Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 1, pp. 37–56
DOI : 10.47974/JIM-2113

On the mixed multifractal dimensions of product measures and sets

Rim Achour
Department of Mathematics
Analysis, Probability and Fractals Laboratory LR18ES17
Faculty of Sciences of Monastir
University of Monastir
5000 Monastir
Tunisia

Abstract

In this paper, our intention is to approximate the mixed multifractal Hausdorff and packing dimensions of product measures. The main results in this article examine the relationships with various mixed multifractal dimensions of measures on $\mathbb{R}^k$ and extend many well-known results concerning Hausdorff and packing of measures and dimensions.

Subject Classification: *28A20, 28A75, 28A78, 28A80, 49Q15.*

Keywords: *Mixed multifractal Hausdorff measures, Mixed multifractal packing measures, Mixed multifractal Hausdorff dimensions, Mixed multifractal packing dimensions.*

1. Introduction

The multifractal analysis of measures has produced a great interest in fractal geometry. It has been used in many fields such as turbulence analysis, earthquake analysis, dynamical systems, financial time series modeling, and rainfall modeling. Therefore, the continued attraction of researchers toward developing mathematical theories and methods for multifractal measures remains highly significant. While mixed multifractal analysis is a prolongation of multifractal analysis of single objects like functions, measures, distributions, statistical data, etc. It was recently developed purely from a mathematical perspective. In statistics and physics, it has been strongly linked to mathematical theory and appeared in different forms (see [7]). In some applications like break fields, each attribute of a data sample can be described by multiple types of measures. Researchers use well-adapted measures for mixed-type data like [7]. The mixed multifractal analysis investigates the simultaneous scaling behavior of several

E-mail: **rim.achour.jeddi@gmail.com**

finite measures and gives the foundation for a much clearer comprehension of the local geometry of fractal and multifractal measures.

In this paper, we analyze the mixed multifractal generalization of Hausdorff and packing measures and dimensions of product measures presented in [20]. Let $\mathbb{R}_+^d$ ($\mathbb{R}_-^d$, respectively) be a vector that contains d components where all the components are positive real numbers (negative real numbers, respectively). Consider $\mathcal{P}(\mathbb{R}^k)$ the family of Borel probability measures on $\mathbb{R}^k$ and $\mathcal{P}_F(\mathbb{R}^k)$ the family of Borel probability measures on $\mathbb{R}^k$ that satisfies Federer condition (see Section 3) and $\mathcal{P}_F^d(\mathbb{R}^k) = \mathcal{P}_F(\mathbb{R}^k) \times \ldots \times \mathcal{P}_F(\mathbb{R}^k)$ d times. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathcal{P}^d(\mathbb{R}^k)$ and x within $supp\ \boldsymbol{\mu}$, where $supp\ \boldsymbol{\mu}$ is the topological support of $\boldsymbol{\mu}$ and $\mathcal{P}^d(\mathbb{R}^k) = \mathcal{P}(\mathbb{R}^k) \times \ldots \times \mathcal{P}(\mathbb{R}^k)$ d times.

The aim is to examine the concurrent scaling behavior of $\boldsymbol{\mu}$ then the lower and upper local dimensions of $\boldsymbol{\mu}$ at x are described as follows

$$\underline{\alpha}_{\boldsymbol{\mu}}(x) = \liminf_{r\downarrow 0} \frac{\log \boldsymbol{\mu}(\mathcal{B}_r(x))}{\log r} = \left(\lim_{r\downarrow 0} \frac{\log \mu_1(\mathcal{B}_r(x))}{\log r}, \ldots, \lim_{r\downarrow 0} \frac{\log \mu_d(\mathcal{B}_r(x))}{\log r}\right),$$

and

$$\overline{\alpha}_{\boldsymbol{\mu}}(x) = \limsup_{r\downarrow 0} \frac{\log \boldsymbol{\mu}(\mathcal{B}_r(x))}{\log r} = \left(\lim_{r\downarrow 0} \frac{\log \mu_1(\mathcal{B}_r(x))}{\log r}, \ldots, \lim_{r\downarrow 0} \frac{\log \mu_d(\mathcal{B}_r(x))}{\log r}\right).$$

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, let

$$\underline{\Gamma_{\boldsymbol{\alpha}}} = \{x \in supp(\boldsymbol{\mu}) \mid \alpha_j \leq \underline{\alpha}_{\mu_j}(x),\ \forall j = 1, \ldots, d\} \text{ and}$$

$$\overline{\Gamma}^{\boldsymbol{\alpha}} = \{x \in supp(\boldsymbol{\mu}) \mid \alpha_j \geq \overline{\alpha}_{\mu_j}(x), \forall j = 1, \ldots, d\}.$$

If $\overline{\alpha}_{\boldsymbol{\mu}}(x)$, $\underline{\alpha}_{\boldsymbol{\mu}}(x)$ are equal, then we indicate their shared value by $\alpha_{\boldsymbol{\mu}}(x)$ and we call this quantity by the local dimension. We defined for $\boldsymbol{\alpha} \in \mathbb{R}_+^d$, the level set $\Gamma_{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = \{x \in supp\ \boldsymbol{\mu} \mid \alpha_{\boldsymbol{\mu}}(x) = \boldsymbol{\alpha}\}$.

L. Olsen is among the first who worked on the mixed multifractal analysis. For example, in [22], he studied a thorough analysis of point sets that are points of divergence for all measures $\mu_1, \ldots, \mu_d$ at the same time, and shows that these points have an amazingly rich structure, while in [23] he provided a thorough explanation of the mixed multifractal theory of finitely many self-similar measures. By the way, L. Olsen worked a lot on the multifractal analysis. For example, in [21] he studied some self-similar measures on the multifractal structure. In [25], he was interested in the self-affine measures and random self-similar measures in [26] as well as the product measures done in [24].

B. Selmi et al. also have several works on the multifractal analysis see for example [3], [31], [32], [34], and on the mixed multifractal analysis for example on the mixed multifractal densities and regularities with respect to gaugs in [2], for vector-valued measures in [33] and for homogeneous Moran measures in [11], and many other authors did analyze and study the multifractal structure of specific measures such as [1], [5], [6], [10], [14], [17], [16], [15], [18], [27], [28], [29] [30].

In our case, we study with the mixed multifractal generalization of packing and Hausdorff measures of product measures in the following manner: Consider $\boldsymbol{\mu}$ as a Borel probability measure, $t \in \mathbb{R}$ and $\mathbf{q} \in \mathbb{R}^d$ then we denote $\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ the mixed multifractal Hausdorff measure and $\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ the mixed multifractal packing measure. Let $\boldsymbol{\mu} \in \mathcal{P}_F^d(\mathbb{R}^k)$, $\boldsymbol{\nu} \in \mathcal{P}_F^d(\mathbb{R}^l)$, $\mathbf{q} \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$ then we can find $c > 0$ such that

$$\int \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(D^y) d\mathcal{H}_{\boldsymbol{\nu}}^{\mathbf{q},t}(y) \le c\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(D),$$

$$\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B) \le c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B),$$

$$\int \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(D^y) d\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(y) \le c\mathcal{P}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(D),$$

$$\mathcal{P}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B) \le c\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B),$$

for $A \subseteq \mathbb{R}^k$, $B \subseteq \mathbb{R}^l$, $D \subseteq \mathbb{R}^{k+l}$ and $D^y = \{x \in \mathbb{R}^k | \quad (x, y) \in D\}$.

Now, we structure the paper in the subsequent manner: In Section 2 we provide various important definitions and some preliminaries necessary for the continuation. The main results can be found in Section 3, while Section 4 includes the demonstrations of the key findings. Finally, we study an example in a particular case within Section 5. We start by defining the mixed multifractal generalizations of packing and Hausdorff measures and we study their dimensions.

2. Definitions and preliminary

Let $A \subseteq \mathbb{R}^k$, $A \neq \emptyset$ and $\delta > 0$. Let also $\boldsymbol{\mu} \in \mathcal{P}^{\mathbf{d}}(\mathbb{R}^k)$, $\mathbf{q} = (q_1, \ldots, q_d) \in \mathbb{R}^d$ and t be a real number. The mixed generalized multifractal Hausdorff and packing measures are detailed in the following manner:

Let

$$\boldsymbol{\mu}(\mathcal{B}_r(x)) \equiv (\mu_1(\mathcal{B}_r(x)), \ldots, \mu_d(\mathcal{B}_r(x))),$$

where $\mathcal{B}_r(x)$ is the closed ball such that $r \ge 0$ is its radius and $x \in \mathbb{R}^k$ is its center. Then the product

$$(\boldsymbol{\mu}(\mathcal{B}_r(x)))^{\mathbf{q}} \equiv (\mu_1(\mathcal{B}_r(x)))^{q_1} \ldots (\mu_d(\mathcal{B}_r(x)))^{q_d}.$$

Next, let

$$\begin{aligned}\overline{\mathcal{H}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(A) &= \inf\left\{\sum_i (\mu_1(\mathcal{B}_{r_i}(x_i)))^{q_1} \ldots (\mu_d(\mathcal{B}_{r_i}(x_i)))^{q_d}(2r_i)^t\right\} \\ &= \inf\left\{\sum_i (\boldsymbol{\mu}(\mathcal{B}_{r_i}(x_i)))^{\mathbf{q}}(2r_i)^t\right\},\end{aligned}$$

where $(\mathcal{B}_{r_i}(x_i))_i$ is a centered δ-covering of A, and the infimum is considered across the set of every centered δ-covering of A. We have $\overline{\mathcal{H}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(\emptyset) = 0, \text{for} A = \emptyset$. Now, define the limit as $\delta \searrow 0$,

$$\overline{\mathcal{H}}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = \sup_{\delta>0} \overline{\mathcal{H}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(A) = \lim_{\delta\to 0} \overline{\mathcal{H}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(A)$$

and let

$$\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = \sup_{B\subseteq A} \overline{\mathcal{H}}_{\boldsymbol{\mu}}^{\mathbf{q},t}(B).$$

$\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ is termed the mixed generalized multifractal Hausdorff measure on $\mathbb{R}^k$ and it is an outer metric measure while the restriction $\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ on Borel sets is a measure.

Now, let

$$\begin{aligned}\overline{\mathcal{P}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(A) &= \sup\left\{\sum_i \left((\mu_1(\mathcal{B}_{r_i}(x_i)))^{q_1} \dots (\mu_d(\mathcal{B}_{r_i}(x_i)))^{q_d}\right)(2r_i)^t\right\}\\ &= \sup\left\{\sum_i (\boldsymbol{\mu}(\mathcal{B}_{r_i}(x_i)))^{\mathbf{q}}(2r_i)^t\right\},\end{aligned}$$

where $(\mathcal{B}_{r_i}(x_i))_i$ is a centered δ-packing of A, and the supremum is considered across the set of all centered δ-packing of A. We have $\overline{\mathcal{P}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(\emptyset) = 0$, for $A = \emptyset$. Define the limit as $\delta \searrow 0$,

$$\overline{\mathcal{P}}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = \inf_{\delta>0} \overline{\mathcal{P}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(A) = \lim_{\delta\to 0} \overline{\mathcal{P}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(A),$$

and we denote

$$\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = \inf_{A\subseteq \bigcup_i A_i} \sum_i \overline{\mathcal{P}}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A_i).$$

$\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ is termed the mixed generalized multifractal packing measure on $\mathbb{R}^k$ and it is an outer metric measure while the restriction $\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ on Borel sets is a measure.

Proposition 2.1: Let $A \subseteq \mathbb{R}^k$ and $t \in \mathbb{R}$ then

(a) We can find an unique number $dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A) \in [-\infty, +\infty]$ such that

$$\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = +\infty \quad \text{for} \quad t < dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A) \quad \text{and}$$
$$\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = 0 \quad \text{for} \quad t > dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A).$$

The cut $dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A)$ is called the dimension of mixed generalized multifractal Hausdorff measure of A.

(b) We can find an unique number $Dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A) \in [-\infty, +\infty]$ such that

$$\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = +\infty \quad \text{for} \quad t < Dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A) \quad \text{and}$$
$$\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = 0 \quad \text{for} \quad t > Dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A).$$

The cut $Dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A)$ is called the dimension of mixed generalized multifractal packing measure of A.

(c) We can find an unique number $\Delta_{\boldsymbol{\mu}}^{\mathbf{q}}(A) \in [-\infty, +\infty]$ such that

$$\overline{\mathcal{P}}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = +\infty \quad \text{for} \quad t < \Delta_{\boldsymbol{\mu}}^{\mathbf{q}}(A) \quad \text{and}$$
$$\overline{\mathcal{P}}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = 0 \quad \text{for} \quad t > \Delta_{\boldsymbol{\mu}}^{\mathbf{q}}(A).$$

The cut $\Delta_{\boldsymbol{\mu}}^{\mathbf{q}}(A)$ is called the dimension of mixed generalized multifractal packing pre-measure of A.

Remark 2.1: If $A = supp\ \boldsymbol{\mu}$ where $supp\ \boldsymbol{\mu}$ is the support of the measure $\boldsymbol{\mu}$. We simply represent

$$b_{\boldsymbol{\mu}}(\mathbf{q}) = dim_{\boldsymbol{\mu}}^{\mathbf{q}}\, supp\ \boldsymbol{\mu}, \quad B_{\boldsymbol{\mu}}(\mathbf{q}) = Dim_{\boldsymbol{\mu}}^{\mathbf{q}}\, supp\ \boldsymbol{\mu} \text{ and } \Delta_{\boldsymbol{\mu}}(\mathbf{q}) = \Delta_{\boldsymbol{\mu}}^{\mathbf{q}}\, supp\ \boldsymbol{\mu}.$$

Proposition 2.2: [20] Consider $\boldsymbol{\mu}$ as a vector valued measure consisting of probability measures on $\mathbb{R}^k$ and $\boldsymbol{q} \in \mathbb{R}^d$. Then $b_{\boldsymbol{\mu},.}(q)$ and $B_{\boldsymbol{\mu},.}(q)$ are σ-stable.

Proposition 2.3: [20] Consider $\boldsymbol{\mu}$ as a vector valued measure consisting of probability measures on $\mathbb{R}^k$ and $\boldsymbol{q} \in \mathbb{R}^d$. Then

(1) $0 \leq b_{\boldsymbol{\mu}}(\mathbf{q}) \leq B_{\boldsymbol{\mu}}(\mathbf{q}) \leq \Delta_{\boldsymbol{\mu}}(\mathbf{q}),$ for all $q_i < 1$ and $i = 1, \dots, d$.

(2) $0 \geq \Delta_{\boldsymbol{\mu}}(\mathbf{q}) \geq B_{\boldsymbol{\mu}}(\mathbf{q}) \geq b_{\boldsymbol{\mu}}(\mathbf{q}),$ for all $q_i > 1$ and $i = 1, \dots, d$.

We also need some technical results which are studied in [20].

Lemma 2.1: *Consider $\langle \boldsymbol{\alpha}, \boldsymbol{q} \rangle$ as the scalar product of $\boldsymbol{\alpha}$ and $\boldsymbol{q}$ where $\boldsymbol{\alpha}, \boldsymbol{q} \in \mathbb{R}^d$. Then we have*

(1) $\forall\ \boldsymbol{\alpha} \in \mathbb{R}^d,\ t \in \mathbb{R},\ \rho > 0$ and $\mathbf{q} \in \mathbb{R}_+^d$ that satisfy $0 \leq \langle \boldsymbol{\alpha}, \mathbf{q} \rangle + t$, one has

$$2^{\langle \boldsymbol{\alpha},\mathbf{q}\rangle + d\rho} \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}\left(\overline{\Gamma}^{\boldsymbol{\alpha}}\right) \geq \mathcal{H}^{\langle \boldsymbol{\alpha},\mathbf{q}\rangle + t + d\rho}\left(\overline{\Gamma}^{\boldsymbol{\alpha}}\right) \quad \text{and} \quad 2^{\langle \boldsymbol{\alpha},\mathbf{q}\rangle + d\rho} \mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}\left(\overline{\Gamma}^{\boldsymbol{\alpha}}\right) \geq \mathcal{P}^{\langle \boldsymbol{\alpha},\mathbf{q}\rangle + t + d\rho}\left(\overline{\Gamma}^{\boldsymbol{\alpha}}\right).$$

(2) $\forall\ \boldsymbol{\alpha} \in \mathbb{R}^d,\ t \in \mathbb{R},\ \rho > 0$ and $\mathbf{q} \in \mathbb{R}_-^d$ that satisfy $0 \leq \langle \boldsymbol{\alpha}, \mathbf{q} \rangle + t$, one has

$$2^{\langle \boldsymbol{\alpha},\mathbf{q}\rangle + d\rho} \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}\left(\underline{\Gamma}_{\boldsymbol{\alpha}}\right) \geq \mathcal{H}^{\langle \boldsymbol{\alpha},\mathbf{q}\rangle + t + d\rho}\left(\underline{\Gamma}_{\boldsymbol{\alpha}}\right) \quad \text{and} \quad 2^{\langle \boldsymbol{\alpha},\mathbf{q}\rangle + d\rho} \mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}\left(\underline{\Gamma}_{\boldsymbol{\alpha}}\right) \geq \mathcal{P}^{\langle \boldsymbol{\alpha},\mathbf{q}\rangle + t + d\rho}\left(\underline{\Gamma}_{\boldsymbol{\alpha}}\right).$$

Let g a real-valued function such that $g: \mathbb{R}^d \to \mathbb{R}$, the Legende transform g^* is defined as follows

$$g^*: \mathbb{R}^d \to [-\infty, +\infty] \quad \text{with} \quad g^*(x) = \inf_y(\langle x, y \rangle + g(y)).$$

Theorem 2.1: *Let $\boldsymbol{\alpha} \in \mathbb{R}_+^d$ and $\boldsymbol{q} \in \mathbb{R}^d$, then the following results are satisfied:*

(1) *If $\langle \boldsymbol{\alpha}, \boldsymbol{q} \rangle + b_{\boldsymbol{\mu}}(\boldsymbol{q}) \geq 0$, then*

$$dim_{\boldsymbol{\mu}}^{\mathbf{q}}\left(\overline{\Gamma}^{\boldsymbol{\alpha}}\right) \leq b_{\boldsymbol{\mu}}^*(\boldsymbol{\alpha}), \quad \text{for all} \quad \mathbf{q} \in \mathbb{R}_+^d \quad \text{and} \quad dim_{\boldsymbol{\mu}}^{\mathbf{q}}\left(\underline{\Gamma}_{\boldsymbol{\alpha}}\right) \leq b_{\boldsymbol{\mu}}^*(\boldsymbol{\alpha}), \quad \text{for all} \quad \mathbf{q} \in \mathbb{R}_-^d.$$

(2) *If $\langle \boldsymbol{\alpha}, \boldsymbol{q} \rangle + B_{\boldsymbol{\mu}}(\boldsymbol{q}) \geq 0$, then*

$$Dim_{\boldsymbol{\mu}}^{\mathbf{q}}\left(\overline{\Gamma}^{\boldsymbol{\alpha}}\right) \leq B_{\boldsymbol{\mu}}^*(\boldsymbol{\alpha}), \quad \text{for all} \quad \mathbf{q} \in \mathbb{R}_+^d \quad \text{and} \quad Dim_{\boldsymbol{\mu}}^{\mathbf{q}}\left(\underline{\Gamma}_{\boldsymbol{\alpha}}\right) \leq B_{\boldsymbol{\mu}}^*(\boldsymbol{\alpha}), \quad \text{for all} \quad \mathbf{q} \in \mathbb{R}_-^d.$$

3. The main results

Before starting the main results, we first define D^y, for $y \in \mathbb{R}^l$ and $D \subseteq \mathbb{R}^{k+l}$, as follows

$$D^y = \{x \in \mathbb{R}^k | (x, y) \in D\}, \tag{1}$$

Second, for $\mu_j \in \mathcal{P}(\mathbb{R}^k)$ and $a > 1$, take $T_a(\mu_j) := \limsup_{r\downarrow 0} \left(\sup_{x\in \operatorname{supp}\mu_j} \frac{\mu_j(\mathcal{B}_{ar}(x))}{\mu_j(\mathcal{B}_r(x))}\right)$ and define the family $\mathcal{P}_F(\mathbb{R}^k)$ of Federer probability measures on $\mathbb{R}^k$ by

$$\mathcal{P}_F(\mathbb{R}^k) = \{\mu_j \in \mathcal{P}(\mathbb{R}^k) \mid T_a(\mu_j) < \infty, \quad \text{for certain} \quad a > 1\}.$$

By using [21], we have $T_a(\mu_j) < \infty$, for all $a > 1$.

Theorem 3.1: *Let* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathcal{P}_F^d(\mathbb{R}^k)$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathcal{P}_F^d(\mathbb{R}^l)$, $\mathbf{q} = (q_1, \ldots, q_d) \in \mathbb{R}^d$ *and* $s, t \in \mathbb{R}$. *Then we can find* $c > 0$ *in order that for* $A \subseteq \mathbb{R}^k$, $B \subseteq \mathbb{R}^l$ *and* $D \subseteq \mathbb{R}^{k+l}$ *we have*

$$\int \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(D^y)\, d\mathcal{H}_{\boldsymbol{\nu}}^{\mathbf{q},t}(y) \le c\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(D), \tag{2}$$

$$\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B) \le c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B), \tag{3}$$

$$\int \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(D^y)\, d\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(y) \le c\mathcal{P}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(D), \tag{4}$$

$$\mathcal{P}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B) \le c\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B). \tag{5}$$

Remark 3.1:

(1) For $d = 1$ and $\mathbf{q} = 0$, the inequality (2) is due from [19] and [4], while the proof of the rest of the inequalities can be found in [12], [8] and [9].

(2) For $d = 1$ and $\mathbf{q} = 1$, this case coincides with the case studied by Olsen in [24].

Since $\operatorname{supp} \boldsymbol{\mu} \times \boldsymbol{\nu} = \operatorname{supp} \boldsymbol{\mu} \times \operatorname{supp} \boldsymbol{\nu}$, it is easy to see the subsequent corollary.

Corollary 3.1: *Let* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathcal{P}_F^d(\mathbb{R}^k)$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathcal{P}_F^d(\mathbb{R}^k)$, $\boldsymbol{q} = (q_1, \ldots, q_d) \in \mathbb{R}^d$ *and* $s, t \in \mathbb{R}$. *Then*

$$\begin{aligned} dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A) + dim_{\boldsymbol{\nu}}^{\mathbf{q}}(B) \quad &\le \quad dim_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q}}(A \times B) \le dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A) + Dim_{\boldsymbol{\nu}}^{\mathbf{q}}(B) \\ &\le \quad Dim_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q}}(A \times B) \le Dim_{\boldsymbol{\mu}}^{\mathbf{q}}(A) + Dim_{\boldsymbol{\nu}}^{\mathbf{q}}(B), \end{aligned}$$

for $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^l$.

A direct result of this last corollary is the subsequent.

Corollary 3.2: *Let* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathcal{P}_F^d(\mathbb{R}^k)$ *and* $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathcal{P}_F^d(\mathbb{R}^k)$. *Then*

$$b_{\boldsymbol{\mu}} + b_{\boldsymbol{\nu}} \le b_{\boldsymbol{\mu}\times\boldsymbol{\nu}} \le b_{\boldsymbol{\mu}} + B_{\boldsymbol{\nu}} \le B_{\boldsymbol{\mu}\times\boldsymbol{\nu}} \le B_{\boldsymbol{\mu}} + B_{\boldsymbol{\nu}}.$$

4. Demonstration of the main findings

4.1 *Some Definitions*

Let

$$\mathcal{U}_n = \left\{\prod_{i=1}^k \left[l_i 2^{-n}, (l_i + 1)2^{-n}\right[\;\middle|\; l_1, \ldots, l_k \in \mathbb{Z}\right\} \quad \text{and}$$

$$\mathcal{W}_n = \left\{\prod_{i=1}^{k} \left[l_i 2^{-(n+1)}, (l_i+2)2^{-(n+1)}\right[\ \mid \ l_1, \dots, l_k \in \mathbb{Z}\right\},$$

for $n \in \mathbb{N}$. We denote $\mathcal{U}_n$ $(\mathcal{W}_n\ respectively)$ the set of half-open dyadic cubes of order n (the set of half-open semi-dyadic cubes of order n, respectively). Let $u_n(x)$ $(v_n(x)$ respectively) be the unique half-open dyadic cube of order n of which x is contained (the unique half-open semi-dyadic cube v of order n of which x is contained, respectively), for all $x \in \mathbb{R}^k$, such as $dist(u_{n+2}(x), \mathbb{R}^k \backslash v) \geq 2^{-(n+2)}$. Let

$$\mathcal{W}_n(A) = \{v_n(x) \ \mid \ x \in A\}, \qquad \text{for} \quad A \subseteq \mathbb{R}^k.$$

Let $\mathbf{G} = \left\{(g_1, \dots, g_k) \ \mid \ g_i = 0 \text{ or } \frac{1}{2}\right\} \subset \mathbb{R}^k$. For each $\mathbf{g} = (g_1, \dots, g_k) \in \mathbf{G}$, we denote

$$\mathcal{W}_{\mathbf{g},n} = \left\{\prod_{i=1}^{k} [2^{-n}(g_i + l_i), 2^{-n}(g_i + l_i + 1)[\ \mid \ l_1, \dots, l_k \in \mathbb{Z}\right\}.$$

From this definition, we notice that $\left(\mathcal{W}_{\mathbf{g},n}\right)_{\mathbf{g}\in\mathbf{G}}$ is a partition of $\mathcal{W}_n$ and $v' \cap v" = \emptyset$, $\forall\ v', v" \in \mathcal{W}_{g,n}$ with $v' \neq v"$. Also if $v', v" \in \mathcal{W}_{\mathbf{g}}$ with $v' \neq v"$ then either v' and $v"$ are disjoint or one is inside the other. Let also $\mathcal{W}_{\mathbf{g},n}(A) = \mathcal{W}_n(A) \cap \mathcal{W}_{\mathbf{g},n}$.

It is much easier to study the measures and the dimensions of Hausdorff by some cover with simple sets like dyadic cubes and balls. Even more, it is much easier to use some semi-dyadic cubes $v_n(x)$ than dyadic cubes $u_n(x)$ because it is less affected by the location of x since the distance of x to the frontier of $v_n(x)$ is limited by 0 and $2^{-(n+2)}$.

4.2 *Proof of inequality* (2)

We first introduce $\mathcal{N}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ as a net measure which is comparable to $\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}$. Let

$$[u] = \bigcup_{v \in \mathcal{W}_n,\ dist(u,v)=0} v, \ \text{for} \ u \in \mathcal{U}_n.$$

That is $[u]$ is the union of v and their adjacent neighboring dyadic cubes of order n. Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathcal{P}^d(\mathbb{R}^k)$, $A \subseteq \mathbb{R}^k$, $\mathbf{q} = (q_1, \dots, q_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Define

$$\mathcal{N}_{\boldsymbol{\mu},n}^{\mathbf{q},t}(A) = \inf\left\{\sum_i (\boldsymbol{\mu}[u_i])^{\mathbf{q}}(2^{-n_i})^t \ \mid \ n \leq n_i, u_i \in \mathcal{U}_{n_i}, A \subseteq \bigcup_{i=1}^{\infty} u_i\right\},\ A \neq \emptyset,$$

$$\mathcal{N}_{\boldsymbol{\mu},n}^{\mathbf{q},t}(\emptyset) = 0 \qquad \text{and} \qquad \mathcal{N}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = \sup_n \mathcal{N}_{\boldsymbol{\mu},n}^{\mathbf{q},t}(A).$$

Lemma 4.1: *Consider $\boldsymbol{\mu} \in \mathcal{P}^d(\mathbb{R}^k)$, $t \in \mathbb{R}$, $\boldsymbol{q} \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Then*

(i) $\mathcal{N}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ is regular metric outer measure.

(ii) $\sup_i \mathcal{N}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A_i) = \mathcal{N}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A)$, for all $A_i \nearrow A$.

(iii) $\sup_i \mathcal{N}_{\boldsymbol{\mu},n}^{\mathbf{q},t}(A_i) = \mathcal{N}_{\boldsymbol{\mu},n}^{\mathbf{q},t}(A)$, for all $A_i \nearrow A$.

Proof: The demonstation is analogous to that of Lemma 5.2.1 in [24].

Lemma 4.2: *Let* $\boldsymbol{\mu} \in \mathcal{P}_F^d(\mathbb{R}^k)$, $\boldsymbol{q} \in \mathbb{R}^d$ *and* $t \in \mathbb{R}$. *We can find a number* $C > 0$ *with*

$$C^{-1}\mathcal{N}_{\boldsymbol{\mu}}^{\mathbf{q},t} \leq \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t} \leq C\mathcal{N}_{\boldsymbol{\mu}}^{\mathbf{q},t}.$$

Proof: Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathcal{P}_F^d(\mathbb{R}^k)$, $j = 1, \dots, d$ and $\mathcal{B}_r(x) \subseteq [u] \subseteq \mathcal{B}_{10r\sqrt{k}}(x)$ for $x \in \operatorname{supp}\mu_j$, $n \in \mathbb{N}$, $\frac{1}{2^{n+2}} \leq r < \frac{1}{2^{n+1}}$ and $u \in \mathcal{U}_n$ with $\mathcal{B}_r(x) \cap u \neq \emptyset$, then we can find a number $a_{1,j} > 0$ with

$$a_{1,j}^{-1} \leq \left(\frac{\mu_j([u])}{\mu_j(\mathcal{B}_r(x))}\right)^{q_j} \leq a_{1,j} \quad \text{and} \quad a_1^{-1} \leq \left(\frac{2^{-n}}{2r}\right)^t \leq a_1, \text{with } a_1 = \prod_{j=1}^d a_{1,j}.$$

Likewise, since $\boldsymbol{\mu} \in \mathcal{P}_F^d(\mathbb{R}^k)$ and $\mathcal{B}_{\frac{1}{2^{n+1}}}(x) \subseteq [u] \subseteq \mathcal{B}_{\frac{3}{2^n}\sqrt{k}}(x)$ for $u \in \mathcal{U}_n$, $n \in \mathbb{N}$ and $x \in u \cap \operatorname{supp}\boldsymbol{\mu}$, then we can find a number $a_{2,j} > 0$ for all $j = 1, \dots, d$ such that

$$a_{2,j}^{-1} \leq \left(\frac{\mu_j\left(\mathcal{B}_{\frac{3}{2^n}\sqrt{k}}(x)\right)}{\mu_j([u])}\right)^{q_j} \leq a_{2,j}.$$

Put $C = \left(a_2\left(6\sqrt{k}\right)^t\right) \vee (a_1^2 3^k)$ where $a_2 = \prod_{i=1}^d a_{2,j}$. Consider $A \subseteq \mathbb{R}^k$, to prove the initial inequality of our Lemma, let $\delta > 0$ and $\left(\mathcal{B}_{r_i}(x_i)\right)_i$ as a centred δ-covering of A. For every i, set n_i the unique integer that satisfies $\frac{1}{2^{n_i+2}} \leq r_i \leq \frac{1}{2^{n_i+1}}$. Moreover, let n be the unique integer such that $\frac{1}{2^{n+2}} \leq \delta < \frac{1}{2^{n+1}}$. We have $n_i \geq n$ and

$$A \subseteq \bigcup_i \mathcal{B}_{r_i}(x_i) \subseteq \bigcup_i \bigcup_{u \in \mathcal{U}_{n_i}, \mathcal{B}_{r_i}(x_i) \cap u \neq \emptyset} u.$$

Then

$$\mathcal{N}_{\boldsymbol{\mu},n}^{\mathbf{q},t}(A) \leq \sum_i \sum_{u \in \mathcal{U}_{n_i}, \mathcal{B}_{r_i}(x_i) \cap u \neq \emptyset} \boldsymbol{\mu}([u])^{\mathbf{q}}(2^{-n_i})^t \leq a_1^2 \sum_i \sum_{u \in \mathcal{U}_{n_i}, \mathcal{B}_{r_i}(x_i) \cap u \neq \emptyset} \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}}(2r_i)^t$$

$$\leq C \sum_i \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}}(2r_i)^t.$$

Therefore

$$\mathcal{N}_{\boldsymbol{\mu},n}^{\mathbf{q},t}(A) \leq C\overline{\mathcal{H}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},t}(A),$$

for $\frac{1}{2^{n+2}} \leq \delta < \frac{1}{2^{n+1}}$. Letting $\delta \searrow 0$ now shows that $\mathcal{N}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) \leq C\overline{\mathcal{H}}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) \leq C\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A)$.

To prove the second inequality, take $B \subseteq A$. Let $n \in \mathbb{N}$, $B \subseteq \bigcup_i u_i$ with $u_i \in \mathcal{U}_{n_i}$ and $n_i \geq n$, and denote $J = \{i | B \cap u_i \neq \emptyset\}$. For $i \in J$ choose $x_i \in u_i \cap B$. Since $u_i \subseteq \mathcal{B}_{\frac{3}{2^{n_i}}\sqrt{k}}(x_i)$, then $\left(\mathcal{B}_{\frac{3}{2^{n_i}}\sqrt{k}}(x_i)\right)_{i \in J}$ is a centred $\left(3k^{\frac{1}{2}}2^{-n}\right)$-covering of B. Therefore,

$$\overline{\mathcal{H}}^{\mathbf{q},t}_{\boldsymbol{\mu},\frac{3}{2^n}\sqrt{k}}(B) \leq \sum_{i\in J} \boldsymbol{\mu}\left(\mathcal{B}_{\frac{3}{2^{n_i}}\sqrt{k}}(x_i)\right)^{\mathbf{q}} \left(2.\frac{3}{2^{n_i}}\sqrt{k}\right)^t \leq C\sum_i \boldsymbol{\mu}([u_i])^{\mathbf{q}}(2^{-n_i})^t.$$

Hence, for all $n \in \mathbb{N}$, $\overline{\mathcal{H}}^{\mathbf{q},t}_{\boldsymbol{\mu},\frac{3}{2^n}\sqrt{k}}(B) \leq C\mathcal{N}^{\mathbf{q},t}_{\boldsymbol{\mu},n}(B)$. Letting $n \to +\infty$ we have $\overline{\mathcal{H}}^{\mathbf{q},t}_{\boldsymbol{\mu}}(B) \leq C\mathcal{N}^{\mathbf{q},t}_{\boldsymbol{\mu}}(B) \leq C\mathcal{N}^{\mathbf{q},t}_{\boldsymbol{\mu}}(A)$ for all $B \subseteq A$, which implies that $\mathcal{H}^{\mathbf{q},t}_{\boldsymbol{\mu}}(A) \leq C\mathcal{N}^{\mathbf{q},t}_{\boldsymbol{\mu}}(A)$.

Lemma 4.3: *Let* $\boldsymbol{\nu} = (\nu_1,\dots,\nu_d) \in \mathcal{P}^d(\mathbb{R}^l)$, $\boldsymbol{q} = (q_1,\dots,q_d) \in \mathbb{R}^d$, *and* $t \in \mathbb{R}$. *Let* $(u_i)_i$ *be a finite or countable subfamily of* $\bigcup_{m\geq n} \mathcal{U}_m$, *for* $n \in \mathbb{N}$ *with* $B \subseteq \bigcup_i u_i$ *and* $u_i \in \mathcal{U}_{n_i}$, *for* $n_i \geq n$. *Let* $c > 0$ *and denote* $(b_i)_i$ *by a family of positive numbers. We suppose*

$$c < \sum_i b_i 1_{u_i}(y),\ for\ \ all\ \ y \in B.$$

Then $c\mathcal{N}^{\mathbf{q},t}_{\boldsymbol{\nu},n}(B) \leq \sum_i b_i\boldsymbol{\nu}([u_i])^{\mathbf{q}}(2^{-n_i})^t$.

Proof: The proof of this lemma follows as [24, lemma 5.2.3].

Lemma 4.4: *Let* $\boldsymbol{\mu} \in \mathcal{P}^d_F(\mathbb{R}^k)$, $\boldsymbol{\nu} \in \mathcal{P}^d_F(\mathbb{R}^l)$, $\boldsymbol{q} \in \mathbb{R}^d$, *and* $s,t \in \mathbb{R}$, *we can find* $c > 0$ *fulfilling the subsequent. Consider* $a \geq 0$, $D \subseteq \mathbb{R}^{k+l}$, $K \subseteq \mathbb{R}^l$ *and suppose*

$$\mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y) > b, \qquad \text{for} \quad y \in K.$$

Then $\frac{b}{c}\mathcal{H}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K) \leq \mathcal{H}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(D)$, where $\boldsymbol{\mu}\times\boldsymbol{\nu} = (\mu_1\times\nu_1,\dots,\mu_d\times\nu_d)$.

Proof: The demonstation is analogous to that of Lemma 5.2.4 in [24].

Proof of inequality (2)*:*

Now, let $K = \{y \in \mathbb{R}^l \mid \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y) > 0\}$ and $\sum_i b_i 1_{K_i}$ be a function with $\bigcup_i K_i = K$, $K_i \cap K_j = \emptyset$ for $i \neq j$, $b_i \geq 0$ and $\sum_i b_i 1_{K_i}(y) < \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y)$ for all $y \in K$.

Observe that $b_i < \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y) = \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}\left(\left(D \cap (\mathbb{R}^k \times K_i)\right)^y\right)$ for $y \in K_i$. Lemma 4.4 implies that

$$\sum_i b_i\mathcal{H}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K_i) \leq \sum_i c\mathcal{H}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(D \cap (\mathbb{R}^k \times K_i)) = c\mathcal{H}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}\left(\bigcup_i \left(D \cap (\mathbb{R}^k \times K_i)\right)\right) \leq c\mathcal{H}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(D).$$

Therefore, we get

$$\begin{aligned}\int \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y)d\mathcal{H}^{\mathbf{q},t}_{\boldsymbol{\nu}}(y) &= \sup\{\textstyle\sum_i b_i\mathcal{H}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K_i) \mid \bigcup_i K_i = K,\ K_i \cap K_j = \emptyset \\ &\qquad \text{for } i \neq j,\ b_i \geq 0,\ \textstyle\sum_i b_i 1_{K_i}(y) < \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y) \\ &\qquad \text{for all } y \in K\} \\ &\leq c\mathcal{H}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(D).\end{aligned}$$

4.3 *Proof of inequality* (3)

We begin by proving that we can find $c > 0$ such that

$$\mathcal{H}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A \times B) \leq c\, \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(A)\, \overline{\mathcal{P}}^{\mathbf{q},t}_{\boldsymbol{\nu}}(B), \text{ for } A \subseteq \mathbb{R}^k \text{ and } B \subseteq \mathbb{R}^l. \quad (6)$$

Assume that $\zeta = 2^{-3} l^{-\frac{1}{2}}$. Since $\mu_j \in \mathcal{P}_F(\mathbb{R}^k)$ and $\nu_j \in \mathcal{P}_F(\mathbb{R}^l)$, for all $j = 1, \ldots, d$, there exist numbers $a_{1,j}, a_{2,j}, a_{3,j} > 0$ such that for $r > 0$, $x \in \mathbb{R}^k$ one has

$$a_{1,j}^{-1} \leq \left(\frac{\mu_j(\mathcal{B}_{4r}(z))}{\mu_j(\mathcal{B}_r(x))}\right)^{q_j} \leq a_{1,j} \text{ for } z \in \mathcal{B}_r(x), \quad a_{2,j}^{-1} \leq \left(\frac{\nu_j(\mathcal{B}_{4r}(y))}{\nu_j(\mathcal{B}_{\zeta r}(y))}\right)^{q_j}$$

$$\leq a_{2,j} \text{ for } y \in \mathbb{R}^l,$$

$$a_{3,j}^{-1} \leq \left(\frac{(\mu_j \times \nu_j)(\mathcal{B}_r(x,y))}{\mu_j(\mathcal{B}_r(x))\nu_j(\mathcal{B}_r(y))}\right)^{q_j} \leq a_{3,j} \qquad \text{for } y \in \mathbb{R}^l.$$

Write $a_k = \prod_{j=1}^{d} a_{k,j}$, for $k = 1,2,3$ and $c_0 = 4^{s+t}\zeta^{-t} a_1 a_2 a_3$. Consider $D \subseteq A \times B$, $\varepsilon > 0$ and $0 < \delta < \varepsilon$. Pick $\left(\mathcal{B}_{r_i}(x_i)\right)_i$ a centred δ-covering of A. For $i \in \mathbb{N}$, let n_i be the unique integer such that $\frac{\sqrt{l}}{2^{n_i}} < r_i \leq \frac{\sqrt{l}}{2^{n_i - 1}}$. For each i and $v \in \mathcal{W}_{n_i}(B)$ with $\left(\mathcal{B}_{r_i}(x_i) \times v\right) \cap D \neq \emptyset$. Take $h'_{i,v} \in \mathcal{B}_{r_i}(x_i)$ and $h''_{i,v} \in v$. Observe that

$$D \subseteq \bigcup_i \left(\bigcup_{v\in\mathcal{W}_{n_i}(B),\left(\mathcal{B}_{r_i}(x_i)\times v\right)\cap D\neq\emptyset} \mathcal{B}_{r_i}(x_i) \times v\right)$$

$$\subseteq \bigcup_i \left(\bigcup_{v\in\mathcal{W}_{n_i}(B),\left(\mathcal{B}_{r_i}(x_i)\times v\right)\cap D\neq\emptyset} \mathcal{B}_{2r_i}(h'_{i,v}) \times \mathcal{B}_{2r_i}(h''_{i,v})\right)$$

$$\subseteq \bigcup_i \left(\bigcup_{v\in\mathcal{W}_{n_i}(B),\left(\mathcal{B}_{r_i}(x_i)\times v\right)\cap D\neq\emptyset} \mathcal{B}_{4r_i}(h'_{i,v}, h''_{i,v})\right).$$

Hence $\left(\mathcal{B}_{4r_i}(h'_{i,v}, h''_{i,v})\right)_{i\in\mathbb{N}, v\in\mathcal{W}_{n_i}(B),\left(\mathcal{B}_{r_i}(x_i)\times v\right)\cap D\neq\emptyset}$ is a centred 4δ-covering of D. Moreover, remark that $\mathcal{B}_{\zeta r_i}(h''_{i,v}) \subseteq \mathcal{B}_{2^{-(n_i+2)}}\left(h''_{i,v}\right) \subseteq v$. Therefore, for each $\mathbf{g} \in \mathbf{G}$, we have $\left(\mathcal{B}_{\zeta r_i}(h'_{i,v}, h''_{i,v})\right)_{i\in\mathbb{N}, v\in\mathcal{W}_{\mathbf{g},n_i}(B),(\mathcal{B}_{r_i}(x_i)\times v)\cap D\neq\emptyset}$ is a centred ε-packing of B. Then we obtain that

$$\overline{\mathcal{H}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},4\delta}(D) \leq \sum_i \left(\sum_{v\in\mathcal{W}_{n_i}(B),\left(\mathcal{B}_{r_i}(x_i)\times v\right)\cap D\neq\emptyset} (\boldsymbol{\mu}\times\boldsymbol{\nu})\left(\mathcal{B}_{4r_i}\left(h'_{i,v}, h''_{i,v}\right)\right)^{\mathbf{q}} (8r_i)^{s+t}\right)$$

$$\leq a_3 \sum_i \left(\sum_{v\in\mathcal{W}_{n_i}(B),\left(\mathcal{B}_{r_i}(x_i)\times v\right)\cap D\neq\emptyset} \boldsymbol{\mu}\left(\mathcal{B}_{4r_i}(h'_{i,v})\right)^{\mathbf{q}} \boldsymbol{\nu}\left(\mathcal{B}_{4r_i}(h''_{i,v})\right)^{\mathbf{q}} (8r_i)^{s+t}\right)$$

$$\leq 4^{s+t}\zeta^{-t} a_1 a_2 a_3$$

$$\sum_i \left(\sum_{v\in\mathcal{W}_{n_i}(B),\left(\mathcal{B}_{r_i}(x_i)\times v\right)\cap H\neq\emptyset} \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}} \boldsymbol{\nu}\left(\mathcal{B}_{\zeta r_i}(h''_{i,v})\right)^{\mathbf{q}} (2r_i)^s (2\zeta r_i)^t\right)$$

$$\leq c_0 \sum_i \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}} (2r_i)^s$$

$$\left(\sum_{\mathbf{g}\in\mathbf{G}} \sum_{v\in\mathcal{W}_{\mathbf{g},n_i}(B),\left(\mathcal{B}_{r_i}(x_i)\times v\right)\cap D\neq\emptyset} \boldsymbol{\nu}\left(\mathcal{B}_{\zeta r_i}(h''_{i,v})\right)^{\mathbf{q}} (2\zeta r_i)^t\right)$$

$\leq c_0 \sum_i \boldsymbol{\mu}\big(\mathcal{B}_{r_i}(x_i)\big)^{\mathbf{q}} (2r_i)^s \left(\sum_{\mathbf{g}\in\mathbf{G}} \overline{\mathcal{P}}_{\boldsymbol{\nu},\varepsilon}^{\mathbf{q},t}(B)\right)$

$\leq c \sum_i \boldsymbol{\mu}\big(\mathcal{B}_{r_i}(x_i)\big)^{\mathbf{q}} (2r_i)^s \overline{\mathcal{P}}_{\boldsymbol{\nu},\varepsilon}^{\mathbf{q},t}(B),$

where $c = c_0 2^l$. Therefore $\overline{\mathcal{H}}_{\boldsymbol{\mu}\times\boldsymbol{\nu},4\delta}^{\mathbf{q},s+t}(D) \leq c\overline{\mathcal{H}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},s}(A)\overline{\mathcal{P}}_{\boldsymbol{\nu},\varepsilon}^{\mathbf{q},t}(B) \leq c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)$ $\overline{\mathcal{P}}_{\boldsymbol{\nu},\varepsilon}^{\mathbf{q},t}(B)$, for all $0 < \delta < \varepsilon$.

Letting $\delta \searrow 0$, we obtain $\overline{\mathcal{H}}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(D) \leq c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\overline{\mathcal{P}}_{\boldsymbol{\nu},\varepsilon}^{\mathbf{q},t}(B)$, for all $D \subseteq A \times B$. Hence

$$\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B) \leq c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\overline{\mathcal{P}}_{\boldsymbol{\nu},\varepsilon}^{\mathbf{q},t}(B),$$

for all $\varepsilon > 0$. Making $\varepsilon \searrow 0$ the inequality (6) yields.

Now we take $(B_i)_i$ a covering of B. Then, by (6) one has

$$\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B_i) \leq c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\overline{\mathcal{P}}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B_i).$$

Hence $\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B) \leq \sum_i \mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B_i) \leq c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A) \sum_i \overline{\mathcal{P}}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B_i)$. The arbitrary on $(B_i)_i$ implies that

$$\mathcal{H}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B) \leq c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A) \inf_{B\subseteq\cup_i B_i} \sum_i \overline{\mathcal{P}}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B_i) = c\mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B).$$

4.4 *Proof of inequality* (4)

We start to point some ordinary results for the mixed multifractal packing measure. The proof of the three subsequent lemmas is analogous to that in [24].

Lemma 4.5: *Let* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathcal{P}^d(\mathbb{R}^k)$, $\boldsymbol{q} = (q_1, \dots, q_d) \in \mathbb{R}^d$ *and* $t \in \mathbb{R}$. *Then* $\overline{\mathcal{P}}_{\boldsymbol{\mu}}^{\boldsymbol{q},t}(\overline{A}) = \overline{\mathcal{P}}_{\boldsymbol{\mu}}^{\boldsymbol{q},t}(A)$, for $A \subseteq \mathbb{R}^k$.

Lemma 4.6: *Let* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathcal{P}^d(\mathbb{R}^k)$, $\boldsymbol{q} = (q_1, \dots, q_d) \in \mathbb{R}^d$ *and* $t \in \mathbb{R}$. *For each set* $A \subseteq \mathbb{R}^k$, *we can find a Borel set* E *with* $A \subseteq E$ *and* E *satisfies* $\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = \mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}(E)$ *i.e. the measure* $\mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}$ *is Borel regular.*

Lemma 4.7: *Let* $\boldsymbol{\mu} \in \mathcal{P}^d(\mathbb{R}^k)$, $\boldsymbol{q} \in \mathbb{R}^d$ *and* $t \in \mathbb{R}$. *Then*

$$\sup_i \mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A_i) = \mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) \text{ for any sequence } A_i \nearrow A \quad \text{and} \quad \mathcal{P}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A) = \inf_{A_i \nearrow A} \sup_i \overline{\mathcal{P}}_{\boldsymbol{\mu}}^{\mathbf{q},t}(A_i), \qquad \text{for } A \subseteq \mathbb{R}^k.$$

Now, in order to prove the inequality (4) we should first give some technical results.

Lemma 4.8: *Let* $\boldsymbol{\mu} \in \mathcal{P}^d(\mathbb{R}^k)$, $\boldsymbol{\nu} \in \mathcal{P}^d(\mathbb{R}^l)$, $\boldsymbol{q} \in \mathbb{R}^d$ *and* $s, t \in \mathbb{R}$. *Then, we can find* $c > 0$ *in order that if* $b \geq 0$, $D \subseteq \mathbb{R}^{k+l}$, $K \subseteq \mathbb{R}^l$ *and* D^y *defined as in* (1) *satisfying*

$$b < \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},t}(D^y), \qquad \text{for} \quad y \in K.$$

Then $b\mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) \leq c\overline{\mathcal{P}}_{\boldsymbol{\mu}\times\mathbf{v}}^{\mathbf{q},s+t}(D)$.

Proof: Assume that $\zeta = 2^{-3}l^{-\frac{1}{2}}$. As $\mu_j \in \mathcal{P}_F(\mathbb{R}^k)$, $\nu_j \in \mathcal{P}_F(\mathbb{R}^l)$ and $q_j \in \mathbb{R}$ for all $j = 1,\dots,d$, then there exist positive numbers $a_{1,j}, a_{2,j}, a_{3,j} > 0$ such that for $r > 0$

$$a_{1,j}^{-1} \leq \left(\frac{\mu_j(\mathcal{B}_{\zeta r}(x))}{\mu_j(\mathcal{B}_r(x))}\right)^{q_j} \leq a_{1,j} \text{ for } x \in \mathbb{R}^k, \quad a_{2,j}^{-1} \leq \left(\frac{\nu_j(\mathcal{B}_{\zeta r}(y))}{\nu_j(\mathcal{B}_r(y))}\right)^{q_j}$$
$$\leq a_{2,j} \text{ for } y \in \mathbb{R}^l,$$
$$a_{3,j}^{-1} \leq \left(\frac{(\mu_j\times\nu_j)(\mathcal{B}_r(x,y))}{\mu_j(\mathcal{B}_r(x))\nu_j(\mathcal{B}_r(y))}\right)^{q_j} \leq a_{3,j} \quad \text{for} \quad x \in \mathbb{R}^k \text{ and } y \in \mathbb{R}^l.$$

Write $a_k = \prod_{j=1}^d a_{k,j}$, where $k = 1,2,3$. As for $y \in K$, we have $b < \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(D^y)$ then we can select also a subset $D(y) \subseteq D^y$ with $b < \overline{\mathcal{H}}_{\boldsymbol{\mu}}^{\mathbf{q},s}(D(y))$.

Denote $K_\delta = \left\{y \in K \mid \overline{\mathcal{H}}_{\boldsymbol{\mu},\delta}^{\mathbf{q},s}(D(y)) > b\right\}$, for $\delta > 0$. Let $\eta > 0$. As $K_\delta \nearrow K$ when $\delta \searrow 0$, Lemma 4.7 gives that there is $\delta_\eta > 0$ where for $0 < \delta < \delta_\eta$, one has

$$\mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K_\delta) > \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) - \eta \quad \text{if} \quad \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) < +\infty \quad \text{and} \quad \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K_\delta) > \frac{1}{\eta}$$
$$\text{if} \quad \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) = +\infty.$$

Fix $0 < \delta < \delta_\eta$, it follows from the last inequalities that

$$\overline{\mathcal{P}}_{\mathbf{v},\delta}^{\mathbf{q},t}(K_\delta) \geq \overline{\mathcal{P}}_{\mathbf{v}}^{\mathbf{q},t}(K_\delta) \geq \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K_\delta) > \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) - \eta, \text{if} \quad \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) < +\infty,$$
$$\overline{\mathcal{P}}_{\mathbf{v},\delta}^{\mathbf{q},t}(K_\delta) \geq \overline{\mathcal{P}}_{\mathbf{v}}^{\mathbf{q},t}(K_\delta) \geq \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K_\delta) > \frac{1}{\eta}, \text{ if } \quad \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) = +\infty.$$

We can also choose $(\mathcal{B}_{r_i}(y_i))_i$ a centred δ-packing of K_δ such that

$$\sum_i \mathbf{v}\left(\mathcal{B}_{r_i}(y_i)\right)^{\mathbf{q}}(2r_i)^t > \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) - \eta, \text{ if } \quad \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) < +\infty, \tag{7}$$
$$\sum_i \mathbf{v}\left(\mathcal{B}_{r_i}(y_i)\right)^{\mathbf{q}}(2r_i)^t > \frac{1}{\eta}, \text{ if } \quad \mathcal{P}_{\mathbf{v}}^{\mathbf{q},t}(K) = +\infty.$$

For $i \in \mathbb{N}$, consider n_i as the unique integer satisfying $2^{-n_i}\sqrt{l} < r_i \leq 2^{-(n_i-1)}\sqrt{l}$. For each i and $v \in \mathcal{W}_{n_i}(D(y_i))$ we can choose a point $x_{i,v} \in v \cap D(y_i)$ satisfying $v_{n_i}(x_{i,v}) = v$.

Now, observe that

$$D(y_i) \subseteq \bigcup_{v\in\mathcal{W}_{n_i}(D(y_i))} v \subseteq \bigcup_{v\in\mathcal{W}_{n_i}(D(y_i))} \mathcal{B}_{r_i}(x_{i,v}).$$

It is clear that $(\mathcal{B}_{r_i}(x_{i,v}))_{v\in\mathcal{W}_{n_i}(D(y_i))}$ is a centred r_i-covering of $D(y_i)$. Hence, we deduce that

$$\overline{\mathcal{H}}_{\boldsymbol{\mu},r_i}^{\mathbf{q},s}(D(y_i)) \leq \sum_{v\in\mathcal{W}_{n_i}(D(y_i))} \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_{i,v})\right)^{\mathbf{q}}(2r_i)^s. \tag{8}$$

Let $r_i < \delta$, then $y_i \in K_\delta \subseteq K_{r_i}$ and

$$\overline{\mathcal{H}}^{\mathbf{q},s}_{\boldsymbol{\mu},r_i}(D(y_i)) > b. \tag{9}$$

We can also observe that $\mathcal{B}_{\zeta r_i}(x_{i,v}, y_i) \subseteq \mathcal{B}_{\zeta r_i}(x_{i,v}) \times \mathcal{B}_{\zeta r_i}(y_i) \subseteq v \times \mathcal{B}_{r_i}(y_i)$. Then for each $\mathbf{g} \in \mathbf{G}$, the set $(\mathcal{B}_{\zeta r_i}(x_{i,v}, y_i))_{i\in\mathbb{N}, v\in\mathcal{W}_{\mathbf{g},n_i}(D(y_i))}$ is a centred δ-packing of D. To conclude, using (7) we have for $0 < \delta < \delta_\eta$ and $D \subseteq \mathbb{R}^{k+l}$

$$\begin{aligned}
\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\delta}(D) &= 2^{-k} \textstyle\sum_{\mathbf{g}\in\mathbf{G}} \overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\delta}(D) \\
&\geq 2^{-k} \textstyle\sum_{\mathbf{g}\in\mathbf{G}} \sum_i \sum_{v\in\mathcal{W}_{\mathbf{g},n_i}(D(y_i))} (\boldsymbol{\mu}\times\boldsymbol{\nu})\left(\mathcal{B}_{\zeta r_i}(x_{i,v}, y_i)\right)^{\mathbf{q}} (2\zeta r_i)^{s+t} \\
&\geq 2^{-k} a_3^{-1} \textstyle\sum_{\mathbf{g}\in\mathbf{G}} \sum_i \sum_{v\in\mathcal{W}_{\mathbf{g},n_i}(D(y_i))} \boldsymbol{\mu}\left(\mathcal{B}_{\zeta r_i}(x_{i,v})\right)^{\mathbf{q}} \boldsymbol{\nu}(\mathcal{B}_{\zeta r_i}(y_i))^{\mathbf{q}} (2\zeta r_i)^{s+t} \\
&\geq 2^{-k}\zeta^{s+t}(a_1a_2a_3)^{-1} \textstyle\sum_i \boldsymbol{\nu}(\mathcal{B}_{r_i}(y_i))^{\mathbf{q}}(2r_i)^t \left(\sum_{v\in\mathcal{W}_{n_i}(D(y_i))} \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_{i,v})\right)^{\mathbf{q}} (2r_i)^s\right) \\
&\geq \tfrac{1}{c} \textstyle\sum_i \boldsymbol{\nu}(\mathcal{B}_{r_i}(y_i))^{\mathbf{q}}(2r_i)^t \overline{\mathcal{H}}^{\mathbf{q},s}_{\boldsymbol{\mu},r_i}(D(y_i)) \\
&\geq \tfrac{b}{c} \textstyle\sum_i \boldsymbol{\nu}(\mathcal{B}_{r_i}(y_i))^{\mathbf{q}}(2r_i)^t \geq \begin{cases} \frac{b}{c}\left(\mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K) - \eta\right) & \text{if } \mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K) < +\infty \\ \frac{b}{c\eta} & \text{if } \mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K) = +\infty, \end{cases}
\end{aligned}$$

where $c = 2^k\zeta^{-s-t}a_1a_2a_3$. Now letting $\delta \searrow 0$ we deduce that

$$\frac{b}{c}\left(\mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K) - \eta\right) \leq \overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(D) \quad \text{if } \mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K) < +\infty \quad \text{and}$$

$$\frac{b}{c\eta} \leq \overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(D) \quad \text{if } \mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K) = +\infty,$$

for $\eta > 0$. When you get the required result.

We can prove the following lemma with similar argument as in [24].

Lemma 4.9: *Let $\boldsymbol{\mu} \in \mathcal{P}_F^d(\mathbb{R}^k)$, $\boldsymbol{\nu} \in \mathcal{P}_F^d(\mathbb{R}^l)$, $\boldsymbol{q} \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Then we can find $c > 0$ that satisfies: Let $b \geq 0$, $D \subseteq \mathbb{R}^{k+l}$, $K \subseteq \mathbb{R}^l$ and suppose that $\mathcal{H}^{\boldsymbol{q},s}_{\boldsymbol{\mu}}(D^y) > b$, for $y \in K$. Then $b\mathcal{P}^{\boldsymbol{q},t}_{\boldsymbol{\nu}}(K) \leq c\mathcal{P}^{\boldsymbol{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(D)$.*

Proof of the inequality (4):

Put $K = \{y \mid \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y) > 0\}$ and $\sum_i b_i 1_{K_i}$ is a function where $\bigcup_i K_i = K$, $K_i \cap K_j = \emptyset$ for $i \neq j$, $b_i \geq 0$ and let $\sum_i b_i 1_{K_i}(y) < \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y)$ for all $y \in K$, we observe that

$$b_i < \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(D^y) = \mathcal{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}\left(\left(D \cap (\mathbb{R}^k \times K_i)\right)^y\right),$$

for all $y \in K_i$. According to the last inequality and Lemma 4.9, we obtain that

$$\begin{aligned}
\textstyle\sum_i b_i \mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(K_i) &\leq \textstyle\sum_i c\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}\left(D \cap (\mathbb{R}^k \times K_i)\right) \\
&= c\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}\left(\textstyle\bigcup_i \left(D \cap (\mathbb{R}^k \times K_i)\right)\right) \leq c\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(D).
\end{aligned}$$

Hence

$$\int \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(D^y)d\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(y) = \sup\Big\{\sum_i b_i\mathcal{P}_{\boldsymbol{\nu}}^{\mathbf{q},t}(K_i) \;\Big|\; \cup_i K_i = K,\ K_i \cap K_j \neq \emptyset \text{ for } i \neq j,$$
$$b_i \geq 0,\ \sum_i b_i 1_{K_i}(y) < \mathcal{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(D^y) \text{ for all } y \in K\Big\}$$
$$\leq c\mathcal{P}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(D).$$

4.5 *Proof of inequality* (5)

To demonstrate the inequality (5) we require some lemmas as follows.

Lemma 4.10: *Consider $\boldsymbol{\mu} \in \mathcal{P}^d(\mathbb{R}^k)$, $\boldsymbol{q} \in \mathbb{R}^d$ and $s \in \mathbb{R}$. Let $\varepsilon > 0$, $A \subset \mathbb{R}^k$, $g \in \boldsymbol{G}$ and $\left(v_{n_i}(x_i)\right)_{i=1,\ldots,n}$ be a finite subfamily of $\cup_l \mathcal{W}_{l,g}$ with $x_i \in A$ and $2^{-(n_i+2)} < \varepsilon$. Take $a_1, \ldots, a_n \in \mathbb{N}$ and $c > 0$ with $\sum_{i=1}^n a_i 1_{v_{n_i}(x_i)}(x) \leq c$ for all $x \in A$. Then*

$$\sum_{i=1}^n a_i \boldsymbol{\mu}\left(\mathcal{B}_{2^{-(n_i+2)}}(x_i)\right)^{\mathbf{q}} \left(2.\,2^{-(n_i+2)}\right)^s \leq \overline{\mathcal{P}}_{\boldsymbol{\mu},\varepsilon}^{\mathbf{q},s}(A).$$

Proof: The demonstation is analogous to that of Lemma 5.5.1 in [24].

Lemma 4.11: *Let $\boldsymbol{\mu} \in \mathcal{P}_F^d(\mathbb{R}^k)$, $\boldsymbol{\nu} \in \mathcal{P}_F^d(\mathbb{R}^l)$, $\boldsymbol{q} \in \mathbb{R}^d$ and $s, t \in \mathbb{R}$. Then we can find a number $c > 0$ that satisfies, for $B \subseteq \mathbb{R}^l$ and $A \subseteq \mathbb{R}^k$, we have*

$$\overline{\mathcal{P}}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{\mathbf{q},s+t}(A \times B) \leq c\overline{\mathcal{P}}_{\boldsymbol{\mu}}^{\mathbf{q},s}(A)\overline{\mathcal{P}}_{\boldsymbol{\nu}}^{\mathbf{q},t}(B).$$

Proof: Since $\mu_j \in \mathcal{P}_F(\mathbb{R}^k)$, for $j = 1, \ldots, d$. Then there exists numbers $a_{1,j}, a_{2,j}, a_{3,j} > 0$ such that for $r > 0$

$$a_{1,j}^{-1} \leq \left(\frac{\mu_j(\mathcal{B}_r(x))}{\mu_j\left(\mathcal{B}_{2^{-n-2}}(x)\right)}\right)^{q_j} \leq a_{1,j}, \text{for } x \in \mathbb{R}^k,\quad 2^{-n+1} \geq r > 2^{-n}$$

$$\text{and}\quad a_1^{-1} \leq \left(\frac{r}{2^{-n-2}}\right)^s \leq a_1, \text{with } a_1 = \prod_{j=1}^d a_{1,j}.$$

$$a_{2,j}^{-1} \leq \left(\frac{\nu_j(\mathcal{B}_r(y))}{\nu_j\left(\mathcal{B}_{2^{-\frac{1}{2}}r}(y)\right)}\right)^{q_j} \leq a_{2,j}, \quad \text{for}\quad r > 0 \text{ and } y \in \mathbb{R}^l,$$

$$a_{3,j}^{-1} \leq \left(\frac{(\mu_j\times\nu_j)(\mathcal{B}_r(x,y))}{\mu_j(\mathcal{B}_r(x))\nu_j(\mathcal{B}_r(y))}\right)^{q_j} \leq a_{3,j}, \quad \text{for}\quad r > 0, x \in \mathbb{R}^k \text{ and } y \in \mathbb{R}^l.$$

Let $\eta, \varepsilon, \delta > 0$ and a centred $\varepsilon \wedge \delta$-packing $\left(\mathcal{B}_{r_i}(x_i, y_i)\right)_i$ of $A \times B$ that satisfy,

$$\overline{\mathcal{P}}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}^{\mathbf{q},s+t}(A \times B) - \eta < \sum_i (\boldsymbol{\mu} \times \boldsymbol{\nu})\left(\mathcal{B}_{r_i}(x_i, y_i)\right)^{\mathbf{q}} (2r_i))^{s+t},$$
$$\text{if}\quad \overline{\mathcal{P}}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}^{\mathbf{q},s+t}(A \times B) < +\infty.$$

One has

$\frac{1}{\eta} < \sum_i (\boldsymbol{\mu} \times \boldsymbol{\nu}) \left(\mathcal{B}_{r_i}(x_i, y_i)\right)^{\mathbf{q}} (2r_i))^{s+t}$, if $\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A \times B) = \infty$.

We can find $n \in \mathbb{N}$ with

$$\begin{aligned} &\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A \times B) - \eta < \sum_{i=1}^{n} (\boldsymbol{\mu} \times \boldsymbol{\nu}) \left(\mathcal{B}_{r_i}(x_i, y_i)\right)^{\mathbf{q}} (2r_i)^{s+t} \\ &\leq a_3 \sum_{i=1}^{n} \boldsymbol{\mu} \left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}} (2r_i)^s \boldsymbol{\nu} \left(\mathcal{B}_{r_i}(y_i)\right) (2r_i)^t, \text{if } \overline{\mathcal{P}}^{q,s+t}_{\mu\times\nu,\delta\wedge\eta}(A \times B) < +\infty, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\frac{1}{\eta} < \sum_{i=1}^{n} (\boldsymbol{\mu} \times \boldsymbol{\nu}) \left(\mathcal{B}_{r_i}(x_i, y_i)\right)^{\mathbf{q}} (2r_i)^{s+t} < \\ &a_3 \sum_{i=1}^{n} \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}} (2r_i)^s \boldsymbol{\nu}\left(\mathcal{B}_{r_i}(y_i)\right)^{\mathbf{q}} (2r_i)^t, \text{ if } \overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A \times B) = +\infty \end{aligned}$$

where $a_3 = \prod_{j=1}^{d} a_{3,j}$. It follows from (10) that we can choose positive integers $z, z_1, \ldots, z_n$ with

$$\frac{z_i}{z} \leq \boldsymbol{\nu}\left(\mathcal{B}_{r_i}(y_i)\right)^{\mathbf{q}} (2r_i))^t$$

and

$$\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A \times B) - \eta \leq a_3 \sum_{i=1}^{n} \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}} (2r_i)^s \frac{z_i}{z},$$

if $\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A \times B) < +\infty$,

$$\frac{1}{\eta} \leq a_3 \sum_{i=1}^{n} \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}} (2r_i)^s \frac{z_i}{z}, \qquad \text{if } \overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A \times B) = +\infty.$$

For each *i*, denote n_i as the unique integer with $\frac{1}{2^{n_i}} < \frac{r_i}{\sqrt{2}} \leq \frac{1}{2^{n_i-1}}$.

Let $J(x) = \left\{i \mid x \in \mathcal{B}_{\frac{r_i}{\sqrt{2}}}(x_i)\right\} \cap \{1, \ldots, n\}$ for $x \in \mathbb{R}^k$. As

$$\mathcal{B}_{\frac{r_i}{\sqrt{2}}}(x_i) \times \mathcal{B}_{\frac{r_i}{\sqrt{2}}}(y_i) \subseteq \mathcal{B}_{\frac{r_i}{\sqrt{2}}}(x_i, y_i) \subseteq \mathcal{B}_{r_i}(x_i, y_i),$$

then $\left(\mathcal{B}_{\frac{r_i}{\sqrt{2}}}(y_i)\right)_{i\in J(x)}$ is a centred δ-packing of B. Whence,

$$\begin{aligned} \sum_{i=1}^{n} z_i 1_{v_{n_i}(x_i)}(x) \leq \sum_{i\in J(x)} z_i &\leq \sqrt{2}^t z a_2 \sum_{i\in J(x)} \boldsymbol{\nu} \left(\mathcal{B}_{\frac{r_i}{\sqrt{2}}}(y_i)\right)^{\mathbf{q}} \left(2.\frac{r_i}{\sqrt{2}}\right)^t \\ &\leq \sqrt{2}^t z a_2 \overline{\mathcal{P}}^{\mathbf{q},t}_{\boldsymbol{\nu},\delta}(B), \end{aligned}$$

for all $x \in A$ where $a_2 = \prod_{i=1}^{d} a_{2,j}$. By using Lemma 4.10 and if $\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A \times B) < \infty$ we obtain that

$$\begin{aligned} \overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A \times B) - \eta \quad &< \quad a_3 \sum_{i=1}^{n} \boldsymbol{\mu}\left(\mathcal{B}_{r_i}(x_i)\right)^{\mathbf{q}} (2r_i)^s \frac{z_i}{z} \leq \frac{1}{z} a_1^2 a_3 \\ &\quad \sum_{i=1}^{n} z_i \boldsymbol{\mu} \left(\mathcal{B}_{2^{-(n_i+2)}}(x_i)\right)^{\mathbf{q}} \left(2.\,2^{-(n_i+2)}\right)^s \\ &\leq \quad \sqrt{2}^t a_1^2 a_2 a_3 \overline{\mathcal{P}}^{\mathbf{q},s}_{\boldsymbol{\mu},\varepsilon}(A) \overline{\mathcal{P}}^{\mathbf{q},t}_{\boldsymbol{\nu},\delta}(B). \end{aligned}$$

Now, if $\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A\times B)=+\infty$, we have for all $\eta>0$

$$\begin{aligned}\frac{1}{\eta} &< a_3\sum_{i=1}^{n}\boldsymbol{\mu}\big(\mathcal{B}_{r_i}(x_i)\big)^{\mathbf{q}}(2r_i)^s\frac{z_i}{z}\\ &\le \frac{1}{z}a_1^2a_3\sum_{i=1}^{n}z_i\boldsymbol{\mu}\left(\mathcal{B}_{2^{-(n_i+2)}}(x_i)\right)^{\mathbf{q}}\left(2.\,2^{-(n_i+2)}\right)^s\\ &\le \sqrt{2}^{\,t}a_1^2a_2a_3\overline{\mathcal{P}}^{\mathbf{q},s}_{\boldsymbol{\mu},\varepsilon}(A)\overline{\mathcal{P}}^{\mathbf{q},t}_{\boldsymbol{\nu},\delta}(B).\end{aligned}$$

Then letting $\eta\searrow 0$ we can infer that $\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A\times B)\le\overline{\mathcal{P}}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu},\varepsilon\wedge\delta}(A\times B)\le c\overline{\mathcal{P}}^{\mathbf{q},s}_{\boldsymbol{\mu},\varepsilon}(A)\overline{\mathcal{P}}^{\mathbf{q},t}_{\boldsymbol{\nu},\delta}(B)$, (with $c=\sqrt{2}^{\,t}a_1^2a_2a_3$) for all $\varepsilon,\delta>0$. We achieve the aim by letting $\varepsilon,\delta\searrow 0$.

Proof of inequality (5):

Let $A\subseteq\bigcup_i A_i$ and $B\subseteq\bigcup_j B_j$. Lemma 4.11 implies that

$$\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A_i\times B)\le\sum_j\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A_i\times B_j)\le c\overline{\mathcal{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(A_i)\sum_j\overline{\mathcal{P}}^{\mathbf{q},t}_{\boldsymbol{\nu}}(B_j).$$

As the covering $(B_j)_j$ of B was arbitrary, then

$$\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A_i\times B)\le c\overline{\mathcal{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(A_i)\inf_{B\subseteq\bigcup_j B_j}\sum_j\overline{\mathcal{P}}^{\mathbf{q},t}_{\boldsymbol{\nu}}(B_j)=c\overline{\mathcal{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(A_i)\mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(B),$$

for all i. Then $\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A\times B)\le\sum_i\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A_i\times B)\le c\sum_i\overline{\mathcal{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(A_i)\mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(B)$.

Finally, the arbitrary on $(A_i)_i$ implies that

$$\mathcal{P}^{\mathbf{q},s+t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(A\times B)\le c\inf_{A\subseteq\bigcup_i A_i}\sum_i\overline{\mathcal{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(A_i)\mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(B)=c\mathcal{P}^{\mathbf{q},s}_{\boldsymbol{\mu}}(A)\mathcal{P}^{\mathbf{q},t}_{\boldsymbol{\nu}}(B).$$

5. An example

We suppose that $Q_1,\dots,Q_n:\mathbb{R}^k\to\mathbb{R}^k$ are contractions and $\lambda_j=(\lambda_{j,1},\dots,\lambda_{j,n})$ is a probability vector for $j=1,2$, it follows from [23] that we can find a unique compact and nonempty set E that satisfies $E=\bigcup_i Q_i(E)$, and a unique Borel probability measure $\boldsymbol{\mu}=(\mu_1,\mu_2)$ with

$$\mu_j=\sum_i\lambda_{j,i}\mu_j\circ Q_i^{-1},\qquad\text{for } j=1,2. \tag{11}$$

E is the self-similar set corresponding with $(Q_i)_i$ if $Q_1,\dots,Q_n$ are similarities. For the same reason, we called μ_j the self similar measure related with $(Q_i,\lambda_{j,i})_i$ for $j=1,2$.

We assume also that $R_1,\dots,R_m:\mathbb{R}^l\to\mathbb{R}^l$ are contractions and $\xi_j=(\xi_{j,1},\dots,\xi_{j,m})$ is a probability vector. Then by using [23], we can find a unique compact and nonempty set F that satisfies $F=\bigcup_{i'}Q_{i'}(F)$, and a unique Borel probability measure $\boldsymbol{\nu}=(\nu_1,\nu_2)$ with

$$\nu_j=\sum_i\xi_{j,i'}\nu_j\circ R_{i'}^{-1},\qquad\text{for } j=1,2. \tag{12}$$

F is the self-similar set corresponding with $(R_{i'})_{i'}$ if $R_1,\dots,R_n$ are similarities. Likewise, ν_j is the self-similar measure related to $(R_{i'},\xi_{j,i'})_{i'}$ for $j=1,2$ for the identical reason.

Let $(i,i') \in \{1,\ldots,n\} \times \{1,\ldots,m\}$. Define

$$\psi_{ii'}: \mathbb{R}^{k+l} \to \mathbb{R}^{k+l} \quad \text{such that} \quad \psi_{ii'}(x,y) = (Q_i(x), R_{i'}(y)).$$

$\psi_{ii'}$ is an affine contraction. We define also a probability vector $(\kappa_{ii'})_{i,i'}$, for $(i,i') \in \{1,\ldots,n\} \times \{1,\ldots,m\}$ as follows

$$\kappa_{ii'} = (\kappa_{ii'}^1, \kappa_{ii'}^2) = (\lambda_{1,i}\xi_{1,i'}, \lambda_{2,i}\xi_{2,i'}).$$

$E \times F$ is the self affine set related with $(\psi_{ii'})_{i,i'}$. Notice that

$$\boldsymbol{\mu} \times \boldsymbol{\nu} = (\mu_1 \times \nu_1, \mu_2 \times \nu_2) = \left(\sum_{i,i'} \kappa_{ii'}^1 (\mu_1 \times \nu_1) \circ \psi_{ii'}^{-1}, \sum_{i,i'} \kappa_{ii'}^2 (\mu_2 \times \nu_2) \circ \psi_{ii'}^{-1}\right).$$

Then $\boldsymbol{\mu} \times \boldsymbol{\nu}$ is the self affine measure corresponding with $(\psi_{ii'}, \kappa_{ii'})_{i,i'}$.

Now, let M and N be compact non-empty sets with non-negative Lebesgue measure with $Q_i(M) \subseteq M$ and $R_{i'}(N) \subseteq N$, $\forall\ i,i' \in \{1,\ldots,n\} \times \{1,\ldots,m\}$. One has $E \times F$ is the invariant set corresponding with $(\psi_{ii'})_{i,i'}$. Then it follows from [24] that $E \times F = \bigcap_r C_r$, where

$$C_r = \bigcup_{(t_1,t_1')} \gamma_{t_1 t_1'} \circ \ldots \circ \gamma_{i_r i_r'}(M \times N).$$

Consider L^k as Lebesgue measure in $\mathbb{R}^k$ and $L^k \llcorner A$ as the restriction of L^k to A, for all $A \in \mathbb{R}^k$. Let

$$\Theta_r^j = \sum_{(i_1 i_1'),\ldots,(i_r i_r')} \lambda_{j,i_1}\xi_{j,i_1'} \ldots \lambda_{j,i_r}\xi_{j,i_r'} \frac{L^{k+l} \llcorner (\gamma_{i_1 i_1'} \circ \ldots \circ \gamma_{i_r i_r'})(M \times N)}{L^{k+l}(\gamma_{i_1 i_1'} \circ \ldots \circ \gamma_{i_r i_r'})(M \times N)}, \text{for } j = 1, 2.$$

It is simple to prove that $\left(\Theta_r^j\right)_r$ is a Cauchy sequence with respect to the weak topology. Then $\left(\Theta_r^j\right)_r$ is convergent with respect to the weak topology. Let Θ^j be the limited of sequence $\left(\Theta_r^j\right)_r$ as follows

$$\Theta^j = \text{weak} - \lim_r \Theta_r^j,$$

where weak-lim is a weak limit. We can prove that the limit Θ^j of the sequence $\left(\Theta_r^j\right)_r$ satisfies

$$\Theta^j = \sum_{i,i'} \kappa_{ii'}^j \Theta^j \circ \psi_{ii'}^{-1}.$$

Therefore, it follows from [13] and $\mu_j \times \nu_j$ is the only measure satisfying $\sum_{i,i'} \kappa_{ii'}^j \Theta^j \circ \psi_{ii'}^{-1}$ that

$$\Theta^j = \mu_j \times \nu_j.$$

For $\mathbf{q} = (q_1, q_2)$ let

$$\underline{a}_{\boldsymbol{\mu}} = \sup_{0<\mathbf{q}} -\frac{b_{\boldsymbol{\mu}}(\mathbf{q})}{\mathbf{q}}, \qquad \overline{a}_{\boldsymbol{\mu}} = \inf_{\mathbf{q}<0} -\frac{b_{\boldsymbol{\mu}}(\mathbf{q})}{\mathbf{q}}.$$

We suppose that $Q_i(E) \cap Q_{i'}(E) = \emptyset$ for $i \neq i'$ and $R_i(F) \cap R_{i'}(F) = \emptyset$, for $i \neq i'$. Let $\theta_{\boldsymbol{\mu}}, \theta_{\boldsymbol{\nu}}: \mathbb{R}^2 \to \mathbb{R}$ such that

$$\sum_i \lambda_{1,i}^{q_1} \lambda_{2,i}^{q_2} (L(Q_i))^{\theta_{\boldsymbol{\mu}}(\mathbf{q})} = 1 \qquad \text{and} \qquad \sum_{i'} \xi_{1,i'}^{q_1} \xi_{2,i'}^{q_2} (L(R_{i'}))^{\theta_{\boldsymbol{\nu}}(\mathbf{q})} = 1,$$

for $\mathbf{q} = (q_1, q_2)$ and where L is Lipschitz constant (see [23]). Then it follows from [21] and [23] that

$$\underline{a}_{\boldsymbol{\mu}} = \left(\min_i \frac{\log \lambda_{1,i}}{\log L(Q_i)}, \min_i \frac{\log \lambda_{2,i}}{\log L(Q_i)}\right), \qquad \overline{a}_{\boldsymbol{\mu}} = \left(\max_i \frac{\log \lambda_{1,i}}{\log s(Q_i)}, \max_i \frac{\log \lambda_{2,i}}{\log L(Q_i)}\right),$$

$$\underline{a}_{\mathbf{v}} = \left(\min_{i'} \frac{\log \xi_{1,i'}}{\log L(R_{i'})}, \min_{i'} \frac{\log \xi_{2,i'}}{\log L(R_{i'})}\right) \text{ and } \overline{a}_{\mathbf{v}} = \left(\max_{i'} \frac{\log \xi_{1,i'}}{\log L(R_{i'})}, \max_{i'} \frac{\log \xi_{2,i'}}{\log L(R_{i'})}\right).$$

It follows also from [11] that

$$b_{\mu} = B_{\mu} = \theta_{\mu} \text{ and } \begin{cases} dim_{\mu}^{\mathbf{q}}(\Gamma_{\mu}(\boldsymbol{\alpha})) = Dim_{\mu}^{\mathbf{q}}(\Gamma_{\mu}(\boldsymbol{\alpha})) = \theta_{\mu}^{*}(\boldsymbol{\alpha}), & \text{for } \boldsymbol{\alpha} \in (\underline{a}_{\mu}, \overline{a}_{\mu}) \\ \Gamma_{\mu}(\boldsymbol{\alpha}) = \emptyset, & \text{for } \boldsymbol{\alpha} \notin [\underline{a}_{\mu}, \overline{a}_{\mu}], \end{cases} \tag{13}$$

$$b_{\mathbf{v}} = B_{\mathbf{v}} = \theta_{\mathbf{v}} \text{ and } \begin{cases} dim_{\mathbf{v}}^{\mathbf{q}}(\Gamma_{\mu}(\boldsymbol{\alpha})) = Dim_{\mathbf{v}}^{\mathbf{q}}(\Gamma_{\mu}(\boldsymbol{\alpha})) = \theta_{\mathbf{v}}^{*}(\boldsymbol{\alpha}), & \text{for } \boldsymbol{\alpha} \in (\underline{a}_{\mathbf{v}}, \overline{a}_{\mathbf{v}}) \\ \Gamma_{\mathbf{v}}(\boldsymbol{\alpha}) = \emptyset, & \text{for } \boldsymbol{\alpha} \notin [\underline{a}_{\mathbf{v}}, \overline{a}_{\mathbf{v}}]. \end{cases} \tag{14}$$

Corollary 3.2, Theorem 2.1, the last equalities (13) and (14) and the similarity implies that

$$\underline{a}_{\mu\times\mathbf{v}} = \underline{a}_{\mu} + \underline{a}_{\mathbf{v}} \quad \text{ and } \quad \overline{a}_{\mu\times\mathbf{v}} = \overline{a}_{\mu} + \overline{a}_{\mathbf{v}}.$$

From the same reason we have firstly $b_{\mu\times\mathbf{v}} = B_{\mu\times\mathbf{v}} = \theta_{\mu\times\mathbf{v}} \leq B_{\mu} + B_{\mathbf{v}} = \theta_{\mu} + \theta_{\mathbf{v}}$, and on the other hand we have $\theta_{\mu\times\mathbf{v}} = B_{\mu\times\mathbf{v}} \geq b_{\mu} + b_{\mathbf{v}} = \theta_{\mu} + \theta_{\mathbf{v}}$. Then

$$b_{\mu\times\mathbf{v}} = B_{\mu\times\mathbf{v}} = \theta_{\mu} + \theta_{\mathbf{v}}.$$

and

$$\begin{cases} dim_{\mu\times\mathbf{v}}^{\mathbf{q}}(\Gamma_{\mu}(\boldsymbol{\alpha})) = dim_{\mu\times\mathbf{v}}^{\mathbf{q}}(\Gamma_{\mu}(\boldsymbol{\alpha})) = (\theta_{\mu} + \theta_{\mathbf{v}})^{*}(\boldsymbol{\alpha}) = \theta_{\mu}^{*}(\alpha) + \theta_{\mathbf{v}}^{*}(\boldsymbol{\alpha}), & \text{for } \boldsymbol{\alpha} \in (\underline{a}_{\mu} + \underline{a}_{\mathbf{v}}, \overline{a}_{\mu} + \overline{a}_{\mathbf{v}}) \\ \Gamma_{\mu\times\mathbf{v}}(\boldsymbol{\alpha}) = \emptyset, & \text{for } \boldsymbol{\alpha} \notin [\underline{a}_{\mu} + \underline{a}_{\mathbf{v}}, \overline{a}_{\mu} + \overline{a}_{\mathbf{v}}]. \end{cases}$$

Acknowledgments

The author expresses their sincere appreciation to the referees for their diligent review and valuable suggestions. This work is supported by the Analysis, Probability & Fractals Laboratory (No. LR18ES17).

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Received January, 2024