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Convergence results for solution of some families of multiple set split feasibility problems in stochastic domain

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Abstract

In this paper, the stochastic equivalence of some families of split feasibility problems has been studied. The work adopted iterative random-type approach using a firmly non-expansive mapping in Hilbert space. For solution of the problems, strong convergence to the random fixed-point was demonstrated. The findings of this study will combine, broaden, and generalize various prior classical findings on the studied families of split feasibility problems.

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1. Introduction

Managerial decisions in a number of organizations are mostly based on the knowledge that is currently accessible regarding past and current observations, as well as perhaps on the process that generates these observations. Making these decisions is a tedious task among mangers due to increase in innovations, technology and competition. Due to the consequences of wrong decisions, decision-makers cannot afford to make decisions based on guesswork, intuition or personal experience only [1]. Hence an understanding of the use of optimization methods to decision making is desirable to managers. One of the basic optimization methods which is widely used in many real life applications due to its ability in handling complexity and intricacies in terms of constraints is the iterative method [2]. When optimization function which idealizes optimization problem is corrupted with noise (observation noise or structural inaccuracy in observed function or both), the deterministic (classical) iterative method will no longer give an efficient estimate, hence a stochastic iterative method which will put into consideration the error (noise) in the observations.

One of such optimization problems with wide applications is split feasibility problems [3]. Multiple-sets split feasibility problem is one os such common families of split feasibility problem.

The classical (deterministic) multiple-sets split equality problem involves locating a point:

$$r \in C = \bigcap_{i=1}^{p} C_i \text{ and } k \in Q = \bigcap_{j=1}^{q} Q_j \text{ such that } Ar = Bk \tag{1.1}$$

where, C and Q are considered as non-empty convex sets from two real Hilbert spaces denoted as H_1 and H_2. It has $A: H_1 \to H_2$ as a linear bounded operator with adjoints $A: H_1 \to H_3$ and $B: H_2 \to H_3$

While the deterministic multiple-sets split feasibility problem involves locating a point

$$r \in C = \bigcap_{i=1}^{p} C_i \text{ such that } Ar \in Q = \bigcap_{j=1}^{q} Q_j \tag{1.2}$$

Where C and Q retains its meaning in (1.1) and $A: H_1 \to H_2$ with adjoint $A^*: H_2 \to H_1$.

The stochastic equivalent of (1.1) and (1.2) are to locate the random point

$$r(\omega) \in C = \bigcap_{i=1}^{p} C_i \text{ and } k(\omega) \in Q = \bigcap_{j=1}^{q} Q_j \text{ such that } Ar(\omega) = Bt(\omega) \tag{1.3}$$

and

$$r(\omega) \in C = \bigcap_{i=1}^{p} C_i \text{ such that } Ar(\omega) \in Q = \bigcap_{j=1}^{q} Q_j \tag{1.4}$$

where, $r(\omega)$ and $t(\omega)$ are random variables define on probability space (Ω, Σ, Ψ), $\omega \in \Omega$, C, Q, A and B retains its meaning as used in (1.1) and (1.2).

The iterative procedure for problems (1.1) and (1.2) would have serve as a good alternative to problems (1.3) and (1.4) except that they (1.3 and 1.4) cannot be fully specified because it is contaminated by noise , hence its gradient may not exist [4-5]. Therefore, this study tends to generate the stochastic equivalence of approximation results of multiple-sets split equality and multiple-sets split feasibility problems with adoption of firmly non-expansive mapping using random-type iterative schemes.

Censor and Elfving [6] was first to introduce the concept of split feasibility and multiple-sets split feasibility problems. Their study was followed up by [5, 7-8,]. In particular, Nweke et al. [9] studied approximations results for stochastic split feasibility problems using random typed iterative scheme in particular perturbed Mann iterative scheme. Split common and equality problems was studied by Moudafi [10] together with Chuang and Chen [11] with some of their references. Stochastic approximation was introduced by Robbins and Monro [12] and further studies using random-type iterative schemes were deployed to solve variational inequality, minimization and convex feasibility problems [4, 13-14]. Areas of application of split feasibility problems could be found in [3, 6, 15].

2. Definitions

This study will adopt the notation and definition of complete probability measure space denoted by (Ω, Σ, Ψ), measurable space, probability measure, random operator, Fejer monotone, Non-expansiveness, Firmly non-expansiveness and metric projection.

Definition 2.1: [3, 5] . Suppose that $A: H_1 \to H_2$ be a bounded linear operator and (Ω, Σ, Ψ) a complete probability measure space, the adjoint $A^*: H_2 \to H_1$ has the following properties.

(i) $\int_{\Omega} \langle Ar(\omega), k(\omega)\rangle_{H_2} \; \mathrm{d}\Psi(\omega) \; = \; \int_{\Omega} \langle r(\omega), \; A^* k(\omega)\rangle_{H_1} \; \mathrm{d}\Psi(\omega), \; \forall \; r(\omega) \in H_1$ and $k(\omega) \in H_2$, where, $\langle . \, , . \rangle_{H_1}$ and $\langle . \, , . \rangle_{H_2}$ are inner products of H_1 and H_2 respectively.

(ii) $\| A \|^2 = \| A^* \|^2 = \| A^* A \|^2 = \| AA^* \|^2$

3. Stochastic approximation based on robbin-monro algorithm

This study will adopt the recursive algorithm proposed by Robboins and Monro [12] to solve the multiple-set split feasibility problem. See Nweke and Udom [5], Nweke et al. [9] as well as Udom and Nweke [14] for details on the application of RM algorithm in solving stochastic feasibility problems.

4. Main Results

Assumption 4.1

(a) Let β_n represents a real sequence on the interval $0 < \beta_n < 1$, and λ be a constant that on the interval $0 < \lambda < 2$.

(b) he mapping $P_C : C \to C$ is a random projection onto C while $P_Q : Q \to Q$ is random projection onto Q, where $C \subset H_1$ and $Q \subset H_2$.

(c) Let the bounded linear random operator and its adjoint be represented thus: $A : H_1 \to H_2$ and $A^* : H_2 \to H_1$ respectively.

(d) Let (Ω, Σ, Ψ) represents a complete probability measure space.

Theorem 4.1: *If assumption 4.1 holds, then the solution of the problem in (1.4) defined by (4.1) and generated by the sequence* $\{r_n(\omega)\}$ *converges almost surely as well as in quadratic mean to the random fixed-point* $w(\omega) \in \Gamma$.

$$r_{n+1}(\omega) = (1 - \beta_n) r_n(\omega) + \beta_n (P_C(\omega, r_n) - \lambda A^* (1 - P_Q) A r_n(\omega)) \tag{4.1}$$

Proof: Establishing the stochastic boundedness of the sequence $\{r_n(\omega)\}$, we have

$$E\|r_{n+1}(\omega)-w(\omega)\|^2=\int_\Omega \|r_{n+1}(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)$$
$$=\int_\Omega \|(1-\beta_n)r_n(\omega)+\beta_n(P_C(\omega,r_n)-\lambda A^*(1-P_Q)Ar_n(\omega))-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)$$
$$=\int_\Omega \|r_n(\omega)-w(\omega)-\beta_n(r_n(\omega)-P_C(\omega,r_n)+\lambda A^*(1-P_Q)Ar_n(\omega))\|^2\,\mathrm{d}\Psi(\omega)$$
$$=\int_\Omega \|r_n(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)-2\beta_n\int_\Omega \langle r_n(\omega)-w(\omega)\ ,\ r_n(\omega)-P_C(\omega,r_n)$$
$$+\lambda A^*(1-P_Q)Ar_n(\omega)\rangle \mathrm{d}\Psi(\omega)+\beta_n^2\int_\Omega \|r_n(\omega)-P_C(\omega,r_n)$$
$$+\lambda A^*(1-P_Q)Ar_n(\omega)\|^2\,\mathrm{d}\Psi(\omega)$$
$$\leq \int_\Omega \|r_n(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)-2\beta_n\int_\Omega \langle r_n(\omega)-z(\omega),r_n(\omega)-P_C(\omega,r_n)\rangle \mathrm{d}\Psi(\omega)$$
$$-2\lambda\beta_n\int_\Omega \langle A^*r_n(\omega)-A^*w(\omega),(1-P_Q)Ar_n(\omega)\rangle \mathrm{d}\Psi(\omega)$$
$$+\beta_n^2\int_\Omega (2\|r_n(\omega)-P_C(\omega,r_n)\|^2+2\lambda^2\|A\|^2\|(1-P_Q)Ar_n(\omega)\|^2)\mathrm{d}\Psi(\omega) \tag{4.2}$$

Adopting the assumption of firmly non-expansivity of the random operators P_C and P_Q, we have

$$\leq \int_\Omega \|r_n(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)-2\beta_n\int_\Omega \|r_n(\omega)-P_C(\omega,r_n)\|^2\,\mathrm{d}\Psi(\omega)$$
$$-2\lambda\beta_n\int_\Omega \|(1-P_Q)Ar_n(\omega)\|^2\,\mathrm{d}\Psi(\omega)+2\beta_n^2\int_\Omega \|r_n(\omega)-P_C(\omega,r_n)\|^2\,\mathrm{d}\Psi(\omega)$$
$$+2\lambda^2\beta_n^2\|A\|^2\int_\Omega \|(1-P_Q)Ar_n(\omega)\|^2\,\mathrm{d}\Psi(\omega)$$
$$=\int_\Omega \|r_n(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)-2\lambda\beta_n(1-\lambda\beta_n\|A\|^2)\int_\Omega \|(1-P_Q)Ar_n(\omega)\|^2\,\mathrm{d}\Psi(\omega)$$
$$-2\beta_n(1-\beta_n)\int_\Omega \|r_n(\omega)-P_C(\omega,r_n)\|^2\,\mathrm{d}\Psi(\omega)$$
$$\therefore E\|r_{n+1}(\omega)-w(\omega)\|^2=\int_\Omega \|r_{n+1}(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)\leq \int_\Omega \|r_n(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega) \tag{4.3}$$

Therefore the sequence $\{r_n(\omega)\}$ is stochastically bounded, hence Fejer monotone

From (4.3) we have

$$2\beta_n(1-\beta_n)\int_\Omega \|r_n(\omega)-P_C(\omega,r_n)\|^2\,\mathrm{d}\Psi(\omega)$$
$$+2\lambda\beta_n(1-\lambda\beta_n\|A\|^2)\int_\Omega \|(1-P_Q)Ar_n(\omega)\|^2\,\mathrm{d}\Psi(\omega)$$
$$\leq \int_\Omega \|r_n(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)-\int_\Omega \|r_{n+1}(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)$$
$$\leq \left[\int_\Omega \|r_n(\omega)-w(\omega)\|\,\mathrm{d}\Psi(\omega)-\int_\Omega \|r_{n+1}(\omega)-w(\omega)\|\,\mathrm{d}\Psi(\omega)\right]$$

$$\times\left[\int_{\Omega}\| r_n(\omega)-w(\omega)\|\,\mathrm{d}\Psi(\omega)+\int_{\Omega}\| r_{n+1}(\omega)-w(\omega)\|\,\mathrm{d}\Psi(\omega)\right] \tag{4.4}$$

Since the sequence is Feje monotone, then

$$\lim_{n\to\infty}\int_{\Omega}\| r_n(\omega)-P_C(\omega,r_n)\|^2\,\mathrm{d}\Psi(\omega)=0$$

and

$$\lim_{n\to\infty}\int_{\Omega}\|(1-P_Q)Ar_n(\omega)\|^2\,\mathrm{d}\Psi(\omega)=0$$

$$\therefore \lim_{n\to\infty}\int_{\Omega}\| y_{n+1}(\omega)-w(\omega)\|\,\mathrm{d}\Psi(\omega)=0$$

and

$$\lim_{n\to\infty}\int_{\Omega}\| y_{n+1}(\omega)-w(\omega)\|^2\,\mathrm{d}\Psi(\omega)=0$$

That ends the proof

Corollary 4.1: *Let define the ρ-relaxation of* P_C *and* P_Q *in Theorem 4.1 by* $P_{C_\rho}=(1-\rho)\mathrm{I}+\lambda P_C$ *and* $P_{Q_\rho}=(1-\rho)\mathrm{I}+\rho P_C$, *where* $0<\rho<2$. *Suppose assumptions 4.1 holds with all the conditions on* P_C *and* P_Q *in Theorem 4.1 holdings for* P_{C_ρ} *and* P_{Q_ρ}, *then the solution of the problem in (1.3) generated by* $\{r_n(\omega)\}$ *and defined by (4.5) converges almost surely to the random fixed-point* $w(\omega)\in\Gamma$. *It also converges in mean square, consequently in probability.*

$$r_{n+1}(\omega)=(1-\beta_n)r_n(\omega)+\beta_n(P_{C_\rho}(\omega,r_n)-\lambda A^*(1-P_{Q_\rho})Ar_n(\omega)) \tag{4.5}$$

Proof:

$$r_{n+1}(\omega)=(1-\beta_n)r_n(\omega)+\beta_n(P_{C_\rho}(\omega,r_n)-\lambda A^*(1-P_{Q_\rho})Ar_n(\omega))$$

Replacing for $P_{C_\rho}=(1-\rho)\mathrm{I}+\lambda P_C$ and $P_{Q_\rho}=(1-\rho)\mathrm{I}+\lambda P_Q$

$$r_{n+1}(\omega)=(1-\beta_n)r_n(\omega)+\beta_n(((1-\rho)\mathrm{I}+\rho P_C)r_n(\omega)-\lambda A^*(1-(1-\rho)\mathrm{I}-\rho P_Q)Ar_n(\omega))$$

$$r_{n+1}(\omega)=(1-\beta_n)r_n(\omega)+\beta_n((1-\rho)r_n(\omega)+\rho P_C(\omega,r_n)-\lambda\rho A^*(1-P_Q)Ar_n(\omega))$$

$$r_{n+1}(\omega)=y_n(\omega)-\rho\beta_n(P_C(\omega,r_n)+\lambda A^*(1-P_Q)Ar_n(\omega))$$

Therefore,

$$E\| r_{n+1}(\omega) - w(\omega)\|^2$$
$$= \int_\Omega \| r_n(\omega) - w(\omega) - \rho\beta_n (r_n(\omega) - P_C(\omega, r_n) + \lambda A^*(1-P_Q)Ar_n(\omega))\|^2 \,\mathrm{d}\Psi(\omega)$$
$$= \int_\Omega \| r_n(\omega) - w(\omega)\|^2 \,\mathrm{d}\Psi(\omega) - 2\rho\beta_n \int_\Omega \langle r_n(\omega) - w(\omega)\ ,\ r_n(\omega) - P_C(\omega, y_n)$$
$$+\lambda A^*(1-P_Q)Ar_n(\omega)\rangle \mathrm{d}\Psi(\omega) + \rho^2\beta_n^2 \int_\Omega \| r_n(\omega) - P_C(\omega, r_n)$$
$$+\lambda A^*(1-P_Q)Ar_n(\omega)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$\le \int_\Omega \| r_n(\omega) - w(\omega)\|^2 \,\mathrm{d}\Psi(\omega) - 2\rho\beta_n \int_\Omega \langle r_n(\omega) - w(\omega), r_n(\omega) - P_C(\omega, r_n)\rangle \mathrm{d}\Psi(\omega)$$
$$-2\rho\lambda\beta_n \int_\Omega \langle Ar_n(\omega) - Aw(\omega), (1-P_Q)Ar_n(\omega)\rangle \mathrm{d}\Psi(\omega)$$
$$+\rho^2\beta_n^2 \int_\Omega (2\| r_n(\omega) - P_C(\omega, r_n)\|^2 + 2\lambda^2 \|A\|^2 \|(1-P_Q)Ar_n(\omega)\|^2)\mathrm{d}\Psi(\omega)$$

Appealing the firmly non-expansivity of the operators P_C and P_Q

$$\le \int_\Omega \| r_n(\omega) - w(\omega)\|^2 \,\mathrm{d}\Psi(\omega) \ - 2\rho\beta_n \int_\Omega \| r_n(\omega) - P_C(\omega, r_n)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$-2\rho\lambda\beta_n \int_\Omega \|(1-P_Q)Ar_n(\omega)\|^2 \,\mathrm{d}\Psi(\omega) + 2\rho^2\beta_n^2 \int_\Omega \| r_n(\omega) - P_C(\omega, y_n)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$+2\rho^2\lambda^2\beta_n^2 \|A\|^2 \int_\Omega \|(1-P_Q)Ar_n(\omega)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$= \int_\Omega \| r_n(\omega) - w(\omega)\|^2 \,\mathrm{d}\Psi(\omega) - 2\rho\lambda\beta_n(1-\lambda\beta_n \|A\|^2)\int_\Omega \|(1-P_Q)Ar_n(\omega)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$-2\rho\beta_n(1-\rho\beta_n)\int_\Omega \| r_n(\omega) - P_C(\omega, r_n)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$\therefore\ E\| r_{n+1}(\omega) - w(\omega)\|^2 = \int_\Omega \| r_{n+1}(\omega) - w(\omega)\|^2 \,\mathrm{d}\Psi(\omega) \le \int_\Omega \| r_n(\omega) - w(\omega)\|^2 \,\mathrm{d}\Psi(\omega) \tag{4.6}$$

This shows that the sequence $\{r_n(\omega)\}$ is stochastically bounded and is Fejer monotone.

From (4.6) we have

$$2\rho\lambda\beta_n(1-\lambda\beta_n \|A\|^2)\int_\Omega \|(1-P_Q)Ar_n(\omega)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$+2\rho\beta_n(1-\beta_n)\int_\Omega \| r_n(\omega) - P_C(\omega, y_n)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$\le \int_\Omega \| r_n(\omega) - w(\omega)\|^2 \,\mathrm{d}\Psi(\omega) - \int_\Omega \| r_{n+1}(\omega) - w(\omega)\|^2 \,\mathrm{d}\Psi(\omega)$$
$$\le \left[\int_\Omega \| r_n(\omega) - w(\omega)\| \,\mathrm{d}\Psi(\omega) - \int_\Omega \| r_{n+1}(\omega) - w(\omega)\| \,\mathrm{d}\Psi(\omega)\right]$$
$$\times\left[\int_\Omega \| r_n(\omega) - w(\omega)\| \,\mathrm{d}\Psi(\omega) + \int_\Omega \| r_{n+1}(\omega) - w(\omega)\| \,\mathrm{d}\Psi(\omega))\right] \tag{4.7}$$

Appealing to the monotone of the sequence

$$\lim_{n\to\infty}\int_{\Omega}\left\|r_n(\omega)-P_C(\omega,r_n)\right\|^2 \mathrm{d}\Psi(\omega)=0$$

$$\lim_{n\to\infty}\int_{\Omega}\left\|(1-P_Q)Ar_n(\omega)\right\|^2 \mathrm{d}\Psi(\omega)=0$$

$$\therefore \quad \lim_{n\to\infty}\int_{\Omega}\left\|r_{n+1}(\omega)-w(\omega)\right\|\mathrm{d}\Psi(\omega)=0$$

and

$$\lim_{n\to\infty}\int_{\Omega}\left\|r_{n+1}(\omega)-w(\omega)\right\|^2 \mathrm{d}\Psi(\omega)=0$$

This completes the proof.

Assumption 4.2:

(a) Let α_n and β_n represents real sequences on the interval (0, 1) and λ_1,λ_2 be constants that lie on the interval (0, 2).

(b) The mapping $P_C:C\to C$ is a random projection onto C while $P_Q:Q\to Q$ is random projection onto Q, where $C\subset H_1$ and $Q\subset H_2$.

(c) Let the bounded random operators $A:H_1\to H_3$ and $B:H_2\to H_3$ have the respective adjoints $A^*:H_3\to H_1$ and $A^*:H_3\to H_2$.

(d) Le the complete probability measure space be denoted by (Ω,Σ,Ψ) and H_1,H_2,H_3 three real finite dimensional Hilbert spaces.

Theorem 4.2: *If assumption 4.2 holds, then the solution of the problem (1.3) generated by $\{r_n(\omega)\}$ and $\{k_n(\omega)\}$ and defined by (4.8) converges in quadratic mean to the random fixed-point.*

$$\begin{cases} r_0=1 \\ r_n(\omega)=(1-\beta_n)r_n(\omega)+\beta_n(P_C(\omega,r_n)-\lambda_1A^*(1-P_Q)Ar_n(\omega)) \\ r_{n+1}(\omega)=(1-\alpha_n)r_n(\omega)+\alpha_n(P_C(\omega,k_n)-\lambda_2B^*(1-P_Q)Bk_n(\omega)) \end{cases} \tag{4.8}$$

Proof: We first show that $\{r_n(\omega)\},\{k_n(\omega)\}$ are stochastically bounded.

$$E\,||\,k_n(\omega)-w(\omega)\,||=\int_{\Omega}||\,k_n(\omega)-w(\omega)\,||\,\mathrm{d}\Psi(\omega)$$

$$=\int_{\Omega}||\,(1-\beta_n)r_n(\omega)+\beta_n(P_C(\omega,r_n)-\lambda_1A^*(1-P_Q)Ar_n(\omega))-w(\omega)\,||\,\mathrm{d}\Psi(\omega) \tag{4.9}$$

Following (4.6), (4.9) becomes

$$\le\int_{\Omega}||\,r_n(\omega)-w(\omega)\,||^2\,\mathrm{d}\Psi(\omega)-2\lambda\beta_n(1-\lambda_1\beta_n\,||\,A\,||^2)\int_{\Omega}||\,(1-P_Q)Ar_n(\omega)\,||^2\,\mathrm{d}\Psi(\omega)$$

$$-2\beta_n(1-\beta_n)\int_\Omega \left\| r_n(\omega) - P_C r_n(\omega) \right\|^2 \mathrm{d}\Psi(\omega)$$

$$\therefore \qquad E\left\| k_n(\omega) - w(\omega) \right\|^2 \le \int_\Omega \left\| r_n(\omega) - w(\omega) \right\|^2 \mathrm{d}\Psi(\omega) \tag{4.10}$$

and

$$\begin{aligned}
&E\| r_{n+1}(\omega) - w(\omega) \| = \int_\Omega \| r_{n+1}(\omega) - w(\omega) \| \,\mathrm{d}\Psi(\omega) \\
&= \int_\Omega \| (1-\alpha_n) k_n(\omega) + \alpha_n (P_C k_n(\omega) - \lambda_2 B^*(1-P_Q) B k_n(\omega)) - w(\omega) \|^2 \,\mathrm{d}\Psi(\omega) \\
&\quad \le \int_\Omega \| k_n(\omega) - w(\omega) \|^2 \,\mathrm{d}\Psi(\omega) \\
&\quad + \alpha_n^2 \int_\Omega (2 \| k_n(\omega) - P_C r_n(\omega) \|^2 + 2\lambda_2^2 \| B \|^2 \| (1-P_Q) B k_n(\omega) \|^2) \mathrm{d}\Psi(\omega) \\
&\quad -2\alpha_n \int_\Omega \langle k_n(\omega) - w(\omega), k_n(\omega) - 1 - P_C k_n(\omega) \rangle \mathrm{d}\Psi(\omega) \\
&\quad -2\lambda_2 \alpha_n \int_\Omega \langle B k_n(\omega) - B w(\omega), (1-P_Q) B k_n(\omega) \rangle \mathrm{d}\Psi(\omega) \\
&\quad \le \int_\Omega \| k_n(\omega) - w(\omega) \|^2 \,\mathrm{d}\Psi(\omega) \\
&\quad -2\alpha_n (1-\alpha_n) \int_\Omega \| k_n(\omega) - P_C r_n(\omega) \|^2 \,\mathrm{d}\Psi(\omega) \\
&\quad -2\lambda_2 \alpha_n (1 - \lambda_2 \alpha_n \| B \|^2) \int_\Omega \| (1-P_Q) B k_n(\omega) \|^2 \,\mathrm{d}\Psi(\omega)
\end{aligned} \tag{4.11}$$

This implies that

$$E\| r_{n+1}(\omega) - w(\omega) \| = \int_\Omega \| r_{n+1}(\omega) - w(\omega) \| \,\mathrm{d}\Psi(\omega) \le \int_\Omega \| k_n(\omega) - w(\omega) \| \,\mathrm{d}\Psi(\omega) \tag{4.12}$$

Then it follows from (4.10) and (4.12) that

$$\begin{aligned}
&E\| r_{n+1}(\omega) - w(\omega) \|^2 \le \int_\Omega \| k_n(\omega) - w(\omega) \|^2 \,\mathrm{dP}(\omega) \\
&\quad + \int_\Omega \| r_n(\omega) - w(\omega) \|^2 \,\mathrm{d}\Psi(\omega) \\
&\quad \le \int_\Omega \| r_n(\omega) - w(\omega) \|^2 \,\mathrm{d}\Psi(\omega) + \int_\Omega \| r_1(\omega) - w(\omega) \|^2 \,\mathrm{d}\Psi(\omega) \\
&\quad \le \max\left(\int_\Omega \| (r_n(\omega) - w(\omega)) \| \,\mathrm{d}\Psi(\omega), \int_\Omega \| (r_1(\omega) - w(\omega)) \| \,\mathrm{d}\Psi(\omega) \right)
\end{aligned} \tag{4.13}$$

Therefore both $\{k_n(\omega)\}$ and $\{r_n(\omega)\}$ are stochastically bounded and Fejer monotone.

From (4.11)

$$2\alpha_n(1-\alpha_n)\int_\Omega \| k_n(\omega)-P_C r_n(\omega)\|^2 \, \mathrm{d}\Psi(\omega)$$
$$+2\lambda_2\alpha_n(1-\lambda_2\alpha_n \|B\|^2)\int_\Omega \|(1-P_Q)Bk_n(\omega)\|^2 \, \mathrm{d}\Psi(\omega)$$
$$\le \int_\Omega \| k_n(\omega)-w(\omega)\|^2 \, \mathrm{d}\Psi(\omega) - \int_\Omega \| r_{n+1}(\omega)-w(\omega)\|^2 \, \mathrm{d}\Psi(\omega)$$
$$\le \int_\Omega \| r_n(\omega)-w(\omega)\|^2 \, \mathrm{d}\Psi(\omega) - \int_\Omega \| r_{n+1}(\omega)-w(\omega)\|^2 \, \mathrm{d}\Psi(\omega)$$
$$\le \left[\int_\Omega \|(r_n(\omega)-w(\omega))\| \, \mathrm{d}\Psi(\omega) - \int_\Omega \|(r_{n+1}(\omega)-w(\omega))\| \, \mathrm{d}\Psi(\omega)\right]$$
$$\left[\int_\Omega \|(r_n(\omega)-w(\omega))\| \, \mathrm{d}\Psi(\omega) + \int_\Omega \|(r_{n+1}(\omega)-w(\omega))\| \, \mathrm{d}\Psi(\omega)\right] \tag{4.14}$$

Therefore, from (4.14)

$\lim_{n\to\infty}\int_\Omega \| r_n(\omega)-z(\omega)\| \, \mathrm{d}\Psi(\omega)=0$, $\lim_{n\to\infty}\int_\Omega \| k_n(\omega)-P_C k_n(\omega)\|^2 \, \mathrm{d}\Psi(\omega)=0$ and

$\lim_{n\to\infty}\int_\Omega \|(1-P_C)By_n(\omega)\|^2 \, \mathrm{dP}(\omega)=0$

$\therefore \lim_{n\to\infty}\int_\Omega \| r_{n+1}(\omega)-P_C k_n(\omega)\|^2 \, \mathrm{d}\Psi(\omega) \le \lim_{n\to\infty}\int_\Omega \| k_n(\omega)-P_C k_n(\omega)\|^2 \, \mathrm{d}\Psi(\omega)=0$

This completes the proof.

Illustrative Example

Suppose that

$H_1 = H_2 = \mathbf{R}^2$, $C=\{r\in\mathbf{R}^2 : r^2+5r-50-e_{1C}\le 0, 2r^2+88r+80-e_{2C}\le 0\}$

and $Q=\{r\in\mathbf{R}^2 : 2r^2-7r-15-e_{1Q}\le 0, 4r^2+16r+15-e_{2Q}\le 0\}$

where, e_{iC}~$\mathrm{N}(0,\sigma^2_{e_C})$ and e_{iQ}~$\mathrm{N}(0,\sigma^2_{e_Q})$

The stochastic split convex feasibility problem is to locate

$$r\in C = r^2+5r-50-e_{1C}\cap 2r^2+88r+80-e_{2C}$$

such that $Ar\in Q = 2r^2-7r-15-e_{1Q}\cap 4r^2+16r+15-e_{2Q}$

Let chose $A=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

Using algorithm (4.28), we have

$$r_{n+1}=(1-\beta_n)r_n+\beta_n(P_C r_n-\lambda A^*(1-P_Q)Ar_n)$$

where,

$$P_C r_n = r_n - \alpha_n z_{n+1} = r_n - \alpha_n \sum_{i=1}^{2}(m_i C_i(r_n) + e_{in+1}) = r_n - \frac{1}{2}\alpha_n(3r_n^2 + 93r_n + 30 - 2e_{Cn+1})$$

and

$$P_Q r_n = r_n - \alpha_n k_{n+1} = r_n - \alpha_n \sum_{i=1}^{2}(m_i Q_i(r_n) + e_{in+1}) = r_n - \frac{1}{2}\alpha_n(6r_n^2 + 9r_n - 2e_{Qn+1})$$

This gives

$$r_{n+1} = (1-\beta_n)r_n + \beta_n(r_n - \frac{1}{2}\alpha_n(3r_n^2 + 93r_n + 30 - 2e_{Cn+1}) - 4\beta_n\lambda r_n$$
$$+ 4\beta_n\lambda(r_n - \frac{1}{2}\alpha_n(6r_n^2 + 9r_n - 2e_{Qn+1})$$

$$r_{n+1} = r_n - \frac{1}{2}\beta_n\alpha_n(3r_n^2 + 93r_n + 30 - 2e_{Cn+1}) - 2\beta_n\lambda\alpha_n(6r_n^2 + 9r_n - 2e_{Qn+1})$$

Table 1

Results of illustrative example when algorithm in (4.1) is adopted.

Iterations	$\delta = 10^{-3}$	$\delta = 10^{-5}$	$\delta = 10^{-7}$	$\delta = 10^{-9}$	$\delta = 10^{-11}$
1	-1.3852	-1.385242	-1.38524233	-1.3852423321	-1.385242332139
2	-0.2945	-0.294506	-0.29450617	-0.2945061705	-0.294506170537
3	-0.1418	-0.141832	-0.14183226	-0.1418322647	-0.141832264663
4	-0.1771	-0.177149	-0.17714872	-0.1771487204	-0.177148720368
5	-0.1705	-0.170532	-0.17053216	-0.1705321570	-0.170532157021
6	-0.1712	-0.171223	-0.17122294	-0.1712229385	-0.171222938452
7		-0.171195	-0.17119452	-0.1711945229	-0.171194522912
8		-0.171194	-0.17119404	-0.1711940431	-0.171194043121
9			-0.17119401	-0.1711940102	-0.171194010198
10				-0.1711940064	-0.171194006406
11				-0.1711940058	-0.171194005809
12				-0.1711940057	-0.171194005692
13					-0.171194005665
14					-0.171194005658
15					-0.171194005656

e follows standard normal distribution, δ is convergence criteria error bound, $\alpha_n = 1/n+5$ represents the step size.

Using algorithm in (4.5)

$$r_{n+1} = r_n - \frac{1}{2}\rho\beta_n\alpha_n(3r_n^2 + 93r_n + 30 - 2e_{Cn+1}) - 2\rho\beta_n\lambda\alpha_n(6r_n^2 + 9r_n - 2e_{Qn+1})$$

Table 2

Results of illustrative example when algorithm in (4.5) is adopted.

Iterations	$\delta = 10^{-3}$	$\delta = 10^{-5}$	$\delta = 10^{-7}$	$\delta = 10^{-9}$	$\delta = 10^{-11}$
1	-3.0549	-3.054912	-3.05491196	-3.0549119646	-3.054911964637
2	-1.9739	-1.973866	-1.97386639	-1.9738663889	-1.973866388926
3	-0.1084	-0.108441	-0.10844105	-0.1084410472	-0.108441047158
4	-0.2380	-0.237996	-0.23799603	-0.2379960295	-0.237996029544
5	-0.1141	-0.114055	-0.11405482	-0.1140548176	-0.114054817633
6	-0.2171	-0.217108	-0.21710833	-0.2171083333	-0.217108333309
7	-0.1415	-0.141476	-0.14147589	-0.1414758919	-0.141475891866
8	-0.1887	-0.188744	-0.18874373	-0.1887437344	-0.188743734417
9	-0.1627	-0.162661	-0.16266095	-0.1626609487	-0.162660948736
10	-0.1748	-0.174801	-0.17480103	-0.1748010340	-0.174801033958
11	-0.1699	-0.169931	-0.16993091	-0.1699309113	-0.169930911334
12	-0.1716	-0.171561	-0.17156111	-0.1715611086	-0.171561108598
13	-0.1711	-0.171108	-0.17110813	-0.1711081270	-0.171108127046
14		-0.171210	-0.17120969	-0.1712096938	-0.171209693819
15		-0.171192	-0.17119188	-0.1711918825	-0.171191882457
16		-0.171194	-0.17119420	-0.1711942003	-0.171194200302
17			-0.17119400	-0.1711939957	-0.171193995681
18				-0.1711940058	-0.171194005792
19				-0.1711940057	-0.171194005658
20					-0.171194005655

$e \sim N(0,1)$, $\delta =$ convergence criteria error bound, $\lambda = 1.5$,
$\alpha_n = 1/n + 5 =$ step size

The results, as shown in tables 1 and 2, indicate that the multiple-sets split convex feasibility problem has a solution of -0.17. The algorithm shown in (4.1) and (4.5) was used to approximate this value, and both examples make it evident that higher error bound leads to faster convergence.

5. Conclusion

Through the use of random-type iterative algorithms, this study has extended the approximation (convergence) results for deterministic (classical) multiple-sets split feasibility and split equality problems to their stochastic equivalence. There was a convergence in both probability and mean square (quadratic mean) to a random fixed-point in Hilbert space. Numerical examples are used to illustrate the applicability of the stochastic multiple-sets split feasibility problems and theorems stated and proved in the study.

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