Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 1, pp. 209–224
DOI : 10.47974/JIM-2441

Construction of tubular surface by infinitesimal bending of a curve

Suyash Parkash Chauhan †
Pradeep Kumar Pandey *
Department of Mathematics
Jaypee University of Information Technology
Solan 173234
Himachal Pradesh
India

Sudhakar Kumar Chaubey §
Department of Information Technology
Section of Mathematics
University of Technology and Applied Sciences
Sinhas 324
Oman

Abstract

This paper investigates the tubular surface generated by the first-order infinitesimal bending of a regular curve in Euclidean space $\mathbb{R}^3$. We also study the focal surfaces derived from the tubular surface. First, the invariants of the mean and Gaussian curvatures are computed. Additionally, conditions are examined for parametric curves to be geodesic or asymptotic lines on these surfaces. Finally, several examples are provided with visual representations.

Subject Classification: *53A04, 53A05.*

Keywords: *Infinitesimal bending, Tubular surface, Gaussian and mean curvatures, Focal surfaces.*

† *ORCID-ID:* `0009-0004-7510-8590`
* *ORCID-ID:* `0000-0003-0872-8904`
§ *ORCID-ID:* `0000-0002-3882-4596`

† *E-mail:* `suyash968@gmail.com`
* *E-mail:* `pandeypkdelhi@gmail.com (Corresponding Author)`
§ *E-mail:* `sudhakar.chaubey@utas.edu.om`

1. Introduction

A key aspect of bending theory is the analysis of infinitesimal bending of curves and surfaces [18]. It has various applications in fields like Physics, computer graphics, and biological sciences, etc. [6, 16, 30]. Infinitesimal bending of a curve is an isometric deformation to some extent, as the arc length remains unchanged within a given precision [11]. It is an active field of research with foundational contributions in [2, 5] and further advancements in [8, 10, 11, 16, 30].

The envelope formed by spheres of constant radius is defined as a tubular surface. The concept of tubular surface was introduced by Monge in 1850 [15]. They have vast applications, such as the designing of blend surfaces, robotic path planning, reconstruction of shapes, and many more [21, 22, 23, 27]. In [3], spherical indicatrices have been used to generate tubular surfaces in $\mathbb{R}^3$. Kim et al. investigated the mean and the Gaussian curvatures of tubes [13], while the Weingarten tube like surfaces are studied in [29]. Talat and Essin described a new method of creating tubes in the Lorentzian Heisenberg group [14].

Hagen et al. introduced the notion of line congruences in 1991 [24]. Line congruences are the surfaces obtained by transforming one surface into another by lines [28]. A focal surface is one of the notable examples of line congruences. Focal surfaces offer a method to examine the surface's nature prior to its processing [9]. They can also be used to visualise the distribution of heat and pressure on an aeroplane; ozone, temperature and rainfall on Earth's surface [9]. The generalised focal surface in $\mathbb{R}^3$ has been investigated in [9]. Additionally, the surface of revolution and its focal surfaces in $\mathbb{G}_3$ are discussed in [12].

As per available data (Google Scholar), tubular surface and its focal surfaces have not been investigated under infinitesimal bending. To address this gap, we study the geometric properties of these surfaces in this context.

2. Preliminaries

This section begins with the Frenet formulae for a regular curve with arbitrary speed in $\mathbb{R}^3$. Then, we recall some basic notions and definitions important throughout this study.

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u): I \to \mathbb{R}^3$ $(I \subseteq \mathbb{R})$ be a regular space curve with arbitrary speed. If $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ represent the vectors of Frenet frame, with κ and τ as its curvature and torsion, respectively; then [19]

$$\mathbf{t} = \frac{\boldsymbol{\alpha}'}{\|\boldsymbol{\alpha}'\|}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \quad \mathbf{b} = \frac{\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}$$

and
$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3}, \qquad \tau = \frac{(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}') \cdot \boldsymbol{\alpha}'''}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|^2}.$$

Lemma 2.1: ([19], [20], [26]) *The derivative of the Frenet frame vectors of* $\boldsymbol{\alpha}$ *is given by*

$$\begin{bmatrix}\mathbf{t}'\\ \mathbf{n}'\\ \mathbf{b}'\end{bmatrix} = v\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{1}$$

where v *denotes the speed of* $\boldsymbol{\alpha}$.

Definition 2.2: ([17, 18]) Consider a non unit speed regular curve

$$\boldsymbol{\zeta}: \boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$$

belonging to the family of the curves

$$\boldsymbol{\zeta}_\varepsilon: \boldsymbol{\alpha}_\varepsilon = \boldsymbol{\alpha}(u) + \varepsilon\mathbf{h}(u), \qquad 0 \leq \varepsilon, \quad \varepsilon \to 0, \quad \varepsilon \in \mathbb{R} \tag{2}$$

such that $u \in \mathbb{R}$, and we obtain $\boldsymbol{\zeta}$ for $\varepsilon = 0$, i.e., $\boldsymbol{\zeta} = \boldsymbol{\zeta}_0$.

Let $\boldsymbol{\zeta}_\varepsilon$ is a family of curves, then it is called the infinitesimal bending of $\boldsymbol{\zeta}$ if

$$ds_\varepsilon^2 - ds^2 = o(\varepsilon),$$

where s denotes the arc length and $\mathbf{h}$ is the field of infinitesimal bending of $\boldsymbol{\alpha}$.

Theorem 2.3: ([17], [18]) For $\mathbf{h}$ to be the field of infinitesimal bending of $\boldsymbol{\zeta}$, it must satisfy the following necessary and sufficient condition

$$d\boldsymbol{\alpha} \cdot d\mathbf{h} = 0. \tag{3}$$

Theorem 2.4: ([17, 18]) *The field of infinitesimal bending of* $\boldsymbol{\zeta}$ *is*

$$\mathbf{h}(u) = \int [\, f(u)\,\mathbf{n} + g(u)\,\mathbf{b}\,]\, du \tag{4}$$

where f *and* g *are the integrable functions chosen arbitrarily.*

Definition 2.5: ([1], [4], [19]) If $\boldsymbol{\alpha}$ is a curve in $\mathbb{R}^3$, then the tube of radius R around the curve $\boldsymbol{\alpha}$ is a parameterised surface given as

$$\boldsymbol{\chi}(u, \phi) = \boldsymbol{\alpha}(u) + R(\cos\phi\,\mathbf{n} + \sin\phi\,\mathbf{b}), \qquad 0 \leq \phi \leq 2\pi \tag{5}$$

Definition 2.6: ([7], [9]) Let $\boldsymbol{\chi}(u, \phi)$ be parametric surface and $\mathbf{N}(u, \phi)$ be its unit normal vector field. Then, the focal surfaces are given by the following parametric equations as

$$\begin{aligned}\mathbf{F}_1(u, \phi) &= \boldsymbol{\chi}(u, \phi) + \frac{1}{\kappa_1(u,\phi)}\,\mathbf{N}\\ \mathbf{F}_2(u, \phi) &= \boldsymbol{\chi}(u, \phi) + \frac{1}{\kappa_2(u,\phi)}\,\mathbf{N}\end{aligned} \tag{6}$$

where κ_1, κ_2 are principal curvatures of $\boldsymbol{\chi}(u, \phi)$.

Definition 2.7: ([25, 29]) For a pair $(K, H), K \neq H$, of the Gaussian and the mean curvatures of $\boldsymbol{\chi}(u, \phi)$ is called a (K, H)- Weingarten surface if $\psi(K, H) = 0$, where ψ denotes the Jacobi function given by $K_u H_\phi - H_u K_\phi = 0$.

Definition 2.8: ([25, 29]) For a pair $(K, H), K \neq H$, of the Gaussian and mean curvatures of the surface $\boldsymbol{\chi}(u, \phi)$ is (K, H)- linear Weingarten surface if

$$aK + bH = c$$

for $a, b, c \in \mathbb{R}$ and at least one of them is non-zero.

3. Tubular Surface Generated by the Infinitesimal Bending of Curve and Their Focal Surfaces

Let $\boldsymbol{\alpha}$ be a space curve and $\boldsymbol{\alpha}_\varepsilon$ denotes the infinitesimal bending of $\boldsymbol{\alpha}$. The tubular surface obtained by $\boldsymbol{\alpha}_\varepsilon$ is given as

$$\boldsymbol{\chi}^\varepsilon(u,\phi) = \boldsymbol{\alpha}_\varepsilon(u) + R\,(\cos\phi\,\mathbf{n}_\varepsilon + \sin\phi\,\mathbf{b}_\varepsilon), \quad 0 \le \phi \le 2\pi \tag{7}$$

where $\mathbf{n}_\varepsilon$ and $\mathbf{b}_\varepsilon$ are the normal and the binormal vectors of $\boldsymbol{\alpha}_\varepsilon$.

Theorem 3.1: *For the tubular surface $\boldsymbol{\chi}^\varepsilon(u,\phi)$, the Gaussian curvature K_ε and the mean curvature H_ε are given by*

$$K_\varepsilon = -\frac{\kappa_\varepsilon\cos\phi}{R(1-R\,\kappa_\varepsilon\cos\phi)}$$

$$H_\varepsilon = \frac{1-2R\,\kappa_\varepsilon\cos\phi}{2R(1-R\,\kappa_\varepsilon\cos\phi)}.$$

Proof: Partial derivatives of $\boldsymbol{\chi}^\varepsilon(u,\phi)$ are given by

$$\boldsymbol{\chi}_u^\varepsilon = v_\varepsilon(1-R\,\kappa_\varepsilon\cos\phi)\mathbf{t}_\varepsilon + R\,v_\varepsilon\,\tau_\varepsilon(-\sin\phi\,\mathbf{n}_\varepsilon + \cos\phi\,\mathbf{b}_\varepsilon) \tag{8}$$

$$\boldsymbol{\chi}_\phi^\varepsilon = R(-\sin\phi\,\mathbf{n}_\varepsilon + \cos\phi\mathbf{b}_\varepsilon) \tag{9}$$

and

$$\begin{aligned}\boldsymbol{\chi}_{uu}^\varepsilon = \; & [R\,v_\varepsilon^2\,\kappa_\varepsilon\,\tau_\varepsilon\sin\phi + v'_\varepsilon\,(1-R\,\kappa_\varepsilon\cos\phi) - R\,v_\varepsilon\,\kappa'_\varepsilon\cos\phi + \;]\mathbf{t}_\varepsilon \\ & +[v_\varepsilon^2\,\kappa_\varepsilon(1-R\,\kappa_\varepsilon\cos\phi) - R\,(v_\varepsilon\,\tau_\varepsilon)'\sin\phi - R\,v_\varepsilon^2\,\tau_\varepsilon\cos\phi]\mathbf{n}_\varepsilon \\ & +[R\,(v_\varepsilon\,\tau_\varepsilon)'\cos\phi - R\,(v_\varepsilon\,\tau_\varepsilon)^2\sin\phi]\mathbf{b}_\varepsilon\end{aligned} \tag{10}$$

$$\boldsymbol{\chi}_{\phi\phi}^\varepsilon = -R(\cos\phi\,\mathbf{n}_\varepsilon + \sin\phi\mathbf{b}_\varepsilon) \tag{11}$$

$$\boldsymbol{\chi}_{u\phi}^\varepsilon = R\,v_\varepsilon\,\kappa_\varepsilon\sin\phi\,\mathbf{t}_\varepsilon - R\,v_\varepsilon(\tau_\varepsilon\cos\phi\,\mathbf{n}_\varepsilon + \tau_\varepsilon\sin\phi\,\mathbf{b}_\varepsilon) \tag{12}$$

Unit normal vector on $\boldsymbol{\chi}^\varepsilon(u,\phi)$ is given by

$$\mathbf{N}_\varepsilon = \frac{\boldsymbol{\chi}_u^\varepsilon\times\boldsymbol{\chi}_\phi^\varepsilon}{\left\|\boldsymbol{\chi}_u^\varepsilon\times\boldsymbol{\chi}_\phi^\varepsilon\right\|} = -(\cos\phi\,\mathbf{n}_\varepsilon + \sin\phi\,\mathbf{b}_\varepsilon) \tag{13}$$

The first fundamental form has following components

$$\begin{aligned}E_\varepsilon &= (v_\varepsilon(1-R\,\kappa_\varepsilon\cos\phi))^2 + (R\,v_\varepsilon\,\tau_\varepsilon)^2 \\ F_\varepsilon &= R^2\,v_\varepsilon\,\tau_\varepsilon \\ G_\varepsilon &= R^2\end{aligned} \tag{14}$$

Also, components of second fundamental form are given by

$$\begin{aligned}L_\varepsilon &= R\,(v_\varepsilon\,\tau_\varepsilon)^2 - v_\varepsilon^2\,\kappa_\varepsilon\,(1-R\,\kappa_\varepsilon\cos\phi)\cos\phi \\ M_\varepsilon &= R\,v_\varepsilon\,\tau_\varepsilon \\ N_\varepsilon &= R\end{aligned} \tag{15}$$

The expression of Gaussian curvature using (14) and (15) is given by

$$K_\varepsilon = \frac{L_\varepsilon\,N_\varepsilon - M_\varepsilon^2}{E_\varepsilon\,G_\varepsilon - F_\varepsilon^2} = -\frac{\kappa_\varepsilon\cos\phi}{R(1-R\,\kappa_\varepsilon\cos\phi)},$$

also the mean curvature is given as

$$H_\varepsilon = \frac{E_\varepsilon N_\varepsilon - 2F_\varepsilon M_\varepsilon + G_\varepsilon L_\varepsilon}{2(E_\varepsilon G_\varepsilon - F_\varepsilon^2)} = \frac{1-2R\,\kappa_\varepsilon \cos\phi}{2R(1-R\,\kappa_\varepsilon\cos\phi)}.$$

Theorem 3.2: $\boldsymbol{\chi}^\varepsilon(u,\phi)$ *is a regular surface iff*

$$1 - R\,\kappa_\varepsilon \cos\phi \neq 0.$$

Proof: A surface is regular if it satisfies $E_\varepsilon\, G_\varepsilon - F_\varepsilon^2 \neq 0$

Using (14), we have

$$E_\varepsilon\, G_\varepsilon - F_\varepsilon^2 = 1 - R\,\kappa_\varepsilon \cos\phi$$

Conversely, if $E_\varepsilon\, G_\varepsilon - F_\varepsilon^2 \neq 0$ holds, this implies the surface is regular.

Theorem 3.3: *Let* $\boldsymbol{\chi}^\varepsilon(u,\phi)$ *denotes a tubular surface, then*

1. s-parametric curves are asymptotic lines on $\boldsymbol{\chi}^\varepsilon(u,\phi)$ iff

$$R = \frac{\kappa_\varepsilon \cos\phi}{\tau_\varepsilon^2 + \kappa_\varepsilon^2 \cos^2\phi}.$$

2. ϕ-parametric curves can not be asymptotic lines on $\boldsymbol{\chi}^\varepsilon(u,\phi)$.

Proof:

1. s-parametric curves are asymptotic lines on a regular surface if it satisfies [1]

$$\boldsymbol{\chi}_{uu}^\varepsilon \cdot \mathbf{N}_\varepsilon = 0$$

equation (10) and (13) yields

$$\kappa_\varepsilon \cos\phi = R\,(\tau_\varepsilon^2 + \kappa_\varepsilon^2 \cos^2\phi)$$

$$R = \frac{\kappa_\varepsilon \cos\phi}{\tau_\varepsilon^2 + \kappa_\varepsilon^2 \cos^2\phi}$$

The converse is also true for the above equation.

2. ϕ-parametric curves are asymptotic lines on a regular surface if it satisfies [1]

$$\boldsymbol{\chi}_{\phi\phi}^\varepsilon \cdot \mathbf{N}_\varepsilon = 0$$

equation (11) and (13) yields

$$R = 0.$$

Theorem 3.4: *Let* $\boldsymbol{\chi}^\varepsilon(u,\phi)$ *denotes a tubular surface, then*

1. *s-parametric curves are geodesic lines on* $\boldsymbol{\chi}^\varepsilon(u,\phi)$ *iff*

$$\kappa_\varepsilon^2 v_\varepsilon^3\,(\cos\phi - v_\varepsilon\,\tau_\varepsilon \sin\phi)\sin\phi = (v_\varepsilon\,\tau_\varepsilon)'\, v'_\varepsilon.$$

2. ϕ-*parametric curves are always geodesic lines on* $\boldsymbol{\chi}^\varepsilon(u,\phi)$.

Proof:

1. s-parametric curves are geodesic lines on a regular surface if it satisfies [1]

$$\boldsymbol{\chi}_{uu}^\varepsilon \times \mathbf{N}_\varepsilon = \mathbf{0}$$

equation (10) and (13) yields

$$\begin{aligned} &\kappa_\varepsilon\, v_\varepsilon^2(1-\ R\,\kappa_\varepsilon\cos\phi)\sin\phi - R(v_\varepsilon\,\tau_\varepsilon)' = 0\\ &(v'_\varepsilon(1-R\,\kappa_\varepsilon\cos\phi) + R\,v_\varepsilon^2\,\kappa_\varepsilon\,\tau_\varepsilon\sin\phi - R\,v_\varepsilon\,\kappa_\varepsilon\cos\phi) = 0\\ &(v'_\varepsilon(1-R\,\kappa_\varepsilon\cos\phi) + R\,v_\varepsilon^2\,\kappa_\varepsilon\,\tau_\varepsilon\sin\phi - R\,v_\varepsilon\,\kappa_\varepsilon\cos\phi) = 0 \end{aligned} \tag{16}$$

Solving the last two equations for $(1 - R\,\kappa_\varepsilon\cos\phi)$ and substituting in the first equation of the equation (16)

this implies, $\kappa_\varepsilon^2 v_\varepsilon^3\ (\cos\phi - v_\varepsilon\,\tau_\varepsilon\sin\phi)\sin\phi = (v_\varepsilon\,\tau_\varepsilon)'\,v'_\varepsilon$

The converse is also true for the above equation.

2. ϕ-parametric curves are geodesic lines on a regular surface if it satisfies [1]

$$\boldsymbol{\chi}^\varepsilon_{\phi\phi} \times \mathbf{N}_\varepsilon = \mathbf{0}$$

equation (11) and (13) yields

$$\boldsymbol{\chi}^\varepsilon_{\phi\phi} \times \mathbf{N}_\varepsilon = \mathbf{0}$$

Therefore, each ϕ-parameter curve of regular tubular surface $\boldsymbol{\chi}^\varepsilon(u,\phi)$ is geodesic curve.

Theorem 3.5: *Let $\boldsymbol{\chi}^\varepsilon(u,\phi)$ denotes a tubular surface, then $\boldsymbol{\chi}^\varepsilon$ is a $(K_\varepsilon\,H_\varepsilon)$-Weingarten surface.*

Proof: The partial derivatives of K_ε and H_ε are given by

$$K_{\varepsilon_u} = \frac{\kappa'_\varepsilon\cos\phi}{R\,(1-R\,\kappa_\varepsilon\cos\phi)^2}$$

$$K_{\varepsilon_\phi} = \frac{\kappa_\varepsilon\ \sin\phi}{R\,(1-R\,\kappa_\varepsilon\cos\phi)^2}$$

and

$$H_{\varepsilon_u} = \left(\frac{\kappa'_\varepsilon\sin\phi}{2\,(1-R\,\kappa_\varepsilon\cos\phi)^2}\right)$$

$$H_{\varepsilon_\phi} = \frac{\kappa_\varepsilon\sin\phi}{2\,(1-R\,\kappa_\varepsilon\cos\phi)^2}$$

Partial derivatives of K_ε and H_ε satisfy the Jacobi function given in (Definition 2.7).

Theorem 3.6: *Consider $\boldsymbol{\chi}^\varepsilon(u,\phi)$ denotes a tubular surface, then $\boldsymbol{\chi}^\varepsilon$ is a $(K_\varepsilon\,H_\varepsilon)$- linear Weingarten surface for $a = -R^2$, $b = 2R$ and $c = 1$.*

Proof: Using (Definition 2.8), we get

$$a\,K_\varepsilon + b\,H_\varepsilon = c$$

$$c = 1,\quad \text{yields}\quad a = -R^2 \quad \text{and} \quad b = 2R.$$

Theorem 3.7: *The principal curvatures associated with $\boldsymbol{\chi}^\varepsilon(u,\phi)$ are given by*

$$\kappa_{\varepsilon 1} = -\frac{\kappa_\varepsilon\cos\phi}{(1-R\,\kappa_\varepsilon\cos\phi)} \qquad \kappa_{\varepsilon 2} = \frac{1}{R}.$$

Proof: Let $A = \begin{bmatrix} E_\varepsilon & F_\varepsilon \\ F_\varepsilon & G_\varepsilon \end{bmatrix}$ and Let $B = \begin{bmatrix} L_\varepsilon & M_\varepsilon \\ M_\varepsilon & N_\varepsilon \end{bmatrix}$, then shape operator at any point on surface is given by

$$S(p) = A^{-1}B$$

At any point on surface, the eigenvalues of $S(p)$ are the magnitudes of principal curvatures [19]. Therefore, we have

$$\kappa_{\varepsilon 1} = -\frac{\kappa_\varepsilon \cos\phi}{(1-R\,\kappa_\varepsilon \cos\phi)}, \qquad \kappa_{\varepsilon 2} = \frac{1}{R}.$$

Focal surfaces of tubular surface $\boldsymbol{\chi}^\varepsilon(u, \boldsymbol{\phi})$

Let $\boldsymbol{\chi}^\varepsilon(u,\phi)$ be a parametric surface and $\mathbf{N}_\varepsilon$ be its unit normal vector field. The parametric equations of focal surfaces are given by

$$\begin{aligned} \mathbf{F}_1^\varepsilon(u,\phi) &= \boldsymbol{\chi}^\varepsilon(u,\phi) + \frac{1}{\kappa_{\varepsilon 1}}\mathbf{N}_\varepsilon \\ \mathbf{F}_2^\varepsilon(u,\phi) &= \boldsymbol{\chi}^\varepsilon(u,\phi) + \frac{1}{\kappa_{\varepsilon 2}}\mathbf{N}_\varepsilon. \end{aligned} \tag{17}$$

We obtain $\mathbf{F}_2^\varepsilon(u,\phi) = \boldsymbol{\alpha}_\varepsilon(u)$, corresponding to the principal curvature $\kappa_{\varepsilon 2}$. The focal surface associated with $\kappa_{\varepsilon 1}$ can be computed using equation (7), (13), and (17).

$$\mathbf{F}_1^\varepsilon(u,\phi) = \boldsymbol{\alpha}_\varepsilon(u) + \frac{1}{\kappa_\varepsilon \cos\phi}(\cos\phi\, \mathbf{n}_\varepsilon + \sin\phi\, \mathbf{b}_\varepsilon). \tag{18}$$

We are now left with $\mathbf{F}_1^\varepsilon(u,\phi)$ only. For simplicity, we denote it as $\mathbf{F}^\varepsilon(u,\phi)$.

Theorem 3.8: *For the focal surface* $\mathbf{F}^\varepsilon(u,\phi)$, *Gaussian curvature* K^ε *and mean curvature* H^ε *are as follows*

$$K^\varepsilon = 0$$

$$H^\varepsilon = -\frac{v_\varepsilon \kappa_\varepsilon^3}{2\,(\kappa_\varepsilon' + v_\varepsilon\, \kappa_\varepsilon\, \tau_\varepsilon \tan\phi)}.$$

Proof: Partial derivatives of equation (18) are given by

$$\boldsymbol{F}_u^\varepsilon = \frac{1}{\kappa_\varepsilon}\left(-\frac{\kappa_\varepsilon'}{\kappa_\varepsilon} - v_\varepsilon\, \tau_\varepsilon \tan\phi\right)\boldsymbol{n}_\varepsilon + \frac{1}{\kappa_\varepsilon}\left(v_\varepsilon\, \tau_\varepsilon - \frac{\kappa_\varepsilon'}{\kappa_\varepsilon}\tan\phi\right)\mathbf{b}_\varepsilon \tag{19}$$

$$\mathbf{F}_\phi^\varepsilon = \frac{1}{\kappa_\varepsilon}(\sec^2\phi)\mathbf{b}_\varepsilon \tag{20}$$

$$\begin{aligned} \mathbf{F}_{uu}^\varepsilon = &\left(\frac{v_\varepsilon\, \kappa_\varepsilon'}{\kappa_\varepsilon} + v_\varepsilon^2\, \tau_\varepsilon \tan\phi\right)\mathbf{t}_\varepsilon \\ &+\left(\frac{2(\kappa_\varepsilon')^2}{\kappa_\varepsilon^3} - \frac{\kappa_\varepsilon''}{\kappa_\varepsilon^2} - \frac{(v_\varepsilon\, \tau_\varepsilon)^2}{\kappa_\varepsilon} + \left(\frac{2 v_\varepsilon\, \kappa_\varepsilon'\, \tau_\varepsilon}{\kappa_\varepsilon^2} - \frac{(v_\varepsilon\, \tau_\varepsilon)'}{\kappa_\varepsilon}\right)\tan\phi\right)\mathbf{n}_\varepsilon \\ &+\left(\frac{(v_\varepsilon\, \tau_\varepsilon)'}{\kappa_\varepsilon} - \frac{2 v_\varepsilon\, \kappa_\varepsilon'\, \tau_\varepsilon}{\kappa_\varepsilon^2} + \left(\frac{2(\kappa_\varepsilon')^2}{\kappa_\varepsilon^3} - \frac{(v_\varepsilon\, \tau_\varepsilon)^2}{\kappa_\varepsilon} - \frac{\kappa_\varepsilon''}{\kappa_\varepsilon^2}\right)\tan\phi\right)\mathbf{b}_\varepsilon \end{aligned} \tag{21}$$

$$\mathbf{F}_{\phi\phi}^\varepsilon = \frac{1}{\kappa_\varepsilon}(2\sec^2\phi\, \tan\phi)\mathbf{b}_\varepsilon \tag{22}$$

$$\mathbf{F}_{u\phi}^\varepsilon = \frac{v_\varepsilon\, \tau_\varepsilon}{\kappa_\varepsilon}\sec^2\phi\ \mathbf{n}_\varepsilon - \frac{\kappa_\varepsilon'}{\kappa_\varepsilon^2}\sec^2\phi\, \mathbf{b}_\varepsilon \tag{23}$$

Unit normal vector on the focal surface $F^\varepsilon(u,\phi)$ is given by

$$\mathbf{N}^{\varepsilon} = -\mathbf{t}_{\varepsilon} \tag{24}$$

Using above equations, the first fundamental form has following components

$$\begin{aligned} \mathbf{E}^{\varepsilon} &= \frac{1}{\kappa_{\varepsilon}^{2}} \sec^{2} \phi \left(\left(\frac{(\kappa')^{2}}{\kappa_{\varepsilon}^{2}} \right) + (v_{\varepsilon}\, \tau_{\varepsilon})^{2} \right) \\ \mathbf{F}^{\varepsilon} &= \frac{1}{\kappa_{\varepsilon}^{2}} \sec^{2} \phi \left(v_{\varepsilon}\, \tau_{\varepsilon} - \frac{\kappa_{\varepsilon}'}{\kappa_{\varepsilon}} \tan \phi \right) \\ \mathbf{G}^{\varepsilon} &= \frac{1}{\kappa_{\varepsilon}^{2}} (\sec^{2} \phi) \end{aligned} \tag{25}$$

Also, components of second fundamental form are given as

$$\begin{aligned} \mathbf{L}^{\varepsilon} &= -\frac{v_{\varepsilon}\, \kappa_{\varepsilon}'}{\kappa_{\varepsilon}} - v_{\varepsilon}^{2}\, \tau_{\varepsilon} \tan \phi \\ \mathbf{M}^{\varepsilon} &= 0 \\ \mathbf{N}^{\varepsilon} &= 0 \end{aligned} \tag{26}$$

From equation (25) and (26), the Gaussian and mean curvatures of focal surface $\mathbf{F}^{\varepsilon}(u, \phi)$ are given by

$$K^{\varepsilon} = 0$$

$$H^{\varepsilon} = -\frac{v_{\varepsilon} \kappa_{\varepsilon}^{3}}{2\,(\kappa_{\varepsilon}' + v_{\varepsilon}\, \kappa_{\varepsilon}\, \tau_{\varepsilon} \tan \phi)}.$$

Theorem 3.9: *Let* $\mathbf{F}^{\varepsilon}(u, \phi)$ *denotes a focal surface of tubular surface* $\boldsymbol{\chi}^{\varepsilon}(u, \phi)$, *then*

1. s-parametric curves are asymptotic lines on $\mathbf{F}^{\varepsilon}(u, \phi)$ iff
$$\phi = \tan^{-1}\left(\frac{\kappa_{\varepsilon}'}{v_{\varepsilon}\, \kappa_{\varepsilon}\, \tau_{\varepsilon}}\right)$$
2. ϕ-parametric curves are always asymptotic lines on $\mathbf{F}^{\varepsilon}(u, \phi)$.

Proof:

1. s-parametric curves are asymptotic lines on a regular surface if it satisfies [1]
$$\mathbf{F}_{uu}^{\varepsilon} \cdot \mathbf{N}^{\varepsilon} = 0$$
From equation (21) and (24), we have
$$\frac{v_{\varepsilon}\, \kappa_{\varepsilon}'}{\kappa_{\varepsilon}} - v_{\varepsilon}^{2}\ \tau_{\varepsilon} \tan \phi = 0$$
This implies, $\phi = \tan^{-1}\left(\frac{\kappa_{\varepsilon}'}{v_{\varepsilon}\, \kappa_{\varepsilon}\, \tau_{\varepsilon}}\right)$
The converse also holds true.
2. ϕ-parametric curves are asymptotic lines on a regular surface if it satisfies [1]
$$\mathbf{F}_{\phi\phi}^{\varepsilon} \cdot \mathbf{N}^{\varepsilon} = 0$$
The proof follows from equation (22) and equation (24)

Theorem 3.10: *Let* $\mathbf{F}^{\varepsilon}(u,\phi)$ *be a focal surface of tubular surface* $\boldsymbol{\chi}^{\varepsilon}(u,\phi)$, *then*

1. s*-parametric curves are geodesic lines on* $\mathbf{F}^{\varepsilon}(u,\phi)$ *iff*
$$2\,(\kappa_{\varepsilon}')^2-\kappa_{\varepsilon}\,\kappa_{\varepsilon}''-(v_{\varepsilon}\,\kappa_{\varepsilon}\,\tau_{\varepsilon})^2=0$$
and $$2\,v_{\varepsilon}\kappa_{\varepsilon}'\,\tau_{\varepsilon}-\kappa_{\varepsilon}(v_{\varepsilon}\,\tau_{\varepsilon})'=0.$$
2. ϕ*-parametric curves are geodesic lines on* $\mathbf{F}^{\varepsilon}(u,\phi)$ *for* $\phi=0,\pi$.

Proof:

1. s-parametric curves are geodesic lines on a regular surface if it satisfies [1]
$$\mathbf{F}_{uu}^{\varepsilon}\times\mathbf{N}_{\varepsilon}=\mathbf{0}$$
equation (21) and (24) yields
$$-\left[\frac{2(\kappa_{\varepsilon}')^2}{\kappa_{\varepsilon}^3}+\left(\frac{2v_{\varepsilon}\,\kappa_{\varepsilon}'\,\tau_{\varepsilon}}{\kappa_{\varepsilon}^2}-\frac{(v_{\varepsilon}\,\tau_{\varepsilon})'}{\kappa_{\varepsilon}}\right)\tan\phi-\frac{(v_{\varepsilon}\,\tau_{\varepsilon})'}{\kappa_{\varepsilon}}-\frac{\kappa_{\varepsilon}''}{\kappa_{\varepsilon}^2}\right]\mathbf{n}_{\varepsilon}$$
$$+\left[\frac{(v_{\varepsilon}\,\tau_{\varepsilon})'}{\kappa_{\varepsilon}}-\frac{2v_{\varepsilon}\,\kappa_{\varepsilon}'\,\tau_{\varepsilon}}{\kappa_{\varepsilon}^2}-\left(\frac{(v_{\varepsilon}\,\tau_{\varepsilon})'}{\kappa_{\varepsilon}}-\frac{2(\kappa_{\varepsilon}')^2}{\kappa_{\varepsilon}^3}+\frac{\kappa_{\varepsilon}''}{\kappa_{\varepsilon}^2}\right)\tan\phi\right]\mathbf{b}_{\varepsilon}=0$$
this implies, $2\,(\kappa_{\varepsilon}')^2-\kappa_{\varepsilon}\,\kappa_{\varepsilon}''-(v_{\varepsilon}\,\kappa_{\varepsilon}\,\tau_{\varepsilon})^2=0$
and $$2\,v_{\varepsilon}\,\kappa_{\varepsilon}'\,\tau_{\varepsilon}-\kappa_{\varepsilon}(v_{\varepsilon}\,\tau_{\varepsilon})'=0$$
The converse is also true for the above equation.
2. ϕ-parametric curves are geodesic lines on a regular surface if it satisfies [1]
$$\mathbf{F}_{\phi\phi}^{\varepsilon}\times\mathbf{N}^{\varepsilon}=\mathbf{0}$$
equation (22) and (24) yields
$$\frac{2}{\kappa_{\varepsilon}}\sec^2\phi\,\tan^2\phi=0$$
this implies, $$\sin\phi=0$$
and $$\phi=0,\pi$$
The converse also holds true.

Theorem 3.11: *Let* $\mathbf{F}^{\varepsilon}(u,\phi)$ *denotes a focal surface, then it is a* $(K^{\varepsilon},H^{\varepsilon})$*-Weingarten surface.*

Proof: The result follows from (Definition 2.7) and (Theorem 3.8).

Theorem 3.12: *The focal surface* $\mathbf{F}^{\varepsilon}(u,\phi)$ *is a* $(K^{\varepsilon},H^{\varepsilon})$*- linear Weingarten surface for* $a\in\mathbb{R}$, and $b=-\frac{(\kappa_{\varepsilon}'+v_{\varepsilon}\,\kappa_{\varepsilon}\,\tau_{\varepsilon}\tan\phi)}{v_{\varepsilon}\,\kappa_{\varepsilon}^3}c$.

Proof: From (Definition 2.8) and (Theorem 3.8), we have
$$-b\left(\frac{v_{\varepsilon}\,\kappa_{\varepsilon}^3}{\kappa_{\varepsilon}'+v_{\varepsilon}\,\kappa_{\varepsilon}\,\tau_{\varepsilon}\tan\phi}\right)=c$$
this implies, $$b\left(\frac{\kappa_{\varepsilon}'+v_{\varepsilon}\,\kappa_{\varepsilon}\,\tau_{\varepsilon}\tan\phi}{v_{\varepsilon}\,\kappa_{\varepsilon}^3}\right)=c.$$

4. Examples

To better understand the results of this study, we consider following examples.

Example 1: Let $\boldsymbol{\alpha}(u)$ be the spine curve of the tubular surface given by

$$\boldsymbol{\alpha}(u) = (\cos u, \quad \sin u, \quad 0) \qquad u \in [0,2\pi].$$

Frenet apparatus of $\boldsymbol{\alpha}$ is given by

$$\mathbf{t} = (-\sin u, \quad \cos u, \quad 0)$$
$$\mathbf{n} = (-\cos u, \quad -\sin u, \quad 0)$$
$$\mathbf{b} = (0, \quad 0, \quad 1)$$
$$\kappa = 1 \qquad \text{and} \qquad \tau = 0.$$

Case I: Choosing $f(u) = 0, \quad g(u) = 1$

$$\Rightarrow \qquad \mathbf{h}(u) = (0, \quad 0, \quad u)$$
$$\boldsymbol{\alpha}_\varepsilon(u) = (\cos u, \quad \sin u, \quad \varepsilon\, u)$$

also,
$$\mathbf{t}_\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2}}(-\sin u, \quad \cos u, \quad \varepsilon)$$
$$\mathbf{n}_\varepsilon = (-\cos u, \quad -\sin u, \quad 0)$$

$$\mathbf{b}_\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2}}(\varepsilon \sin u, \quad -\varepsilon \cos u, \quad 1)$$
$$\kappa_\varepsilon = \frac{1}{1+\varepsilon^2} \qquad \text{and} \qquad \tau_\varepsilon = \frac{1}{1+\varepsilon^2}.$$

The tubular surface is given by the following parametric equation

$$\boldsymbol{\chi}^\varepsilon(u,\phi) = (\cos u, \qquad \sin u, \qquad \varepsilon\, u) + R\,(\cos\phi\; \mathbf{n}_\varepsilon + \sin\phi\; \mathbf{b}_\varepsilon)$$

Principal curvatures of tubular surface $\boldsymbol{\chi}_\varepsilon$ are

$$\kappa_{\varepsilon 1} = -\frac{\cos\phi}{1+\varepsilon^2 - R\cos\phi} \qquad \text{and} \qquad \kappa_{\varepsilon 2} = \frac{1}{R}$$

Focal surface of tubular surface $\boldsymbol{\chi}_\varepsilon$ has following equation

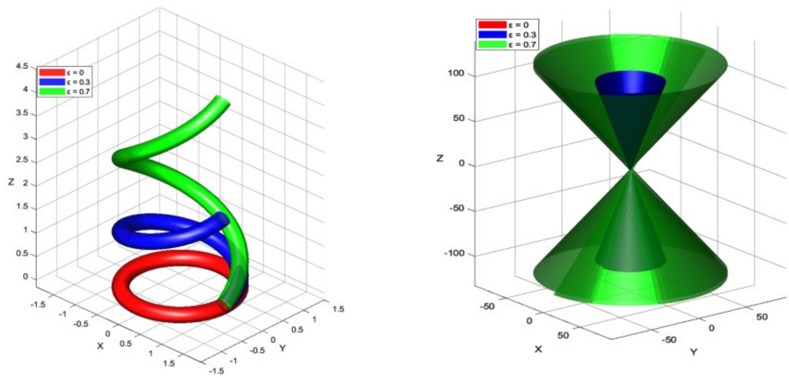
$$\mathbf{F}^\varepsilon(u,\phi) = (\cos u, \quad \sin u, \quad \varepsilon\, u) + (1+\varepsilon^2)\,(\mathbf{n}_\varepsilon + \tan\phi\; \mathbf{b}_\varepsilon).$$

For Case I, the tubular and focal surfaces generated by the parametric equations are displayed in Figure 1(a) and Figure 1(b) respectively. These surfaces are obtained for $R = 0.15$ and $\varepsilon = 0, 0.3, 0.7$.

Case II: For $f(u) = 0, \quad g(u) = 2\pi - u$, we get

$$\mathbf{h}(u) = (0, \quad 0, \quad u\,(2\pi - u))$$

and
$$\boldsymbol{\alpha}_\varepsilon(u) = (\cos u, \sin u, \varepsilon\, u\,(2\pi - u))$$



(a) Spine curves lie on the surface of a unit radius cylinder.

(b) Focal surfaces corresponding to $\boldsymbol{\chi}_{\varepsilon}(u, \boldsymbol{\phi})$.

Figure 1

Obtain the Frenet apparatus of the curve $\boldsymbol{\alpha}_{\varepsilon}$ as computed in Case I. The tubular surface is given by the following parametric equation

$$\boldsymbol{\chi}^{\varepsilon}(u, \phi) = (\cos u, \quad \sin u, \quad \varepsilon\, u\,(2\pi - u)) + R\,(\cos\phi\, \mathbf{n}_{\varepsilon} + \sin\phi\, \mathbf{b}_{\varepsilon})$$

and the corresponding equation of focal surface is given by

$$\mathbf{F}^{\varepsilon}(u, \phi) = \ (\cos u, \quad \sin u, \quad \varepsilon\, u\,(2\pi - u)) + \frac{\omega_2^3}{\omega_1}(\mathbf{n}_{\varepsilon} + \tan\phi\, \mathbf{b}_{\varepsilon})$$

where

$$\omega_1 = [1 + 4\,\varepsilon^2\,(1 + (\pi - u)^2)]^{\frac{1}{2}} \quad \text{and} \quad \omega_2 = [1 + 4\,\varepsilon^2 (\pi - u)^2]^{\frac{1}{2}}$$

Figure 2 represents the tubular surfaces obtained for $R = 0.15$, $f(u) = 0$, $g(u) = 2\pi - u$ and $\varepsilon = 0, 0.3, 0.7$.

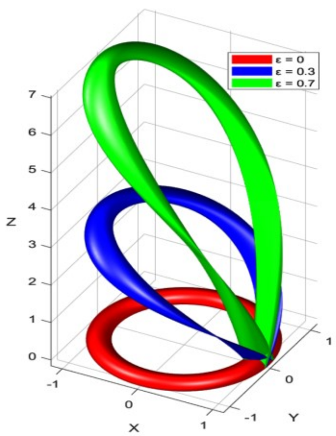


Figure 2

Figure 3 represents the focal surfaces obtained for $f(u) = 0$, $g(u) = 2\pi - u$ and $\varepsilon = 0, 0.3, 0.7$.

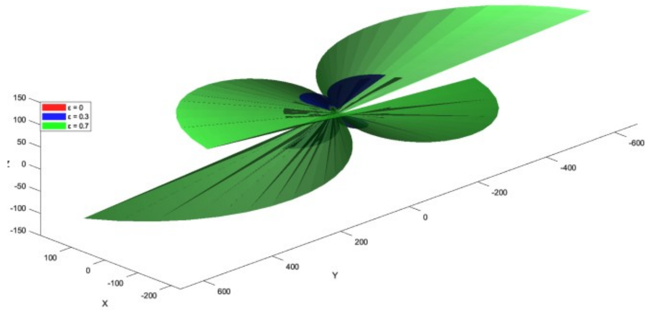


Figure 3

Case III: For $f(u) = 1$ and $g(u) = 1$, we have

$$\mathbf{h}(u) = (-\sin u, \quad \cos u, \quad u)$$

and $$\boldsymbol{\alpha}_\varepsilon(u) = (\cos u - \varepsilon \sin u,\ \sin u + \varepsilon \cos u,\ \varepsilon\, u)$$

The tubular surface is given by the following parametric equation

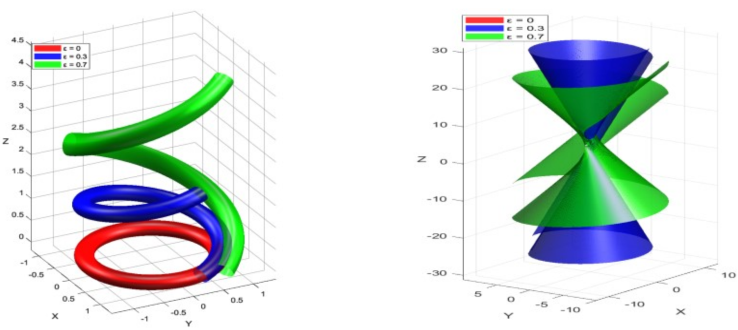
$$\boldsymbol{\chi}^\varepsilon(u, \phi) = (\cos u - \varepsilon \sin u, \sin u + \varepsilon \cos u, \varepsilon\, u) + R\,(\cos\phi\, \mathbf{n}_\varepsilon + \sin\phi\, \mathbf{b}_\varepsilon)$$

and the corresponding equation of focal surface is given by

$$\mathbf{F}^\varepsilon(u, \phi) = (\cos u - \varepsilon \sin u, \sin u + \varepsilon \cos u, \varepsilon\, u) + \frac{\delta_2}{\delta_1}(\mathbf{n}_\varepsilon + \tan\phi\, \mathbf{b}_\varepsilon)$$

where $$\delta_1 = (1 + \varepsilon^2)^{\frac{1}{2}} \quad \text{and} \quad \delta_2 = 1 + 2\,\varepsilon^2$$

For Case III, the tubular and focal surfaces generated by the parametric equations are displayed in Figure 4(a) and Figure 4(b) respectively. We obtain these surfaces for $R = 0.15$ and $\varepsilon = 0, 0.3, 0.7$



(a) Spine curve at $\boldsymbol{\varepsilon = 0.7}$ doesn't lie on the surface of a unit radius cylinder.

(b) Focal surfaces corresponding to $\boldsymbol{\chi_\varepsilon(u, \phi)}$.

Figure 4

Example 2: Consider a regular space curve $\boldsymbol{\alpha}(u)$ given by

$$\boldsymbol{\alpha}(u) = \left(\cos\frac{u}{\sqrt{2}},\quad \sin\frac{u}{\sqrt{2}},\quad \frac{u}{\sqrt{2}}\right) \qquad u \in [0,2\pi].$$

Choosing $f(u) = 1$ and $g(u) = 0$, we have

$$\mathbf{h} = \left(-\sqrt{2}\sin\frac{u}{\sqrt{2}}, \sqrt{2}\quad \cos\frac{u}{\sqrt{2}},\quad 0\right)$$

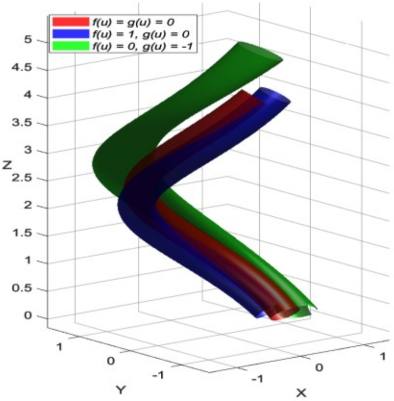
and

$$\boldsymbol{\alpha}_\varepsilon = \left(\cos\frac{u}{\sqrt{2}} - \sqrt{2}\,\varepsilon\sin\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}} + \sqrt{2}\,\varepsilon\cos\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}\right).$$

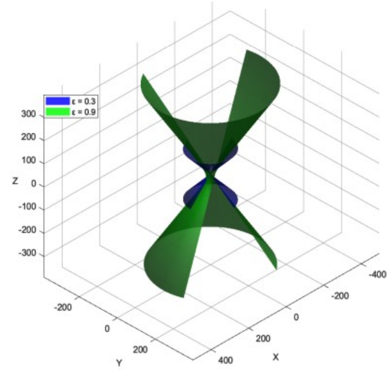
Obtain the Frenet apparatus of the curve $\boldsymbol{\alpha}_\varepsilon$ as computed in Case I of Example 1.

Tubular surface obtained from the spine curve $\boldsymbol{\alpha}_\varepsilon$ is given by

$$\boldsymbol{\chi}^\varepsilon(u) = \quad \boldsymbol{\alpha}_\varepsilon + R(\cos\phi\ \ \mathbf{n}_\varepsilon + \sin\phi\ \ \mathbf{b}_\varepsilon)$$



(a) Tubular surfaces at R= 0.15.



(b) Focal surfaces corresponding to $\boldsymbol{\chi}_\varepsilon(\boldsymbol{u}, \boldsymbol{\phi})$.

Figure 5

Figure 5(a) and Figure 5(b) represents the tubular and focal surfaces respectively. These surfaces are obtained for different choices of the functions $f(u)$ and $g(u)$.

Conclusion

This paper introduces new type of tubular surface, we compute the first and second fundamental forms to analyse their Gaussian and mean curvatures. Additionally, some conditions are investigated under which the tubular surface can be classified as Weingarten and linear Weingarten surfaces. Furthermore, parametric curves are examined to be an asymptotic or a geodesic line on these surfaces. Also, we derive the focal surfaces of these tubes. We give some illustrative examples along with their graphical representation. For future work, infinitesimal bending of curves can be studied using different frames.

Acknowledgement

The first author acknowledges with gratitude the financial support received through a PhD fellowship from Jaypee University of Information Technology.

References

[1] E. Abbena, S. Salamon, and A. Gray, Modern Differential Geometry of curves and surfaces with Mathematica. Chapman and Hall/CRC (2017).

[2] A. D. Aleksandrov, "On infinitesimal bendings of irregular surfaces," *Matematicheskii Sbornik*, vol. 1, no. 3, pp. 307–321 (1936).

[3] F. Ateş, E. Kocakuşakli, İ. Gök, and Y. Yayli, "A study of the tubular surfaces constructed by the spherical indicatrices in Euclidean 3-space," *Turkish Journal of Mathematics*, vol. 42, no. 4, pp. 1711–1725 (2018).

[4] M. P. Do Carmo, Differential geometry of curves and surfaces: revised and updated second edition. Dover Publications (2016).

[5] N. V. Efimov, "Qualitative problems of the theory of deformation of surfaces," *Uspekhi Matematicheskikh Nauk*, vol. 3, no. 2, pp. 47–158 (1948).

[6] Z. Guan, J. Ling, N. Tao, X. Ping, and T. Rongxi, "Study and application of Physics-based deformable curves and surfaces," *Computers and Graphics*, vol. 21, no. 3, pp. 305–313 (1997).

[7] F. Güler, "The focal surfaces of offset surface," *Optik*, vol. 271, p. 170053 (2022).

[8] U. Gözütok, H. A. Çoban, and Y. Sağiroğlu, "Ruled surfaces obtained by bending of curves," *Turkish Journal of Mathematics*, vol. 44, no. 1, pp. 300–306 (2020).

[9] H. Hagen and S. Hahmann, "Generalized focal surfaces: A new method for surface interrogation," in Proceedings Visualization '92. *IEEE Computer Society*, pp. 70–76 (1992).

[10] H. Han, A. S. Shin, and Y. Jang, "Ruled surfaces obtained by infinitesimal bendings of curves in Minkowski 3-space," *International Journal of Geometric Methods in Modern Physics*, vol. 22, no. 1, pp. 2 450 255–71 (2025).

[11] L. H. Kauffman, L. S. Velimirović, M. S. Najdanović, and S. R. Rančić, "Infinitesimal bending of knots and energy change," *Journal of Knot Theory and Its Ramifications*, vol. 28, no. 11, pp. 1940009 (2019).

[12] İ. Kişi, "A new approach to revolution surface with its focal surface in the Galilean 3-space G3," *Hacettepe Journal of Mathematics and Statistics,* vol. 50, no. 6, pp. 1722–1737 (2021).

[13] Y. H. Kim, H. Liu, and J. Qian, "Some characterizations of canal surfaces," *Bulletin of the Korean Mathematical Society,* vol. 53, no. 2, pp. 461–477 (2016).

[14] T. Korpinar and E. Turhan, "Tubular surfaces around timelike biharmonic curves in Lorentzian Heisenberg group Heis3," *Analele stiintifice ale Universitatii Ovidius Constanta,* vol. 20, no. 1, pp. 431–445 (2012).

[15] T. Maekawa, N. M. Patrikalakis, T. Sakkalis, and G. Yu, "Analysis and applications of pipe surfaces," *Computer Aided Geometric Design*, vol. 15, no. 5, pp. 437–458 (1998).

[16] M. Maksimovic, L. Velimirovic, and M. Najdanovic, "Infinitesimal bending of DNA helices," *Turkish Journal of Mathematics,* vol. 45, no. 1, pp. 520–528 (2021).

[17] M. S. Najdanović and L. S. Velimirović, "Infinitesimal bending of curves on the ruled surfaces," *Bulletin of Natural Sciences Research,* vol. 8, no. 1 (2018).

[18] M. S. Najdanović and L. S. Velimirović, "Second order infinitesimal bending of curves," *Filomat*, vol. 31, no. 13, pp. 4127–4137 (2017).

[19] B. O'Neill, Elementary Differential Geometry. Elsevier (2006).

[20] P. K. Pandey and S. Mohammad, "Magnetic and slant curves in Kenmotsu manifolds," *Surveys in Mathematics and its Applications*, vol. 15, pp. 139–151 (2020).

[21] J. Pegna and D. J. Wilde, "Spherical and circular blending of functional surfaces," *Journal of Offshore Mechanics and Arctic Engineering*, vol. 112, no. 2, pp. 134–142 (1990).

[22] K. Percin, O. Zoellner, D. Rall, and M. Wessling, "A tubular electrochemical reactor for slurry electrodes," *ChemElectroChem*, vol. 7, no. 12, pp. 2665–2671 (2020).

[23] M. Poddar and S. Sarkar, "A class of torus manifolds with nonconvex orbit space," *Proceedings of the American Mathematical Society,* vol. 143, no. 4, pp. 1797–1811 (2015).

[24] H. Pottmann, H. Hagen, and A. Divivier, "Visualizing functions on a surface," *Journal of Visualization and Computer Animation*, vol. 2, no. 2, pp. 52–58 (1991).

[25] M. K. Saad, N. Yüksel, N. Oğraş, F. Alghamdi, and A. A. Abdel-Salam, "Geometry of tubular surfaces and their focal surfaces in Euclidean 3-space," *AIMS Mathematics*, vol. 9, no. 5, pp. 12 479–12 493 (2024).

[26] Sameer and P. K. Pandey, "Differential equations for indicatrices, spacelike and timelike curves," *Australian Journal of Mathematical Analysis and Applications*, vol. 20, no. 2, pp. 1–9 (2023).

[27] U. Shani and D. H. Ballard, "Splines as embeddings for generalized cylinders," *Computer Graphics and Image Processing*, vol. 27, no. 2, pp. 129–156 (1984).

[28] M. D. Shepherd, "Line congruences as surfaces in the space of lines," *Differential Geometry and its Applications*, vol. 10, no. 1, pp. 1–26 (1999).

[29] A. H. Sorour, "Weingarten tube-like surfaces in Euclidean 3-space," Studia Universitatis Babeş-Bolyai Mathematica, vol. 61, no. 2 (2016).

[30] L. S. Velimirović, M. S. Ćirić, and M. D. Cvetković, "Change of the Willmore energy under infinitesimal bending of membranes," *Computers and Mathematics with Applications*, vol. 59, no. 12, pp. 3679–3686 (2010).

Received May, 2025