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On generalization of biharmonic curves in a Walker 3-manifold

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Abstract

This paper examines *f*-biharmonic curves within a three-dimensional Walker manifold, focusing on their geometric properties. It derives explicit parametric equations for these curves and establishes the necessary and sufficient conditions for a curve to be *f*-biharmonic. In particular, we proved that when the function *f* remains constant, the f-biharmonic curve reduces to a non-geodesic biharmonic curve.

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1. Introduction

Walker manifold is a Lorentzian manifold admitting a parallel null vector field. Walker metrics have proven to be a valuable tool for constructing interesting indefinite metrics that reveal geometric properties which are not found in any positive definite metrics. In this paper, we consider a Walker 3-manifold. For more details, see [2, 3, 8, 12, 13, 15].

The geometry of harmonic maps is naturally an important subject that is of interest to many geometers [6], [4], [7], [9], [10], [11]. These functions have a wide-ranging field of study because of their applications in mathematics, physics, and engineering. In [7], Eells and Sampson are the first to study harmonic maps between two Riemannian manifolds. In 1862 Airy and Maxwell introduced the biharmonic maps to identify a mathematical model of elasticity. The concept of f-biharmonic maps extends the notion of biharmonic maps. In [10], the author studied $f-$ biharmonic maps between Riemannian manifolds. The authors in [11] characterize an inextensible flow of biharmonic curves according to the Sabban frame in 3 dimensional Heisenberg group.

Moreover, in [14], $f-$biharmonic maps and $f-$biharmonic submanifolds are obtained. The author classified the f-biharmonic curves in Euclidean 3-space and described their properties in n-dimensional space forms.

Niang, Ndiaye, and Diallo [12] classified the strict Walker 3-manifold by construction of two special ruled surfaces.Recently, the authors in [8] have investigated biharmonic curves in a three-dimensional strict Walker manifold. They obtain explicit parametric equations for both biharmonic curves and time-like biharmonic curves.

Motivated by the above works, in this paper we generalize the last results of [8] by studying the geometric properties of f-biharmonic curves in a strict Walker 3-manifold.

2. Preliminaries

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimensions m_1 and m_2. A map $\varphi: M_1 \to M_2$ is harmonic if it is a critical point of the given energy functional

$$E(\varphi) = \frac{1}{2}\int_D \mathrm{P} d\varphi \mathrm{P}^2 dv_1,$$

with D a compact domain of M_1 and dv_1 the volume element of (M_1, g_1). The function $\mathrm{P} d\varphi \mathrm{P}$ is called the Hilbert-Schimidt's norm of $d\varphi$ and defined by

$$\mathrm{P} d\varphi \mathrm{P}^2 := \sum_{i=1}^{n} g_1(d\varphi(e_i), d\varphi(e_i))$$

where $\{e_i\}$ is an orthonormal basis of tangent space of M_1 for all $i = \{1, \cdots, m_1\}$. Let φ_t be a variation of φ and let $V = \left(\frac{\partial}{\partial t}\varphi_t\right)|_{t=0}$ be a variational vector field, then we have

$$\frac{\partial}{\partial t}E(\varphi_t)|_{t=0} = \frac{1}{2}\int_D \left(\frac{\partial}{\partial t}g_1(d\varphi_t, d\varphi_t)\right)|_{t=0} dv_1 = \int_D g_1(tr(\nabla d\varphi), V) dv_1.$$

We define the Euler-Lagrange equation of the functional $E(\varphi)$ by

$$\tau(\varphi) = tr(\nabla d\varphi) = 0.$$

Here $\tau(\varphi)$ is called tension field and $\nabla d\varphi$ the second fundamental form of φ given by

$$\nabla d\varphi(X,Y) = \nabla_X d\varphi(Y) - d\varphi(\nabla_X^{M_1} Y).$$

For more detail, see in in [5]. A map φ is called biharmonic if it minimizes the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_D \mathrm{P}\tau(\varphi)\mathrm{P}^2 dv_1.$$

We have for the bienergy, the following variation formula :

$$\frac{d}{dt}E_2(\varphi,V)|_{t=0} = \int_D g_1(\tau_2(\psi),V)dv_1,$$

and

$$\tau_2(\varphi) = -J(\tau_2(\varphi)) = -\Delta^{\varphi}_{\tau(\varphi)} - trR^{M_2}(d\varphi,\tau(\varphi))d\varphi$$

with J the E_f Jacobi operator and $\tau_2(\varphi)$ is the field of bitension of φ. Here, Δ^φ called the rough Laplacian on the section of the pull-back bundle $\varphi^{-1}TM_2$ that is defined by

$$\Delta^\varphi V = -\sum_{i=1}^n (\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla^{M_2}_{e_i}e_i}V), \quad V \in \Gamma(\varphi^{-1}TM_2),$$

where ∇^φ is the connection induced on the pull-back bundle $\varphi^{-1}TM_2$. If M_1 is a riemannian manifold and $\gamma: I \to M$ be a non nul curve of parameter s, then we have

$$\tau(\gamma) = \nabla_{\frac{\partial}{\partial s}} d\gamma\left(\frac{\partial}{\partial s}\right) = \nabla_T T,$$

if we denote $T = \gamma'$. The biharmonic equation of the curve γ becomes

$$\tau_2(\gamma) = \nabla_T\nabla_T\nabla_T T - R(T,\nabla_T T)T = 0.$$

A map φ is said to be f-harmonic with $f: M \to \mathbb{R}$, if it is a critical point of the energy functional

$$E_f(\varphi) = \frac{1}{2}\int_D f\mathrm{P}d\varphi\mathrm{P}^2 dv_1.$$

The f-tension field $\tau_f(\varphi)$ of φ is a function defined by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(\nabla).$$

The f-bitension field denoted by $\tau_{2,f}(\varphi)$ is given by the following

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \Delta f\tau(\varphi) + 2\nabla_{\nabla f}\tau(\varphi). \tag{1}$$

If the f-bitension field vanishes, then the maps φ is f-biharmonic [1], [10]. The f-biharmonic curves is biharmonic curve if the function f is constant [1].

3. Walker manifolds

A Walker manifold is a Lorentzian manifold admitting a parallel null vector fields. The canonical forms of Walker metrics were studied by Walker in [13]. In adapted coordinates (x_1, x_2, x_2) of $\mathbb{R}^3$, he derived the canonical forms as:

$$g_h^\varepsilon = 2dx_1 \circ dx_3 + \varepsilon dx_2^2 + h(x_1,x_2,x_3)dx_3^2 \tag{2}$$

written in matrix form, we have

$$g_h^\varepsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & h \end{pmatrix}$$

with inverse

$$(g_h^\varepsilon)^{-1} = \begin{pmatrix} -h & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $h(x_1,x_2,x_3)$ is mooth function and $\varepsilon = \pm 1$. In this case, the parallel degenerate line field is given by $X = Span\{\partial_{x_1}\}$. If $\varepsilon = 1$ and $\varepsilon = -1$ the signature of Walker manifold is given respectively by $(2,1)$ and $(1,2)$. In both cases the Walker manifold is Lorentzian.

Using , the Levi-Civita connection associated with the metric (**Error! Reference source not found.**) is computed as follows:

$$\nabla_{\partial_{x_1}}\partial_{x_3} = \frac{1}{2}h_{x_1}\partial_{x_1}, \quad \nabla_{\partial_{x_2}}\partial_{x_3} = \frac{1}{2}h_{x_2}\partial_{x_1}, \quad \nabla_{\partial_{x_3}}\partial_{x_3} = \frac{1}{2}(hh_{x_1} + h_{x_3})$$
$$\partial_{x_1} + \frac{1}{2}h_{x_2}\partial_{x_2} - \frac{1}{2}h_{x_1}\partial_{x_3}, \tag{3}$$

where ∂_{x_1}, ∂_{x_2} and ∂_{x_3} are the coordinate vector fields $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial x_2}$, respectively. If (M, g_h^ε) is a strict Walker manifold that is $h(x_1,x_2,x_3) = h(x_2,x_3)$, then the Levi-Civita connection becomes

$$\nabla_{\partial_{x_2}}\partial_{x_3} = \frac{1}{2}h_{x_2}\partial_{x_1} \qquad \nabla_{\partial_{x_3}}\partial_{x_3} = \frac{1}{2}h_{x_3}\partial_{x_1} - \frac{1}{2}h_{x_2}\partial_{x_2}. \tag{4}$$

and the Christoffel symbols satisfy

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2}h_{x_2}, \qquad \Gamma_{33}^1 = \frac{1}{2}h_{x_3}, \qquad \Gamma_{33}^2 = -\frac{\varepsilon}{2}h_{x_2}. \tag{5}$$

Consider local coordinates (x_1,x_2,x_3) such that (2) holds, an easy computation shows that

$$e_1 = \partial_{x_2}, \quad e_2 = \frac{2-h}{2\sqrt{2}}\partial_{x_1} + \frac{1}{\sqrt{2}}\partial_{x_3}, \quad e_3 = \frac{2+h}{2\sqrt{2}}\partial_{x_1} - \frac{1}{\sqrt{2}}\partial_{x_3} \tag{6}$$

We can show that $\{e_1,e_2,e_3\}$ is a pseudo-orthonormal frame on a Walker manifold (M, g_h^ε) where

$$g_h^\varepsilon(e_1,e_1) = 1, \quad g_h^\varepsilon(e_2,e_2) = \varepsilon, \quad g_h^\varepsilon(e_3,e_3) = -1.$$

The Levi-Civita connection of the Walker metric is given by the following

$$\nabla_{e_i}e_j = \begin{pmatrix} 0 & \frac{1}{4}h_{x_2}(e_2+e_3) & -\frac{1}{4}h_{x_2}(e_2+e_3) \\ \frac{1}{4}h_{x_2}(e_2+e_3) & -\frac{\varepsilon}{4}h_{x_2}e_1 & \frac{\varepsilon}{4}h_{x_2}e_1 \\ -\frac{1}{4}h_{x_2}(e_2+e_3) & \frac{\varepsilon}{4}h_{x_2}e_1 & -\frac{\varepsilon}{4}h_{x_2}e_1 \end{pmatrix} \tag{7}$$

The curvature tensor R of the Levi-Civita connetion ∇ is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{8}$$

where X, Y and Z are vector fields in M and we denote by

$$R_{ijk} = R(e_i, e_j)e_k \tag{9}$$

where $i, j, k \in \{1,2,3\}$. Through a straightforward calculation, we determined the components of the curvature tensor as follows

$$\begin{aligned} R_{121} &= -R_{131} = -\frac{1}{4}h_{x_2x_2}(e_2+e_3) \\ R_{122} &= R_{133} = -R_{123} = -R_{133} = \frac{1}{4}h_{x_2x_2}e_1 \end{aligned} \tag{10}$$

The vector product of two vectors u and v in (M, g_h^ε) is the vector denoted by $u \times v$ and defined by

$$g_h^\varepsilon(u \times v, w) = det(u, v, w) \tag{11}$$

where w is a vector in M and $det(u, v, w)$ is the determinant associeted the canonical basis of $\mathbb{R}^3$. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, using (11), we get :

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - h\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}\right)e_1 - \varepsilon\begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix}e_2 + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}e_3. \tag{12}$$

The frame given in (6) verifies

$$e_1 \times_h e_2 = -e_3, \quad e_2 \times_h e_3 = -e_1, \quad e_3 \times_h e_1 = e_2. \tag{13}$$

4. On f-biharmonic Curves in 3-diemensional Walker manifold

Let β be a unit speed curve of parameter s in the strict Walker 3-manifold (M, g_h^ε). Then the Frenet frame of the curve β is given by

$$\begin{cases} \nabla_T T = \varepsilon_2 kN \\ \nabla_T N = -\varepsilon_1 kT - \varepsilon_3 \tau B \\ \nabla_T B = \varepsilon_2 \tau N \end{cases} \tag{14}$$

Here, T represents the tangent vector, N is the principal normal, and B denotes the binormal vector of β. The quantities k and τ correspond to the curvature and torsion of the curve β, respectively, with:

$$g_h^\varepsilon(T,T) = \varepsilon_1, \qquad g_h^\varepsilon(N,N) = \varepsilon_2, \qquad g_h^\varepsilon(B,B) = \varepsilon_3.$$

Using (14) yields to

$$\nabla_T\nabla_T T = -\varepsilon_1\varepsilon_2 k^2 T + \varepsilon_2 k' N - \varepsilon_2\varepsilon_3 k\tau B \tag{15}$$

and

$$\nabla_T\nabla_T\nabla_T T = (-3\varepsilon_1\varepsilon_2 k'k)T + (-\varepsilon_1 k^3 + \varepsilon_2 k'' - \varepsilon_3 k\tau^2)N - \varepsilon_2\varepsilon_3(2k'\tau + k\tau')B. \tag{16}$$

We put $T = T_1e_1 + T_2e_2 + T_3e_3$, $N = N_1e_1 + N_2e_2 + N_3e_3$ and $B = B_1e_1 + B_2e_2 + B_3e_3$, by using (10) and (13), we have

$$R(T, \nabla_T T)T = \frac{\varepsilon_2 k h_{x_2x_2}(B_2 - B_3)}{4}[(B_2 - B_3)N + (N_2 - N_3)B]. \tag{17}$$

An easy computation gives

$$\tau_2(\beta) = (-3\varepsilon_1\varepsilon_2\kappa'\kappa)T + \left(\varepsilon_1\kappa^3 - \varepsilon_2\kappa'' + \varepsilon_3\kappa\tau^2 - \frac{\varepsilon_2\kappa f_{x_2x_2}(B_2 - B_3)^2}{4}\right)N$$

$$+\left(\frac{\varepsilon_2\kappa f_{x_2x_2}(B_3-B_2)(N_2-N_3)}{4}+\varepsilon_2\varepsilon_3(2\kappa'\tau+\kappa\tau')\right)B. \quad (18)$$

Hence, we get $\tau_{2,f}(\beta)=A_1T+A_2N+A_3B$ where

$$A_1 = -\varepsilon_1\varepsilon_2(3k'kf+2k^2f')$$

$$A_2 = f\left(\varepsilon_1k^3-\varepsilon_2k''+\varepsilon_3k\tau^2-\frac{\varepsilon_2kh_{x_2x_2}}{4}(B_2-B_3)^2\right)+\varepsilon_2(2k'f'+2kf'')$$

$$A_3 = \varepsilon_2\left[\varepsilon_3(2k'\tau+k\tau')f+2\varepsilon_3k\tau f'+f\frac{\varepsilon_2kh_{x_2x_2}}{4}(B_3-B_2)(N_2-N_3)\right]$$

Proposition 4.1: 1 Let $\beta: I\to M$ be a unit speed curve of parameter s and $f: I\mapsto(0,\infty)$ is a function. Then the necessary and sufficient condition for β to be a f-biharmonic curve is

$$f(\nabla_{\beta'}\nabla_{\beta'}\nabla_{\beta'}\beta'-R(\beta',\nabla_{\beta'}\beta')\beta')+2f\nabla_{\beta'}\nabla_{\beta'}\beta'+f''\nabla_{\beta'}\beta'=0 \quad (19)$$

Proof: Let β be a unit speed curve of parameter s and $e_1=\partial x_1$ be an orthonormal frame in (I,ds^2) and $d\alpha(e_1)=d\beta\left(\frac{\partial}{\partial_{x_1}}\right)=\beta'$. The tensor field of β is given by

$$\tau(\beta)=\nabla^{\beta}_{e_1}d\beta(e_1)=\nabla_{\beta'}\beta'.$$

One can verify that for any function $f: I\to(0,+\infty)$,

$$\Delta f=f''$$

and

$$\nabla^{\beta}_{\nabla f}\tau(\beta)=f'\nabla_{\beta'}\nabla_{\beta'}\beta'.$$

Theorem 4.2: 2*A unit speed curve β is an f-biharmonic curve with a function $f: I\mapsto(0,\infty)$ if the following differential system is satisfy*

$$\begin{cases}3k'kf+2k^2f'=0\\ \left(\varepsilon_1k^3-\varepsilon_2k''+\varepsilon_3k\tau^2-\frac{\varepsilon_2kh_{x_2x_2}(B_2-B_3)^2}{4}\right)f+2\varepsilon_2k'f'+\varepsilon_2kf''=0\\ \left(\frac{kh_{x_2x_2}(B_3-B_2)(N_2-N_3)}{4}+\varepsilon_3(2k'\tau+k\tau')\right)f+2\varepsilon_3k\tau f'=0\end{cases} \quad (20)$$

Proof: In deed, $\tau_{2,f}(\gamma)=0\Leftrightarrow$

$$\begin{cases}\varepsilon_1\varepsilon_2(3k'kf+2k^2f')=0\\ f\left(\varepsilon_1k^3-\varepsilon_2k''+\varepsilon_3k\tau^2-\frac{\varepsilon_2kh_{x_2x_2}}{4}(B_2-B_3)^2\right)+\varepsilon_2(2k'f'+2kf'')=0\\ \varepsilon_2\left[\varepsilon_3(2k'\tau+k\tau')f+2\varepsilon_3k\tau f'+f\frac{\varepsilon_2kh_{x_2x_2}}{4}(B_3-B_2)(N_2-N_3)\right]=0\end{cases}$$

As $\varepsilon_1,\varepsilon_2,\varepsilon_3\neq0$, we have

$$\begin{cases} 3k'kf + 2k^2f' = 0 \\ f(\varepsilon_1 k^3 - \varepsilon_2 k'' + \varepsilon_3 k\tau^2 - \dfrac{\varepsilon_2 k h_{x_2x_2}}{4}(B_2 - B_3)^2) + \varepsilon_2(2k'f' + 2kf'') = 0 \\ \varepsilon_3(2k'\tau + k\tau')f + 2\varepsilon_3 k\tau f' + f\dfrac{\varepsilon_2 k h_{x_2x_2}}{4}(B_3 - B_2)(N_2 - N_3) = 0 \end{cases}$$

Corollary 4.3: **3***Let* $\beta: I \to M$ *be unit speed curve of parameter* s. *If* β *is* f*-biharmonic curve then*

$$f(s) = c|k(s)|^{-\frac{3}{2}},$$

where c *is a positive constant.*

Proof: By using the first equation of (20), we get the result

Corollary 4.4: **4***If* β *is geodesic curve, then it is* f*-biharmonic for all positive fonction* f.

Corollary 4.5: **5***If* k *is constant, then* β *is* f*-harmonic if* f *is constant.*

Theorem 4.6: **6***If the function* f *is constant then curve* β *is a nongeodesic biharmonic curve. That is the following relation holds*

$$\begin{cases} k = Constant \neq 0 \\ \varepsilon_1\varepsilon_3 k^2 + \tau^2 = \frac{\varepsilon_2\varepsilon_3(B_2 - B_3)^2 h_{x_2x_2}}{4} \\ \tau' = h_{x_2x_2}(N_2 - N_3)(B_2 - B_3) \end{cases} \tag{21}$$

Proof: By using the formula in (20) and the fact that f is constant, we obtain

$$\begin{cases} k' = 0 \\ \varepsilon_1 k^3 + \varepsilon_3 k\tau^2 - \frac{\varepsilon_2 k h_{x_2x_2}(B_2 - B_3)^2}{4} = 0 \\ \frac{\varepsilon_2 k h_{x_2x_2}(B_3 - B_2)(N_3 - N_2)}{4} + \varepsilon_2\varepsilon_3(k\tau') = 0. \end{cases} \tag{22}$$

That gives

$$\begin{cases} k = Constant \neq 0 \\ \varepsilon_1\varepsilon_3 k^2 + \tau^2 = \frac{\varepsilon_2\varepsilon_3(B_2 - B_3)^2 h_{x_2x_2}}{4} \\ \tau' = h_{x_2x_2}(N_2 - N_3)(B_2 - B_3). \end{cases} \tag{23}$$

5. Conclusion

In this paper, we explored f-biharmonic curves between pseudo-Riemannian 3-manifolds with a parallel null vector field. We established a sufficient condition for a curve to be f-biharmonic and showed that it is a non-geodesic biharmonic curve if and only if the function f remains constant.

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