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Best multiplier co-approximation of unbounded functions in space $\mathbf{L}_{p,\lambda_n}(\mathbf{X})$

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Abstract

The space $\mathbf{L}_{p,\lambda_n}(\mathbf{X})$ which is the space of all unbounded functions on the interval X = (0, 1] is defined. A special norm has been defined on the space $\mathbf{L}_{p,\lambda_n}(\mathbf{X})$. Furthermore, the concept of the best multiplier approximation has been established. Additionally, new sets have been defined such as, the set of best multiplier approximation, a set for proximinal, the set for coproximinal , and a set for semi-Cochebyshev. All the previous sets are defined on $\mathbf{L}_{p,\lambda_n}(\mathbf{X})$. In the space $\mathbf{L}_{p,\lambda_n}(\mathbf{X})$ some results of the best multiplier co-approximation of the unbounded functions defined on X = (0, 1] have been obtained. Some special sets in $\mathbf{L}_{p,\lambda_n}$

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(**X**) have been studied. Several concepts have been built and employed to lead to the best multiplier co- approximation.

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1. Introduction

The best co- approximations were established in the normed linear space by [3] to analyze the some crusul features for Real Hilbert space. Moreever, C.Franchetti and M.Furi have worked on this concept to build the large extant in norm linear space & subsequently in Hilbert space. Furthermore, research on metric linear spaces has focused on approximation, structural properties, and duality, with key contributions from Albinus [1], Lorentz [4], Narang [7], Pantelidis [11], Rolewicz [13], Schnatz [14], and Vasilev [15]. Worked on extending several result on the best approximations defined on norm linear space to the metric linear space. Several authors tried to analyze best coppoximation in metric linear spaces. the theory in the metric linear spaces has not been developed as much as it was modified to the best approximation. Another type of modern approximation is the "coapproximation", this type was provided by [2] and study some on best coapproximation in normed space. Also Kavikumar, Manian, and Moorthy, Study this type of best coapproximations and the best simultaneous co-approximations in the fuzzy norm space ([5], [6]). Also Rao, Abu-Sirhan and Fawalbeh got results of best coapproximation in normed spaces ([8], [9] [10],[12]). In this work we introduce some result on the existences of functions for the best coapproximaton in space L_{p,λ_n} (X). The plan of this work will be presented as follows: In the section one; we presents brief introduction about what the other researcher have done. In section two; main definitions will presented which will be an effective tool in the proof of the main theorems. In section three; three propositions is introduced and proved by utilizing the main definitions presented in section three.

2. New concept via the spaces $\mathbf{L_{p,\lambda_n}}$ (X)

Definition 2.1: Let X = (0, 1], & L_{p,λ_n} (X) the space for each unbounded function, $P \in [1, \infty)$, $f \in L_{p,\lambda_n}$ (X) with norm:

$\|f\|_{p,\lambda_n} = \left(\int_0^1 |f(x)\lambda n|^P \, dx\right)^{1/p} < \infty$, where λ_n be sequences for real number s.t: $\int_0^1 f(x)\lambda n \, dx < \infty$

Definition 2.2: If E is subspace for L_{p,λ_n} (X), $f_0 \in E$ named best multiplier approximation to $f \in L_{p,\lambda_n}$ (X) if, $\|f - f_0\|_{p,\lambda_n} \le \|f - g\|_{p,\lambda_n}$ $\forall\, g \in E$

Moreover, $f_0 \in E$ named best mumultiplier coapproximations to $f \in L_{p,\lambda_n}$ (X) if the below inequality holds: $\|f_0 - g\|_{p,\lambda_n} \le \|f - g\|_{p,\lambda_n}$ $\forall\, g \in E$

Furthermore, a set for each best multiplier approximation denote as $Q_E(f)$ and defined as shown below:

$$Q_E(f) = \{f_0 \in E: \|f - f_0\|_{p,\lambda_n} \leq \|f - g\|_{p,\lambda_n} \ \forall\, g \in E\}$$

Also the set for every best multiplier coapproximations denote as $\Omega_E\,(f)$ and define by introduced below:

$$\Omega_E\,(f) = \{f_0 \in E: \|f_0 - g\|_{p,\lambda_n} \leq \|f - g\|_{p,\lambda_n} \ \forall\, g \in E\} \ .$$

A set E is called proximinal if $Q_E(f)$ include at least one elements $\forall f \in L_{p,\lambda_n}(X)$ and E is called coproximinal if $\Omega_E\,(f)$ includes at least one elements $\forall f \in L_{p,\lambda_n}(X)$. If $\forall f \in L_{p,\lambda_n}(X), \Omega_E(f)$ have exactly one elements, so E named semi-cochebyshev in $L_{p,\lambda_n}(X)$.

Definition 2.3: Assume $\forall f \in L_{p,\lambda_n}(X)$, the set $Q_E(f)$ have exactly one element, the set E named Chebyshev sets in the space $L_{p,\lambda_n}(X)$. Furthermore, if $\forall f \in L_{p,\lambda_n}(X)$, the set $\Omega_E\,(f)$ have exact one elements, the set E call Co- Chebyshev sets in the space $L_{p,\lambda_n}(X)$. Moreover, if $Q_E(f)$ has at most one element, so E call Semi- Chebyshev in the space $L_{p,\lambda_n}(X)$ $\quad \forall f \in L_{p,\lambda_n}(X)$.

Definition 2.4: If E closed subspaces for $L_{p,\lambda_n}(X)$, the

$$L_{p,\lambda_n}(X) \big/ _E = \{f + E : f \in L_{p,\lambda_n}(X)\}$$

With the below properties:

(i) $(f + E) + (h + E) = (f + h) + E \qquad , \forall f, g \in L_{p,\lambda_n}(X)$

(ii) $\beta(f + E) = \beta F + E, \qquad \forall\, f \in L_{p,\lambda_n}(X), \forall$ scalar β

(iii) $\|f + E, h + E\|_{p,\lambda_n} = \inf_{g\in E} \|f - h, g\|_{p,\lambda_n} = \|h + E, f + E\|_{p,\lambda_n}$

Proposition: A proximinal, coproximinal, chrebyshev and Co-Chebyshev subspaces of the space $L_{p,\lambda_n}(X)$

Proof: By using definition of (2.2) we get the desired results.

Remark 1: If E the subspaces for $L_{p,\lambda_n}(X)$, then

$$Q_E^*\,(f_0) = \{f \in L_{p,\lambda_n}(X): f_0 \in Q_E(f), \forall\, f_0 \in E\}$$

$$\Omega_E^*\,(f_0) = \{f \in L_{p,\lambda_n}(X): f_0 \in \Omega_E(f), \forall\, f_0 \in E\}$$ Are both closed sets.

Proof: We first show that the set $Q_E^*\,(f_0)$ closed subsets for $L_{p,\lambda_n}(X)$. if $Q_E^*\,(f_0)$ finite dimensional subsets for normed space $L_{p,\lambda_n}(X)$. To show that $Q_E^*\,(f_0)$ closed subsets for $L_{p,\lambda_n}(X)$, we need to show that: $Q_E^*\,(f_0) = \overline{Q_E^*\,(f_0)}$. Let $f \in \overline{Q_E^*\,(f_0)}$, then there must exist (f_n) in $Q_E^*\,(f_0)$ such that f_n converges to f. This yields that (f_n) is a convergence sequence in $Q_E^*\,(f_0)$. Then (f_n) Cauchy sequences in $Q_E^*\,(f_0)$ & since $Q_E^*\,(f_0)$ the finite dimensional this means that $Q_E^*\,(f_0)$ is complete normed space which then suggests that $f_n \to f \in Q_E^*\,(f_0)$

which then leads to the desired equality $Q^*_E(f_0) = \overline{Q^*_E(f_0)}$. Then $Q^*_E(f_0)$ the closed in the $L_{p,\lambda_n}(X)$. Similar argument can be utilized to show that $\Omega^*_E(f_0)$ is also a closed subsets in $L_{p,\lambda_n}(X)$.

Remark 2:

(i) If E subspace for $L_{p,\lambda_n}(X)$, then $Q^*_E(0) \cap E = \{0\}$, $\Omega^*_E(0) \cap E = \{0\}$

(ii) If E is a subspaces for the $L_{p,\lambda_n}(X)$, so $f_0 \in Q_E(f)$.

A proof for (i) follows from the fact that all elements of $Q^*_E(0)$ are positive. The proof of (ii) follows from the fact that every Banach space is a metric space and by utilizing remark (7) page 15 in [Sahil Gupta and Tulsi Dass Narang].

Remark 3: [Sahil Gupta and Tulsi Dass Narang].

$$f_0 \in \Omega_E(f) \text{ iff } f - f_0 \in Q^*_E(0)$$
$$f_0 \in \Omega_E(f) \text{ iff } f - f_0 \in \Omega^*_E(0)$$

Remark 4: $Q_E(f+g) = Q_E(f) + g$ & $\Omega_E(f+g) = \Omega_E(f) + g \quad \forall g \in L_{p,\lambda_n}(X)$.

3. Main results:

In this section; three main propositions will be proved.

Proposition 3.1: If E is close subspaces for $L_{p,\lambda_n}(X)$, & G a co-approximation subspaces for $L_{p,\lambda_n}(X)$, $E \subseteq G$, then G/E coproximinal in the $L_{p,\lambda_n}(X)/E$.

Proof: If $f + E \in L_{p,\lambda_n}(X)/E$ & w_1 the best multiplier coapproximations to f in G. Suppose it be not, then there exist $w_2 + E \in G/E$, that is

$$\|w_1 + E, w_2 + E\|_{p,\lambda_n} > \|f + E, w_2 + E\|_{p,\lambda_n}$$

Since $\inf_{g\in E}\|f - w_{w2}, g\|_{p,\lambda_n} < \|w_1 - w_2, E\|_{p,\lambda_n}$. Then $\exists f_0 \in E$ s.t: $\|f - w_2, f_0\|_{p,\lambda_n} < \|w_1 - w_2, E\|_{p,\lambda_n} \le \|w_1 - w_2, f_0\|_{p,\lambda_n}$

But $\|f, w_2 + f_0\|_{p,\lambda_n} < \|w_1, w_2 + f_0\|_{p,\lambda_n}$ Thus w_1 not best multiplier coapproximations for f from G. Hence $w_1 + E$ best multiplier coapproximations to $f + E$, that is G/E is coproximind in $L_{p,\lambda_n}(X)/E$.

Proposition 3.2: If E co-chebyshev subspaces for $L_{p,\lambda_n}(X)$ & G closed subspaces for $L_{p,\lambda_n}(X)$, $E \subseteq G$, where G/E the semi-cochebyshev in the $L_{p,\lambda_n}(X)/E$ then G be semi co-chebyshev in the $L_{p,\lambda_n}(X)$.

Proof: Assume $f \in L_{p,\lambda_n}(X)$, $ƀ_1, ƀ_2 \in \Omega_G(f)$ We then have: $ƀ_1 + E, ƀ_2 + E \in \Omega_{G/E}(f + E)$ Since G/E the semi co-chebyshevs in the $L_{p,\lambda_n}(X)/E$, then:

$$ƀ_1 + E = ƀ_2 + E, \text{ we get } ƀ_1 - ƀ_2 \in E$$

Since $ƀ_1, ƀ_2 \in \Omega_G(f) \Rightarrow f - ƀ_1, f - ƀ_2 \in \Omega^*_G(0)$

Then $f - ƀ_1,\ f - ƀ_2 \in \Omega_G^*(0)$ (since $E \subseteq G$)

Since $0 \in \Omega_E\,(f - ƀ_1)$, we have from definition: $\|0 - g\|_{p,\lambda_n} \leq \|f - ƀ_1, g\|_{p,\lambda_n}$ $\forall g \in E$. That is

$$\|ƀ_1 - ƀ_2, g + ƀ_1 - ƀ_2\|_{p,\lambda_n} \leq \|f - ƀ_1 + ƀ_1 - ƀ_2\,, g + ƀ_1 - ƀ_2\|_{p,\lambda_n} \quad \forall\, g \in E$$

Since $g' = g + ƀ_1 - ƀ_2$. Then we get:

$$\|ƀ_1 - ƀ_2, g'\|_{p,\lambda_n} \leq \|f - ƀ_2\,, g'\|_{p,\lambda_n} \ \forall\, g' \in E$$

This implies that $ƀ_1 - ƀ_2 \in \Omega_E\ (f - ƀ_2)$

$0 \in \Omega_E\,(f - ƀ_2)$, E is co-chebyshev in $L_{p,\lambda_n}(X)$.

We have $ƀ_1 - ƀ_2 = 0$, $ƀ_1 = ƀ_2$, we get G is semi co-chebyshev in $L_{p,\lambda_n}(X)$.

Proposition 3.3: Let E be coproximinal subspaces for the $L_{p,\lambda_n}(X)$ & $\Omega_E^*(0)$ be convex sets, so E be co-chebyshev in the $L_{p,\lambda_n}(X)$.

Proof: Let $f \in L_{p,\lambda_n}(X), g_1, g_2 \in \Omega_E(f)$. Since $g_1, g_2 \in \Omega_E\,(f)$, we have from remark (3) that: $f - g_1\,, f - g_2 \in \Omega_E^*\,(0)$. We claim that: $g_1 - f \in \Omega_E^*(0)$.

Since $0 \in \Omega_E\,(f - g_1)$, we get:

$$\|0 - g\|_{p,\lambda_n} \leq \|f - g_1, g\|_{p,\lambda_n} \quad , \forall g \in E$$

So, $\|-g - 0\|_{p,\lambda_n} \leq \|-g, g_1 - f\|_{p,\lambda_n} \ \ \forall\, g \in E$

Then we get the below inequality:

$$\|0 - g'\|_{p,\lambda_n} \leq \|g_1 - f, g'\|_{p,\lambda_n} \quad \forall\, g' \in E$$

We get $g_1 - f \in \Omega_E^*(0)$. Now, $f - g_2, g_1 - f \in \Omega_E^*(0)$ and $\Omega_E^*\,(0)$ is convex, we get: $\frac{1}{2}\,[(f - g_2 + (g_1 - f)] \in \Omega_E^*(0)$, This implies that $\frac{1}{2}\,[g_1 - g_2] \in \Omega_E^*(0)$

Also we get: $\frac{1}{2}\,[g_1 - g_2] \in E$ and $\frac{1}{2}\,[g_1 - g_2] \in \{\,\Omega_E^*\,(0)\ \cap\ E\} = \{0\}$

This then yields that: $\frac{1}{2}\,[g_1 - g_2] = 0$ and then $g_1 = g_2$ which then asserts that: E is co-chebyshev.

Conclusion

It shown that if G co-approximation subspaces for $L_{p,\lambda_n}(X)$, hence for every closed sub subspace E of $L_{p,\lambda_n}(X)$, the quotient G/E is also is co-approximation in $L_{p,\lambda_n}(X)/E$. Furthermore, if E be co-chebyshev subspaces for the $L_{p,\lambda_n}(X)$, and if G closed sub space for $L_{p,\lambda_n}(X)$ with $E \subseteq G$, it is proved that if G/E ,so G the semi co-chebyshev in the $L_{p,\lambda_n}(X)/E$, so we have G the semi co-chebyshev in the $L_{p,\lambda_n}(X)$, Moreover, for every co- approximation E of $L_{p,\lambda_n}(X)$ and for every $\Omega_E^*(0)$ convex subset of $L_{p,\lambda_n}(X)$, it is proved that E is co-chebyshev in the space $L_{p,\lambda_n}(X)$.

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