

Convergence regarding EM schemes over neutral type SFDEs at space C_u with undetermined delay

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Abstract

This work investigates the convergence of the EM approximation for neutral-type stochastic FDEs with infinite delay in the space C_u . To ensure numerical accuracy and stability for systems with memory effects, we extend the analysis to demonstrate the squared mean convergence of the EM scheme to the exact solution under Lipschitz conditions. Asker's work on distribution stability is expanded upon here. A numerical illustration of the impact of time step size on accuracy empirically supported the theoretical conclusions.

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1. Introduction

Understanding the convergence of numerical solutions of SDEs ensures their credibility and accuracy in real-world applications such as engineering and economic mathematics. Changed system revised by random behavior, where analytical solutions are usually impractical, and often use SDE [3, 7, 12]. Searching for the convergence characteristics of these solutions is essential for theoretical understanding and application in the real world of the EM method because it provides effective convergence techniques. This is especially important for neutral types of SFDEs, which have unlimited delays. These

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control systems, nervous networks, and population dynamics are required to simulate the long-term memory effect system [4, 5, 11]. Recent developments have mainly focused on the stability and convergence of calculation techniques for SFDEs, for example, Bao et al. [8]. While Kumar and Sabnis [6] examined the significant convergence of SDE with a delay in local Lipschitz conditions, Mao [9] demonstrated a convergence rate for truncated EM approach under Hölder circumstances. Yanting [10] improved this work by suggesting a convergence rate for neutral stochastic delays using the EM schemes.

In this paper, we construct at Asker's [1, 2], which works on stability to check the average convergence of the EM scheme for neutral SFDE with long-term delays in the space $\mathbf{C}_u$. This work will divide as the following sections: Section 2 introduces preliminaries. Section 3 proves the convergence of the EM schemes for NSFDEs with long-term delay, which is the main result. Finally, section 4 involves an example of detailed solutions to clarify the convergence of the EM schemes for NSFDEwID.

2. Preliminary

This section provides some notations and conditions used in the following sections. The space $\mathbf{C}_u$ with a specific positive integer u is:

$$\mathbf{C}_u = \left\{ Y \in \mathbf{C}((0, +\infty]; R^m) : \|Y\|_u = \sup_{-\infty \leq s \leq 0} e^{us} |Y| < \infty \right\},$$

$\mathbf{C}((0, +\infty]; R^m)$ the class for every continuous R^m -value function Y on $(-\infty, 0]$ to R^m with norm $\|Y\|_u$ and $\|Y\|_u = \sup_{-\infty \leq s \leq 0} e^{us} |Y| < \infty$ see [2]. M_0 refers to the family of probability measures on $\Omega = (-\infty, 0]$, to be precise, for any $\theta \in M_0$, $\int_{-\infty}^0 \theta(d\varphi) = 1$. Let us further define M_u for any $u > 0$, as shown in [1]. $M_u = \{\theta \in M_0; \theta^{(u)} := \int_{-\infty}^0 e^{-u\varphi} \theta(d\varphi) < \infty\}$

Examine the SDE with neutral terms and underlying delay in m-dimensions.

$$d\{X(\mathfrak{b}) - H(X_{\mathfrak{b}})\} = b(X_{\mathfrak{b}})d\mathfrak{b} + a(X_{\mathfrak{b}})d\mathcal{W}(\mathfrak{b}), \text{ when } \mathfrak{b} \geq 0, \quad (1)$$

with $\xi = X_0 = \{\xi(0); -\infty < \varphi \leq 0\} \in \mathbf{C}_u$ where $X_{\mathfrak{b}}(\varphi) = X(\mathfrak{b} + \varphi); -\infty < \varphi \leq 0, X(\mathfrak{b}) = \xi(\mathfrak{b})$ where $\mathfrak{b} < 0$ & $b, H: \mathbf{C}_u \rightarrow R^m; a: \mathbf{C}_u \rightarrow R^{m \times n}$ a Borel measurable, $\mathcal{W}(\mathfrak{b})$ is n-dimensional Wiener process. It is important to note $X(\mathfrak{b}) \in R^m$ is point, as though $X_{\mathfrak{b}} \in \mathbf{C}_u$ functions continuously on $(-\infty, \mathfrak{b}]$ with value in R^m . Next, we introduce a definition for a solution to equations (1) in the following definition.

Definition 2.1: [2] "if $(\Omega, \mathbb{F}, \mathbb{P})$ is complete probability spaces. Set $\mathbb{F}_s = \mathbb{F}_0$ for $-\infty < s \leq 0$ and for a given stopping time η_e let $X(s), -\infty < s \leq \eta_e$ be continuous R^m -value and $\mathbb{F}_s$ -adapted process".

By employing reasoning akin to (Theorem (3.11) in [2]), under Lipschitz conditions, one can show that the unique solution $X(\mathfrak{b})$ to the Equation (1) is exact. It also has a unique numerical solution in lemma (3.6) in [1] by using Euler-Maruyama (EM), where the EM scheme that is related to (2.1) is as

follows [1]. Maintaining generality without losing it, assume that for some integer $n > T$ ($T > 0$), a step $\lambda \in (0, 1)$ be fraction of T (i.e., $\lambda = \frac{T}{r}$). With initial data $\xi(J\lambda), -\infty < J \leq u$, and the solution of the discrete EM scheme $M(J\lambda)$ for Equation (2.1) is described as follows:

$$\begin{cases} M(J\lambda) = \xi(J\lambda), -\infty < J \leq 0, \\ M((J+1)\lambda) = M(J\lambda) + H_u(M_{J\lambda}) - H_u(M_{(J-1)\lambda}) \\ \quad + b(M_{J\lambda}) + a(M_{J\lambda})\Delta W_J, 0 \leq J \leq r-1. \end{cases} \quad (2)$$

where $\Delta M_J = M((J+1)\lambda) - M(J\lambda)$ & $M_{J\lambda} = M_{J\lambda}(\varphi) : -\infty < \varphi \leq 0$ be $C((-\infty, 0); R^m)$ random variables. We have define the EM approximation for a continuous type of $X(\mathfrak{b})$ in following: consider $X(\mathfrak{b}) = \xi(\mathfrak{b}), -\infty < \mathfrak{b} \leq 0$, and $\mathfrak{b} \in [J\lambda, (J+1)\lambda), J = 0, 1, 2, \dots, u-1$, thus

$$\begin{aligned} M(\mathfrak{b}) &= \bar{\xi} + H_u \left(M_{(J-1)\lambda} + \frac{\mathfrak{b}-J\lambda}{\lambda} (M_{J\lambda} - M_{(J-1)\lambda}) \right) - H_u(M_{(J-1)\lambda}) + \\ &\quad \int_0^{\mathfrak{b}} b(M_s)ds + \int_0^{\mathfrak{b}} a(M_s)dW_s \end{aligned} \quad (3)$$

To tackle the Numerical solution convergence of NSFDEwID to their exact solution using mean square convergence under Lipschitz conditions, we use the hypotheses in [1] and [2].

3. Main Result

This section is dedicated to studying the numerical solution's convergence of NSFDEwID to its exact solution using the mean square convergence method.

Theorem 1: *The approximated solution $M(\mathfrak{b})$ of (2) converges in mean-square to $X(\mathfrak{b})$. i.e.*

$$E \left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} |X(s) - M(s)|^2 \right) \leq CE \|\xi - \bar{\xi}\|_u^2.$$

Proof:

$$\begin{aligned} &|X(s) - M(s) - (H_u(X_s) - H_u(M_s))|^2 \leq |\xi - \bar{\xi} - (H_u(\xi) - H_u(\bar{\xi}))|^2 \\ &+ \int_0^s [|X(v) - M(v) - (H_u(X_v) - H_u(M_v))|^2 \\ &+ 2[X(v) - M(v) - (H_u(X_v) - H_u(M_v))]^T b(X_v) - b(M_v) \\ &+ |a(X_v) - a(M_v)|^2] dv \\ &+ 2 \int_0^s \{[X(v) - M(v) - (H_u(X_v) - H_u(M_v))]^T a(X_v) - a(M_v)\} dW_v, \end{aligned}$$

by taking the supremum and expected value and using Lemma (3.7) in reference [2], we can derive the following:

$$\begin{aligned} &E \left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} e^{\beta u} |X(s) - M(s) - (H_u(X_s) - H_u(M_s))|^2 \right) \leq ME \|\xi - \bar{\xi}\|_u^2 \\ &+ E \left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} \int_0^s e^{\beta u} [\beta |X(v) - M(v) - (H_u(X_v) - H_u(M_v))|^2 \right. \\ &+ 2[X(v) - M(v) - (H_u(X_v) - H_u(M_v))]^T b(X_v) - b(M_v) \\ &+ |a(X_v) - a(M_v)|^2] dv \end{aligned}$$

$$+2E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} \int_0^s e^{\beta u} [X(v) - M(v) - (H(X_v) - H(M_v))]^T a(X_v) - a(M_v) d\mathcal{W}v\right), \quad (4)$$

Where M equal $\mathfrak{b}_0(1+k)(1+\theta^{(2r)})$. By using Lemma (3.4) in [2] with $\varepsilon=k$, we get $|X(v) - M(v) - (H(X_v) - H(M_v))|^2 \leq (1+k)|X(v) - M(v)|^2$

$$+(1+k) \int_{-\infty}^0 |X(v+\varphi) - M(v+\varphi)|^2 \theta(d\varphi), \quad (5)$$

With BDG inequality, condition (2.6), and Lemma (3.4) from [2], we now have

$$\begin{aligned} & 2E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} \int_0^s e^{\lambda u} [X(v) - M(v) - (H(X_v) - H(M_v))]^T a(X_v) - a(M_v) d\mathcal{W}v\right) \\ & \leq \frac{1}{2}E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} e^{\beta u} |X(v) - M(v) - (H(X_v) - H(M_v))|^2\right) \\ & + 72\beta_3 E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} \int_0^s e^{\beta u} |X(v) - M(v)|^2 dv\right) + 72\beta_4 E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} \int_0^s \int_{-\infty}^0 e^{\beta u} |X(v+\varphi) \right. \\ & \left. - M(v+\varphi)|^2 \theta(d\varphi) dv\right) \end{aligned} \quad (6)$$

By using inequalities (4) and (5) in (6) yields

$$\begin{aligned} & E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} e^{\beta u} \left|X(s) - M(s) - (H(X_s) - H(M_s))\right|^2\right) \leq ME \|\xi - \bar{\xi}\|_u^2 \\ & + E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} \int_0^s e^{\beta u} [\beta(1+k)|X(v) - M(v)|^2 \right. \\ & - 2\beta_1 |X(v) - M(v)|^2 + 2\beta_2 \int_{-\infty}^0 |X(v+\varphi) - M(v+\varphi)|^2 \theta(d\varphi) \\ & + \beta_3 |X(v) - M(v)|^2 + \lambda_4 \int_{-\infty}^0 |X(v+\varphi) - M(v+\varphi)|^2 \theta(d\varphi)] dv \\ & + \frac{1}{2}E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} e^{\beta u} \left|X(s) - M(s) - (H(X_s) - H(M_s))\right|^2\right) \\ & + 72\beta_3 E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} \int_0^s e^{\beta u} |X(v) - M(v)|^2 dv\right) \\ & + 72\beta_4 E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} \int_0^s \int_{-\infty}^0 e^{\beta u} |X(v+\varphi) - M(v+\varphi)|^2 \theta(d\varphi) dv\right) \\ & + \beta(1+k) \int_{-\infty}^0 |X(v+\varphi) - M(v+\varphi)|^2 \theta(d\varphi) \end{aligned}$$

then

$$\begin{aligned} & E\left(\sup_{\mathfrak{b}_0 \leq s \leq \mathfrak{b}} e^{\beta u} |X(s) - M(s) - (H(X_s) - H(M_s))|^2\right) \leq ME \|\xi - \bar{\xi}\|_u^2 \\ & + E\left(\int_0^{\mathfrak{b}} e^{\beta u} [\beta(1+k)|X(s) - M(s)|^2 + \beta(1+k) \int_{-\infty}^0 |X(s+\varphi) \right. \\ & - M(s+\varphi)|^2 \theta(d\varphi) - 2\beta_1 |X(s) - M(s)|^2 + 2\beta_2 \int_{-\infty}^0 |X(s+\varphi) \\ & - M(s+\varphi)|^2 \theta(d\varphi) + \beta_3 |X(s) - M(s)|^2 + \beta_4 \int_{-\infty}^0 |X(s+\varphi) \\ & \left. - M(s+\varphi)|^2 \theta(d\varphi)] ds\right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} E \left(\sup_{\mathfrak{h}_0 \leq s \leq \mathfrak{h}} e^{\beta u} \left| X(s) - M(s) - (\mathbf{H}_u(X_s) - \mathbf{H}_u(M_s)) \right|^2 \right) \\
& + 72\beta_3 E \left(\int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds \right) + 72\beta_4 E \left(\int_0^{\mathfrak{h}} \int_{-\infty}^0 e^{\beta u} |X(s + \varphi) \right. \\
& \left. - M(s + \varphi)|^2 \theta(d\varphi) ds \right). \text{ So, we have} \\
& E \left(\sup_{\mathfrak{h}_0 \leq s \leq \mathfrak{h}} e^{\beta u} |X(s) - M(s) - (\mathbf{H}_u(X_s) - \mathbf{H}_u(M_s))|^2 \right) \leq ME \parallel \xi - \bar{\xi} \parallel_u^2 \\
& + \beta(1+k) E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds + \beta(1+k) E \int_0^{\mathfrak{h}} \int_{-\infty}^0 e^{\beta u} |X(s + \varphi) \\
& - M(s + \varphi)|^2 \theta(d\varphi) ds - 2\beta_1 E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds \\
& + 2\beta_2 E \int_0^{\mathfrak{h}} \int_{-\infty}^0 e^{\beta u} |X(s + \varphi) - M(s + \varphi)|^2 \theta(d\varphi) ds \\
& + \beta_3 E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds + \beta_4 E \int_0^{\mathfrak{h}} \int_{-\infty}^0 e^{\beta u} |X(s + \varphi) \\
& - M(s + \varphi)|^2 \theta(d\varphi) ds + \frac{1}{2} E \left(\sup_{\mathfrak{h}_0 \leq s \leq \mathfrak{h}} e^{\beta u} |X(s) - M(s) \right. \\
& \left. - (\mathbf{H}_u(X_s) - \mathbf{H}_u(M_s)) \right|^2) + 72\beta_3 E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds \\
& + 72\beta_4 E \left(\int_0^{\mathfrak{h}} \int_{-\infty}^0 e^{\beta u} |M(s + \varphi) - X(s + \varphi)|^2 \theta(d\varphi) ds \right), \tag{7}
\end{aligned}$$

by (3.22) in [2], we obtain

$$\begin{aligned}
& \int_0^{\mathfrak{h}} \int_{-\infty}^0 e^{\beta u} |X(s + \varphi) - M(s + \varphi)|^2 \theta(d\varphi) ds \leq \frac{1}{2r-\beta} \parallel \xi - \bar{\xi} \parallel_u^2 \theta^{(2r)} \\
& + \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds. \tag{8}
\end{aligned}$$

Substitution (7) in the inequality (8), we get

$$\begin{aligned}
& E \left(\sup_{\mathfrak{h}_0 \leq s \leq \mathfrak{h}} e^{\beta u} |X(s) - M(s) - (\mathbf{H}_u(X_s) - \mathbf{H}_u(M_s))|^2 \right) \leq ME \parallel \xi - \bar{\xi} \parallel_u^2 \\
& + \beta(1+k) E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds + \frac{\theta^{(2r)}}{2r-\beta} (1+k) \lambda E \parallel \xi - \bar{\xi} \parallel_u^2 \\
& + \beta(1+k) E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds - 2\beta_1 E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds \\
& + 2\beta_2 \frac{\theta^{(2r)}}{2r-\beta} E \parallel \xi - \bar{\xi} \parallel_u^2 + 2\beta_2 E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds \\
& + \beta_3 E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds + \beta_4 \frac{\theta^{(2r)}}{2r-\beta} E \parallel \xi - \bar{\xi} \parallel_u^2 + \beta_4 E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) \\
& - M(s)|^2 ds + \frac{1}{2} E \left(\sup_{\mathfrak{h}_0 \leq s \leq \mathfrak{h}} e^{\beta u} |X(s) - M(s) - (\mathbf{H}_u(X_s) - \mathbf{H}_u(M_s))|^2 \right) \\
& + 72\beta_3 E \left(\int_0^{\mathfrak{h}} v |X(s) - M(s)|^2 ds \right) + 72\beta_4 \frac{\theta^{(2r)}}{2r-\beta} E \parallel \xi - \bar{\xi} \parallel_u^2 \\
& + 72\beta_4 E \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds
\end{aligned}$$

then

$$E \left(\sup_{\mathfrak{h}_0 \leq s \leq \mathfrak{h}} |X(s) - M(s)|^2 \right) \leq (k_3 + 2k_4 [M + \frac{\theta^{(2r)}}{2r-\beta} ((1+k)\beta$$

$$+2\beta_2 + 73\beta_4))E \|\xi - \bar{\xi}\|_u^2 - 2k_4[(2\beta_1 - 2(1+k)\beta - 2\beta_2 - 73\beta_3 - 73\beta_4] \int_0^{\mathfrak{h}} e^{\beta u} |X(s) - M(s)|^2 ds$$

Not that $2\beta_1 > 2\beta_2 + 73\beta_3 - +73\beta_4$ and $\beta \in (0, \frac{1}{M}[2\beta_1 - 2\beta_2 - 73\beta_3 - 73\beta_4])$, and $0 < k < 1$. So, we reach

$$E(\sup_{\mathfrak{h}_0 \leq s \leq \mathfrak{h}} |X(s) - M(s)|^2) \leq CE \|\xi - \bar{\xi}\|_u^2$$

where, $C = \left(k_3 + 2k_4 \left[M + \frac{\theta^{(2r)}}{2r-\beta} ((1+k)\beta + 2\beta_2 + 73\beta_4) \right] \right) []$

4. Numerical Example, Solutions, and Convergence Analysis

We validate Section 3's theoretical results with a numerical example of an NSFDEwID, solved using the Euler-Maruyama scheme. The solution is visualized, convergence is analyzed, and findings are presented with figures and a discussion of convergence behavior.

4.1 Example Description

Consider the NSFDEwID model given in the model (1), represented by the neutral term, drift term, diffusion term, and initial condition as follows, respectively: $H(X_{\mathfrak{h}}) = 0.5 \int_{-\infty}^0 X(\mathfrak{h} + \Phi) e^{\varphi} (d\varphi)$, $b(X_{\mathfrak{h}}) = -X(\mathfrak{h})$, $a(X_{\mathfrak{h}}) = 1$, $X(\mathfrak{h}) = 1$ for $\mathfrak{h} \leq 0$. The delay measure is chosen as $\theta(d\varphi) = e^{\varphi} (d\varphi)$, which satisfies $\theta^{(2u)} < \infty$ for $u > 0$.

4.2 Numerical Algorithm

The Euler-Maruyama (EM) scheme computes the numerical solution as follows: Discretize $[0, T]$ into N steps with the time step $\Delta \mathfrak{h} = T/N$. Initialize $X(\mathfrak{h}) = 1$ for $\mathfrak{h} \leq 0$. Approximate the neutral term $H(X_{\mathfrak{h}})$ via numerical integration over $[\Phi_{min}, 0]$, using the trapezoidal rule. Update X at each step: $X_{j+1} = X_j + H(X_{\mathfrak{h}_j}) - H(X_{\mathfrak{h}_{j-1}}) + b(X_{\mathfrak{h}_j})\Delta \mathfrak{h} + a(X_{\mathfrak{h}_j})\Delta W_j$, where ΔW_j is Wiener increment.

Compute a reference solution with $(\Delta \mathfrak{h}_{ref} = .001)$, calculate the mean square error (MSE) for varying $\Delta \mathfrak{h}$, and plot MSE vs. $\Delta \mathfrak{h}$ on a log-log scale to analyze convergence.

4.3 Numerical Results

Numerical solutions are computed in steps $\Delta \mathfrak{h} = 0.1, 0.01, 0.001$, with $\Delta \mathfrak{h} = 0.001$ as the reference solution approximating the exact solution (Figure 1).

4.4 Discussion of Results

Based on Figure 1, we will discuss the numerical solution behavior as follows: the reference solution (smallest $\Delta \mathfrak{h}$) exhibits smooth stochastic fluctuations around the equilibrium point $X(\mathfrak{h}) = 1$, consistent with the

stabilizing drift term $b(X_{\mathfrak{h}}) = -X_{\mathfrak{h}}$. Numerical solutions with larger time steps (e.g., $\Delta \mathfrak{h} = 0.1$) show more pronounced deviations from the reference solution due to discretization errors.

4.5 Convergence Analysis

The MSE decreases linearly with $\Delta \mathfrak{h}$ on a log-log scale, confirming the EM scheme's mean square convergence under Lipschitz conditions (Theorem 1). Smaller $\Delta \mathfrak{h}$ yields smoother trajectories and reduced error, with the slope verifying the $O(\Delta \mathfrak{h})$ convergence rate.

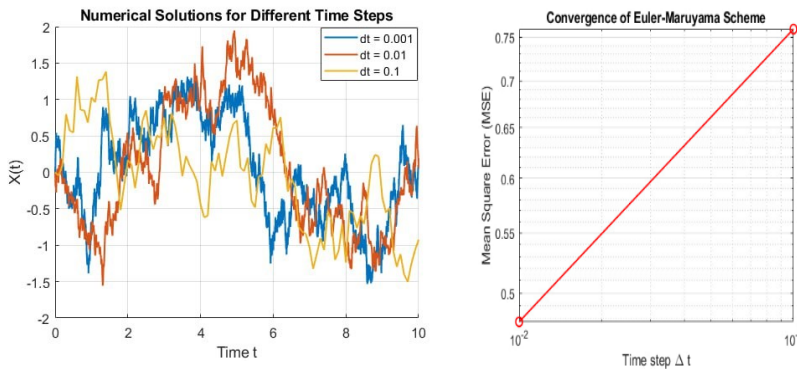


Figure 1
The numerical solutions of NSFDEwID

5. Conclusion

In this study, we have rigorously established the mean square convergence of the Euler-Maruyama (EM) scheme for NSFDEwID in the state space C_u . By proving that the numerical solution converges to the exact solution under Lipschitz conditions, we have validated the reliability and robustness of the EM scheme for solving NSFDEwID. This result extends the foundational work of Asker [2, 1], bridging the gap between theoretical stability in distribution and practical numerical convergence for such equations. Including a numerical example further reinforces our theoretical findings, demonstrating that reducing the time step $\Delta \mathfrak{h}$ systematically decreases the mean square error (MSE), consistent with the derived convergence rates.

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