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Indecomposable decomposition of the A_n quiver

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Abstract

In representation theory, the decomposition of modules over path algebras of quivers plays a crucial role in understanding their structure. This study focuses on the decomposition of representations of type A_n quivers into indecomposable direct summands.

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1. Introduction and Preliminaires

A quiver representation is a concept in mathematics, particularly in the field of representation theory, which combines elements of linear algebra and graph theory. A quiver is essentially a directed graph composed of vertices and arrows (or directed edges). A representation of a quiver consists of the assignment of a vector space to each vertex of the quiver, together with a linear transformation associated to each arrow, where the transformation maps from the vector space assigned to the source vertex of the arrow to the vector space assigned to its target vertex. This approach helps in studying how these vector spaces interact through the directed structure of the graph.

More formally, given a quiver Q with vertices in Q_0 and arrows in Q_1, a representation V of Q consists of:

- A vector space $V(x)$ for each vertex x in Q_0,
- A linear transformation $f(a): V(x) \to V(y)$ for each arrow $a: x \to y \in Q_1$.

Quiver representations are significant in many areas, including algebraic

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structure and study of algebra representation, as they allow for the study about complex systems and categories that have inherent directional or relational structures.

Understanding the decomposition of quiver representations helps in classifying and analyzing them more efficiently. It allows mathematicians to focus on simpler, indecomposable components and explore how these interact or combine to form more complex representations. This approach is fundamental in exploring module theory, algebraic structures, and analyzing systems with relational data represented by directed graphs.

A finite-dimensional representation of a quiver can always split into indecomposable components. According to the Krull-Schmidt Theorem, if a representation can be decomposed into indecomposable representations, this decomposition is unique up to isomorphism and the order of the summands. This is an essential result in representation theory as it guarantees that the decomposition of a representation into indecomposables is well-defined. However, the process of finding the specific decomposition and the indecomposables involved can be complex.

The study of the decomposition of quiver representations is an important area in representation theory, with many influential researchers contributing to this field. Pierre Gabriel was a pioneering figure in the theory of quiver representations. His seminal work in the 1970s introduced the concept of quivers as a tool to study representations of algebras. Gabriel's Theorem, which classifies the indecomposable representations of quivers with underlying graphs as Dynkin diagrams (such as A_n, D_n, E_6, E_7, E_8), is foundational. This result demonstrated that the quivers in question admit finitely many isomorphism classes of indecomposables, up to isomorphism, which can be explicitly described. Claus Ringel has made significant contributions to the representation theory of finite-dimensional algebras and quivers. Idun Reiten's work, particularly in collaboration with Maurice Auslander, has been influential in the development of the theory of almost split sequences and Auslander-Reiten theory. This theory is crucial for studying the relationships between different indecomposable representations and understanding their decompositions. Harm Derksen and Jerzy Weyman have contributed significantly to the broader study of quiver representations and their decompositions, they also give a decomposition for A_2 in [1]. "You can find the necessary fundamental information about representation theory [2]-[13].

In this note we give a decomposition of the A_n quiver for any $n \in \mathbb{Z}$. We also provide a Python code to compute the decompositon of the A_n quiver.

2. Decomposition of the A_n quiver

A quiver is said to be of finite representation type if it has only finitely many non-isomorphic indecomposable representations. A foundational theorem established by Pierre Gabriel asserts that a quiver admits only finitely many isomorphism classes of indecomposable representations that is, it has finite representation type if and only if its underlying undirected graph is a Dynkin diagrams (e.g. A_n, D_n, E_6, E_7, E_8). These graphs have only a finite number of distinct indecomposable representations up to isomorphism. The number of indecomposable representation as in Table 1:

Table 1

Number of indecomposable representations for Dynkin quivers.

Dynkin Type	Number of Indecomposable Representations
A_n	$\frac{n(n+1)}{2}$
D_n	$n(n-1)$
E_6	36
E_7	63
E_8	120

Example 2.1: Let A be a quiver of type A_2

$$A: 1 \overset{\alpha}{\rightarrow} 2$$

consists of two finite dimensional vector spaces V_1 and V_2 in conjunction with a linear transformation $\alpha: V_1 \rightarrow V_2$. There are three indecomposable representations S_1, S_2 and M_{12} namely,

$$S_1: k \rightarrow 0, S_2: 0 \rightarrow k, M_{12}: k \rightarrow k.$$

S_1 and S_2 are simple representations and M_{12} is called string representation.

Example 2.2: Let B be a quiver of type A_n

$$B: k^{n_1} \overset{f_1}{\rightarrow} k^{n_2} \overset{f_2}{\rightarrow} \dots \overset{f_{n-1}}{\longrightarrow} k^{n_n}$$

is a collection of n finite dimensional vector spaces $k^{n_1}, k^{n_2}, \dots, k^{n_n}$ with linear maps $f_i: k^{n_i} \rightarrow k^{n_{i+1}}$. There are $n(n+1)/2$ indecomposable representations. The representations S_i are simple representations associated with the $i-th$ vertex in the quiver. Specifically:

S_i has dimension 1 at the $i-th$ vertex (V_i) and zero at all other vertices $(V_j = 0 \quad i \neq j)$ .These simple representations serve as the fundamental constituents of quiver representations and correspond to the individual vertices of the underlying graph.

The representations M_{ij} are indecomposable representations supported along the directed path from vertex i to vertex j in the quiver are called string. Specifically: for any k such that $i \leq k \leq j$, M_{ij} has dimension 1 at the vertex V_k. At all vertexes $V_l = 0, l \notin [i,j]$, M_{ij} has dimension 0. These representations has the structure of directed paths in the quiver and are key in decomposing arbitrary quiver representations into indecomposables.

For example

$$M_{46}: 0 \xrightarrow{f_1} 0 \xrightarrow{f_2} 0 \xrightarrow{f_3} k \xrightarrow{f_4} k \xrightarrow{f_5} k \xrightarrow{f_6} 0 \dots 0 \xrightarrow{f_{n-1}} 0$$

We also use this notation $rank f_i f_j \dots f_k = r_{ik}$.

The Krull-Schmidt Theorem is a fundamental result in representation theory that guarantees the uniqueness of direct sum decompositions for certain modules and representations. Specifically, it states that if a module (or representation) over a ring can be decomposed into a direct sum of indecomposable submodules, then decomposition is unique modulo permutation and isomorphism. In [1], they give a decomposition for $n = 2$.

Example 2.3: [1] Let us go back to Example 2.1. Any representation V of type A_2 is isomorphic to

$$V \cong S_1^{d_1-r} \oplus S_2^{d_2-r} \oplus M_{12}^r$$

where $d_1 = dimV_1$, $d_2 = dimV_2$ and $r = rank\alpha$, where $\alpha: V_1 \to V_2$ is a linear transformation.

We provide an explicit decomposition of a representation of the A_n quiver in this section.

Theorem 2.4: *Any representation* V *of* A_n *(as in stated in Example 2.2) is isomorphic to*

$V \cong [\bigoplus^{n_1-r_{12}} S_1] \oplus [\bigoplus^{n_2-r_{12}-r_{23}+r_{13}} S_2] \oplus [\bigoplus^{n_3-r_{23}-r_{34}+r_{24}} S_3] \oplus \dots$

$\oplus [\bigoplus^{n_n-r_{1(n-1)}} S_n] \oplus$

$[\bigoplus^{r_1-r_{13}} M_{12}] \oplus [\bigoplus^{r_2-r_{13}} M_{23}] \oplus [\bigoplus^{r_3-r_{32}} M_{24}] \oplus \dots \oplus [\bigoplus^{r_{1(n-1)}-r_{(n-2)n}} M_{(n-1)n}]$

$\oplus \dots \oplus [\bigoplus^{r_{1n}} M_{1n}]$.

Proof: We use induction on n. For $n = 2$ is in [1, Example 6]. The statement remains valid in the case of all values of n less than given n. To clarifty we give the decomposition for $n = 3$. Let V_3 be any representation of A_3

$$V_3: k^{n_1} \xrightarrow{f_1} k^{n_2} \xrightarrow{f_2} k^{n_3},$$

where $rank f_1 = r_{12}$, $rank\ f_2 = r_{23}$ and $rank\ f_2 f_1 = r_{13}$. And $S_1: k \xrightarrow{0} 0 \xrightarrow{0} 0$, $S_2: 0 \xrightarrow{0} k \xrightarrow{0} 0$, $S_3: 0 \xrightarrow{0} 0 \xrightarrow{0} k$ are simple and $M_{12}: k \xrightarrow{1} k \xrightarrow{0} 0$, $M_{23}: 0 \xrightarrow{0} k \xrightarrow{1} k$, $M_{13}: k \xrightarrow{1} k \xrightarrow{1} k$ are string indecomposable representations of A_3 quiver. Let $V_3{}'$ be another representation of A_3 and $g = (g_1, g_2, g_3)$ be an isomorphism between V_3 and $V_3{}'$.

$$\begin{tikzcd}
V_3 \arrow[d, "g"] & k^{n_1} \arrow[r, "f_1"] \arrow[d, "g_1"] & k^{n_2} \arrow[r, "f_2"] \arrow[d, "g_2"] & k^{n_3} \arrow[d, "g_3"] \\
V_3' & k^{n_1} \arrow[r, "g_2 f g_1^{-1}"'] & k^{n_2} \arrow[r, "g_3 f_2 g_2^{-1}"'] & k^{n_3}
\end{tikzcd}.$$

Clearly g_1, g_2 and g_3 are isomorphisms. Matrix forms of $g_2 f_1 g_1^{-1}$, $g_3 f_2 g_2^{-1}$ and $g_3(f_2 f_1)g_1^{-1}$ are

$$B_1 = \begin{bmatrix} I_{r_{12}} & 0_{r_{12}\times(n_1-r_{12})} \\ 0_{(n_2-r_{12})\times r_{12}} & 0_{(n_2-r_{12})\times(n_1-r_{12})} \end{bmatrix}_{n_2\times n_1},$$

$$B_2 = \begin{bmatrix} I_{r_{23}} & 0_{r_{23}\times(n_2-r_{23})} \\ 0_{(n_3-r_{23})\times r_{23}} & 0_{(n_3-r_{23})\times(n_2-r_{23})} \end{bmatrix}_{n_3\times n_2}$$

and

$$B_3 = \begin{bmatrix} I_{r_{13}} & 0_{r_{13}\times(n_1-r_{13})} \\ 0_{(n_3-r_{13})\times r_{13}} & 0_{(n_3-r_{13})\times(n_1-r_{13})} \end{bmatrix}_{n_3\times n_1}$$

respectively. Hence we get decomposition from B_1, B_2 and B_3 are

$$(k\to k)^{r_{12}} \oplus (k\to 0)^{n_1-r_{12}} \oplus (0\to k)^{n_2-r_{12}},$$

$$(k\to k)^{r_{23}} \oplus (k\to 0)^{n_2-r_{23}} \oplus (0\to k)^{n_3-r_{23}}$$

and

$$(k\to k)^{r_{13}} \oplus (k\to 0)^{n_1-r_{13}} \oplus (0\to k)^{n_3-r_{13}}$$

respectively.

Hence we get an indecomposable decomposition of any representation V_3 of A_3 as in the following:

$$V_3 \cong (\oplus_{i=1}^{n_1-r_{12}} S_1) \oplus (\oplus_{i=1}^{n_2-r_{12}-r_{23}+r_{13}} S_2) \oplus (\oplus_{i=1}^{n_3-r_{23}} S_3) \oplus (\oplus_{i=1}^{r_{12}-r_{13}} M_{12}) \oplus (\oplus_{i=1}^{r_{23}-r_{13}} M_{23}) \oplus (\oplus_{i=1}^{r_{13}} M_{13})$$

Consider the representation V_n of A_n

$$A: k^{n_1} \xrightarrow{f_1} k^{n_2} \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} k^{n_n}$$

By an induction assumption, any representation V_{n-1} of A_{n-1} is isomorphic to

$$V_{n-1} \cong [\oplus^{n_1-r_{12}} S_1] \oplus [\oplus^{n_2-r_{12}-r_{23}+r_{13}} S_2] \oplus [\oplus^{n_3-r_{23}-r_{34}+r_{24}} S_3] \oplus \dots \oplus [\oplus^{n_n-r_{1(n-1)}} S_n] \oplus [\oplus^{r_{12}-r_{13}} M_{12}] \oplus [\oplus^{r_{23}-r_{13}} M_{23}] \oplus [\oplus^{r_{34}-r_{24}} M_{34}] \oplus \dots \oplus [\oplus^{r_{1(n-1)}-r_{(n-2)n}} M_{(n-1)n}] \oplus \dots \oplus [\oplus^{r_{1(n-2)}} M_{1(n-1)}].$$

Now consider this representation as follows:

$$\begin{array}{ccccc} k^{n_1} & \xrightarrow{f} & k^{n_{(n-1)}} & \xrightarrow{f_{n-1}} & k^{n_n} \\ \downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 \\ k^{n_1} & \xrightarrow[g_2 f g_1^{-1}]{} & k^{n_{(n-1)}} & \xrightarrow[g_3 f_{n-1} g_2^{-1}]{} & k^{n_n} \end{array}.$$

, where $f = f_{n-2}f_{n-3} \dots f_2 f_1$ and the ranks of f, f_{n-1}, and $f_{n-1}f$ are $r_{1(n-1)}$, $r_{(n-1)n}$, and r_{1n} respectively and take isomorphisms $g_1 : k^{n_1} \to k^{n_1}$, $g_2 : k^{n(n-1)} \to k^{n(n-1)}$, and $g_3 : k^{n_n} \to k^{n_n}$. By Smith Normal Form, we get the matrices of $g_2 f g_1^{-1}$ and $g_3 f_{n-1} g_2^{-1}$ have block forms $\begin{pmatrix} I_{r_{1(n-2)}} & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} I_{r_{1(n-1)}} & 0 \\ 0 & 0 \end{pmatrix}$, where $I_{r_{1(n-2)}}$ and $I_{r_{1(n-1)}}$ are identity matrices. We have another representation V' which are $g_2 f g_1^{-1}$ and $g_3 f_{n-1} g_2^{-1}$ and a morphism $g = (g_1, g_2, g_3): V \to V'$. g is an isomorphism, since g_1 , g_2 and g_3 are isomorphisms. Then $g_2 f g_1^{-1}$ and $g_3 f_{n-1} g_2^{-1}$ have suitable block diagonal form then we get desired decomposition.

Example 2.5: Let M be a representation of type A_5 as in the following:

$$M: k^2 \xrightarrow{g_1} k^3 \xrightarrow{g_2} k \xrightarrow{g_3} k^2 \xrightarrow{g_4} k^2$$

Let $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $g_2 = (1 \quad 1)$, $g_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $g_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then we have $rank g_1 = r_{12} = 2$, $rank g_2 = r_{23} = 1$, $rank g_3 = r_{34} = 1$, $rank g_4 = r_{45} = 2$, $rank g_2 g_1 = r_{13} = 1$, $rank g_3 g_2 = r_{24} = 1$, $rank g_4 g_3 = r_{35} = 1$, $rank g_3 g_2 g_1 = r_{14} = 1$, $rank g_4 g_3 g_2 = r_{25} = 1$, $rank g_4 g_3 g_2 g_1 = r_{15} = 1$. Then we have decomposition

$M \cong (\oplus^{2-2} S_1) \oplus (\oplus^{3-2-1+1} S_2) \oplus (\oplus^{1-1-1+1} S_3) \oplus (\oplus^{2-1-2+1} S_4) \oplus (\oplus^{2-2} S_5)$

$\oplus (\oplus^{2-1} M_{12}) \oplus (\oplus^{1-1} M_{23}) \oplus (\oplus^{1-1} M_{34}) \oplus (\oplus^{2-1} M_{45})(\oplus^{1-1} M_{13}) \oplus (\oplus^{1-1} M_{24}) \oplus (\oplus^{1-1} M_{35})$

$\oplus (\oplus^{1-1} M_{14}) \oplus (\oplus^{1-1} M_{24}) \oplus (\oplus^{1} M_{15})$.

$M \cong S_2 \oplus M_{12} \oplus M_{45} \oplus M_{15}$.

Hence we get

$$M \cong (0 \xrightarrow{0} k \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0) \oplus (k \xrightarrow{1} k \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0)$$
$$\oplus (0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} k \xrightarrow{1} k) \oplus (k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k).$$

3. Application of the Decomposition

This result contribute to a deeper comprehension of the representation theory of some algebras and their module categories, offering a systematic approach to computing decompositions in finite-dimensional cases. By using this result, This result is used to provide a Python code. Listing 1 is the Python code for decomposing a quiver of type A_n into its indecomposable components.

Listing 1

Python Code for Decomposition of A_n Quiver Representation

```
import numpy as np

def decompose_an_quiver_with_compositions(n_values, matrices):
"""
Decomposes an An quiver representation into indecomposable components,
and computes ranks for direct maps, composite maps, string modules,
and specifically M_1n modules.

Args:
n_values: List of dimensions of vector spaces [n1, n2, ..., nn].
matrices: List of matrices representing the maps f1, f2, ..., f(n-1).

Returns:
A tuple containing:
- Simple modules: Dictionary of Si with their dimensions.
- Direct ranks: Dictionary of ranks r_{i(i+1)}.
- Composite ranks: Dictionary of ranks r_{ij...k}.
- String modules: Dictionary of M_{i(i+1)} and M_{ij}.
- Special M_1n modules: Dictionary of M_{1n}.
"""
n = len(n_values) # Number of vertices in the quiver

# Step 1: Compute direct ranks (r_{i(i+1)})
direct_ranks = {}
for i, matrix in enumerate(matrices):
direct_ranks[f"r_{i+1}{i+2}"] = np.linalg.matrix_rank(matrix)

# Step 2: Compute composite ranks (r_{ij...k})
composite_ranks = {}
for i in range(len(matrices)):
composite_map = np.eye(n_values[i]) # Start with identity map
for j in range(i, len(matrices)):
composite_map = np.dot(matrices[j], composite_map)
composite_ranks[f"r_{i+1}{j+2}"] =
    np.linalg.matrix_rank(composite_map)

# Step 3: Compute string modules (M_{i(i+1)} and M_{ij})
string_modules = {}

# Direct string modules (M_{i(i+1)})
for i in range(len(n_values) - 1):
if i == 0:
M = direct_ranks.get(f"r_{i+1}{i+2}", 0) - direct_ranks.get
    (f"r_{i+2}{i+3}", 0)
else:
M = direct_ranks.get(f"r_{i+1}{i+2}", 0) - composite_ranks.get
    (f"r_{i}{i+2}", 0)
if M > 0:
string_modules[f"M{i+1},{i+2}"] = M
# Composite string modules (M_{ij} where j > i+1)
for i in range(len(n_values) - 1):
for j in range(i + 2, len(n_values)):
if i == 0:
M = composite_ranks.get(f"r_{i+1}{j+2}", 0)
```

```
    - composite_ranks.get(f"r_{i+2}{j+2}", 0)
else:
M = composite_ranks.get(f"r_{i+1}{j+2}", 0) -
    composite_ranks.get(f"r_{i}{j+2}", 0)
if M > 0:
string_modules[f"M{i+1},{j+1}"] = M

# Step 4: Compute simple modules (Si)
simple_modules = {}
for i in range(n):
if i == 0: # First vertex (i=1)
Si = n_values[i] - direct_ranks.get(f"r_{i+1}{i+2}", 0)
elif i == n - 1: # Last vertex (i=n)
Si = n_values[i] - direct_ranks.get(f"r_{i}{i+1}", 0)
else: # Middle vertices (1 < i < n)
Si = (n_values[i]
- direct_ranks.get(f"r_{i}{i+1}", 0)
- direct_ranks.get(f"r_{i+1}{i+2}", 0)
+ composite_ranks.get(f"r_{i}{i+2}", 0))
if Si > 0:
simple_modules[f"S{i+1}"] = Si

# Step 5: Compute special M_{1n} modules
m_1n_modules = {}
if n > 1:
composite_map_1n = np.eye(n_values[0]) # Start with identity map
for i in range(len(matrices)):
composite_map_1n = np.dot(matrices[i], composite_map_1n)
m_1n_rank = np.linalg.matrix_rank(composite_map_1n)
m_1n_modules[f"M1,{n}"] = m_1n_rank

return simple_modules, direct_ranks, composite_ranks, string_modules,
    m_1n_modules

# Example usage for A6 quiver
n_values = [2, 3, 1, 2, 2, 1] # Dimensions of V1, V2, ..., V6

# Define the transformations as matrices
f1 = np.array([[1, 1], [1, 0], [0, 1]]) # 3x2 matrix
f2 = np.array([[1, 0, 1]]) # 1x3 matrix
f3 = np.array([[1], [1]]) # 2x1 matrix
f4 = np.array([[1, 1], [1, 1]]) # 2x2 matrix
f5 = np.array([[1, 1]]) # 1x2 matrix

matrices = [f1, f2, f3, f4, f5]

# Compute the decomposition and ranks
simple_modules, direct_ranks, composite_ranks,
    string_modules, m_1n_modules =
    decompose_an_quiver_with_compositions(n_values, matrices)



# Print the decomposition
print("Simple Modules:")
for module, dimension in simple_modules.items():
print(f"{module}: {dimension}")
```

```
print("\nDirect Ranks (r_{i(i+1)}):")
for key, value in direct_ranks.items():
print(f"{key}: {value}")

print("\Composite Ranks (r_{ij...k}):")
for key, value in composite_ranks.items():
print(f"{key}: {value}")

print("\nString Modules (M_{ij}):")
for key, value in string_modules.items():
print(f"{key}: {value}")

print("\nSpecial M_{1n} Modules:")
for key, value in m_1n_modules.items():
print(f"{key}: {value}")
```

We apply for $n = 7$ as an application of this code,

Example 3.1: Let M be a representation of type A_7 as in the following:

$$M: k^2 \overset{g_1}{\to} k^3 \overset{g_2}{\to} k \overset{g_3}{\to} k^2 \overset{g_4}{\to} k^2 \overset{g_5}{\to} k \overset{g_6}{\to} k^3$$

Let $g_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$, $g_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $g_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $g_5 = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $g_6 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Then we have

Simple Representations

S_2 : 1,
S_4 : 1,
S_5 : 1,
S_7 : 2.

Direct Ranks ($r_{i(i+1)}$)

r_{12} : 2,
r_{23} : 1,
r_{34} : 1,
r_{45} : 1,
r_{56} : 1,
r_{67} : 1.

Composite Ranks (r_{ij})

$$\begin{array}{llllllllll} r_{12} & :1, & r_{13} & :1, & r_{14} & :1, & r_{15} & :1, & r_{16} & :1, \\ r_{17} & :1, \\ r_{23} & :1, & r_{24} & :1, & r_{25} & :1, & r_{26} & :1, & r_{27} & :1, \\ r_{34} & :1, & r_{35} & :1, & r_{36} & :1, & r_{37} & :1, \\ r_{45} & :1, & r_{46} & :1, & r_{47} & :1, \\ r_{56} & :1, & r_{57} & :1, \\ r_{67} & :1. \end{array}$$

String Representations (M_{ij})

$$M_{12} \quad :1.$$

Special M_{1n} Representation

$$M_{17} \quad :1.$$

Then we get

$$\begin{aligned} M &\cong (0 \xrightarrow{0} k \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0) \oplus (0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} k \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0) \\ &\oplus (0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} k \xrightarrow{0} 0 \xrightarrow{0} 0) \oplus (0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} k)^2 \\ &\oplus (k \xrightarrow{1} k \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0) \oplus\oplus (k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k) \\ &\cong S_2 \oplus S_4 \oplus S_5 \oplus S_7 \oplus S_7 \oplus M_{12} \oplus M_{17} \end{aligned}$$

Conflict of interest

Authors have disclosed no financial or personal conflicts.

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