$ The NCO is $\vec{\tilde{r}} = -\frac{1}{\beta} \vec{z}$ and for $\delta \vec{r} = \vec{r} - \vec{\tilde{r}}$

$$(\vec{z} + \beta \vec{u}, \vec{r})_{\Sigma, Q} \geq (\vec{z} + \beta \vec{r}, \vec{\tilde{r}})_{\Sigma, Q}, \forall \vec{r} \in L^2(\Sigma, Q) \Rightarrow (\vec{z} + \beta \vec{r}, \delta \vec{r})_{\Sigma, Q} \geq 0.$$

6. Conclusions

The Galerkin approach was utilized successfully to demonstrate the existence for a unique TSVS for the weak formulation system of the TSPE when the mixed control vector is known. The “Lipschitz” operator between the mixed control and its conforming triple state was established and was proved continuous. Under suitable suppositions, the existence for optimal mixed control vector governing by the TSVS was demonstrated. The Fréchet derivative for the objective functional is derived. The derivation of the adjoint triple system associated to the TSVS was found. The necessity optimality theorem “conditions” of the proposed problem was found.

References

- [1] M. T. Hoang and K. Q. N. Thi, "Global asymptotic stability of a basic virus dynamics model with feedback controls," *J. Interdiscip. Math.*, vol. 25, no. 4, pp. 1127–1141 (2021).
- [2] I. Syahrini, R. Masabar, A. Aliasuddin, S. Munzir, and Y. Hazmi, "The application of optimal control through fiscal policy on the Indonesian economy," *J. Asian Finance Econ. Bus.*, vol. 8, no. 3, pp. 741–750 (2021).
- [3] S. Khan, R. K. Tailor, R. Pareek, R. Gujrati, and H. Uygun, "Application of robotic process automation in the education sector," *J. Inf. Optim. Sci.*, vol. 43, no. 7, pp. 1815–1834 (2022).
- [4] A. Khtil and N. Fezans, "Gust load alleviation for flexible aircraft using discrete-time H_∞ preview control," *Aeronaut. J.*, vol. 125, no. 1284, pp. 341–364 (2021).
- [5] A. Manzoni, A. Quarteroni, and S. Salsa, *Optimal Control of Partial Differential Equations, Analysis, Approximation, and Applications, Applied Mathematical Sciences*, vol. 207. New York, NY, USA: Springer (2021).
- [6] L. Kong and M. Wang, "Optimal control for an ordinary differential equation online social network model," *Differ. Equ. Appl.*, vol. 14, no. 2, pp. 205–214 (2022).
- [7] Y. Duan, P. Wu, and Y. Zhou, "An approximating approach to an optimal control problem for an elliptic variational inequality with mixed boundary conditions," *Numer. Funct. Anal. Appl.*, vol. 43, no. 9, pp. 1095–1113 (2022).
- [8] Y. Gu and Y. Zhu, "Optimal control for parabolic uncertain systems based on wavelet transformation," *Axioms*, vol. 11, no. 9, pp. 1–17 (2022).
- [9] A. Arguchintsev and V. P. Poplevko, "An optimal control problem for a hyperbolic system with boundary delay," *J. Math. Sci.*, vol. 35, pp. 3–17 (2021).
- [10] G. Turinici and H. Rabitz, "Quantum wave function controllability," *Chem. Phys.*, vol. 267, nos. 1–3, pp. 1–9 (2001).
- [11] S. J. Al-Qaisi, G. M. Kadhem, and J. A. Al-Hawasy, "Mixed optimal control vector for a boundary value problem of coupled nonlinear elliptic equations," *Iraqi J. Sci.*, vol. 63, no. 9, pp. 3861–3866 (2022).

- [12] G. M. Kadhém, A. A. H. Naeif, and J. A. Al-Hawasy, "The classical continuous mixed optimal control of coupled nonlinear parabolic partial differential equations with state constraints," *Iraqi J. Sci.*, vol. 62, no. 12, pp. 4859–4874 (2021).
- [13] L. H. Ali and J. A. Al-Hawasy, "Boundary optimal control for triple nonlinear hyperbolic boundary value problems," *Iraqi J. Sci.*, vol. 62, no. 6, pp. 2009–2021 (2021).
- [14] N. A. Thyab, "The continuous classical boundary optimal control for some boundary value problems of triple elliptic type," M.S. thesis, Dept. Math., Mustansiriyah Univ., Baghdad, Iraq (2020).
- [15] T. M. Rasheed, "The continuous classical optimal control for nonlinear boundary value problems of triple parabolic type," M.S. thesis, Mustansiriyah Univ., Baghdad, Iraq (2021).
- [16] R. Kamel, M. Kareem, and A. Hussain, "On the n -normed space of L^∞ measurable functions," *J. Interdiscip. Math.*, vol. 25, no. 5, pp. 1453–1460 (2022).
- [17] R. Temam, Navier–Stokes Equations. Amsterdam, The Netherlands: North-Holland (1977).

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