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A study on Pythagorean fuzzy digital feeble centred structure compactification

N. Preethi †
Department of Mathematics
Sri Krishna College of Engineering and Technology
Coimbatore 641008
Tamil Nadu
India

G. K. Revathi *
Department of Mathematics
School of Advanced Sciences
Vellore Institute of Technology
Chennai 600127
Tamil Nadu
India

Abstract

In a topological space, compactification serves as a significant extension. To effectively describe compactness and convergence within topological spaces, nets and filters are utilized. This article primarily explores the compactification of Pythagorean fuzzy digital feeble centered structures. It introduces the concept of Pythagorean fuzzy digital feeble centered structure space along with novel operators. Additionally, the study examines Pythagorean fuzzy digital feeble centered nets and Pythagorean fuzzy digital feeble centered open filters.

Subject Classification: *54A40, 03E72.*

Keywords: *Pythagorean fuzzy digital topology, Pythagorean fuzzy digital feeble topology (PFDFTS), Pythagorean fuzzy digital feeble centred structure (PGDFCS), PFDFCS continuous function, PFDFCS compact, PFDFCS filter and PFDFCS net.*

† *E-mail:* `preethi.mphil@gmail.com`
* *E-mail:* `gk revathi@yahoo.co.in (Corresponding Author)`

1. Introduction

Zadeh [15] formulated the theory of fuzzy sets in 1965 to address the issues of imprecision and vagueness within a universal set. Each element in a fuzzy set is associated with a membership value ranging from 0 to 1, often defined using linguistic variables. Fuzzy sets have proven useful in numerous domains, including image processing, decision-making, fuzzy logic systems, and inference models. Atanassov and Stoeva [3] later introduced intuitionistic fuzzy sets (IFS), which have been widely studied and applied to deal with hesitation and ambiguity in real-world environments. A distinctive feature of IFS [4] is that every element is characterized by three components: a membership degree, a non-membership degree, and an indeterminacy degree, with their total equalling one. Literature shows extensive applications of IFS in fields like financial analysis, pattern detection, complex decision processes, medical diagnosis, and service systems.
Yager [14] introduced Pythagorean fuzzy sets (PFS) as an advanced form of intuitionistic fuzzy sets, allowing membership and non-membership degrees whose squared sum is no greater than one. In practical applications, especially in multi-criteria decision-making, decision-makers may provide evaluations where the total of membership and non-membership degrees exceeds one. This scenario makes the use of intuitionistic fuzzy sets impractical. PFS overcomes this challenge and proves to be more effective in capturing and processing the inherent uncertainty, imprecision, and vagueness present in many decision-making tasks.
To improve the expressiveness and computing efficiency of PFS, academics have been actively investigating and developing novel operations in the field in recent years. These innovative operations are designed to tackle certain uncertainty quantification problems, like managing inaccurate data fusion and supporting complicated system decision-making. To promote wider adoption of Pythagorean fuzzy sets (PFS) [1] in fields like image processing, pattern recognition, and decision support systems, researchers [2, 13] are working to enhance their standard functions. We examine the most recent developments in Pythagorean fuzzy operations in this succinct review, clarifying their theoretical underpinnings and applications. We hope to show how these new operations may be used to more accurately and faithfully capture and measure uncertainty through clear explanations and vivid examples.

To comprehend compact spaces and their properties, one needs to understand the relationship between nets and filters in topological spaces. Nets and filters can be used together to measure compactness and convergence. The relationship among nets, filters, and compactifications is also related to one another. In order to examine the compact spaces and create compactifications, filters are employed. To guarantee convergence in compressed space, nets are also laborious. Insightistic fuzzy-centered net and intuitionistic fuzzy-centered open filter were researched by R. Dhavaseelan, E. Roja, and M. K. Uma. Also, they had proven that an intuitionistic fuzzy-centered structure space could be compactified.
Atanassov and Tsvetkov [4] proposed new intuitionistic fuzzy operations. By utilizing the new operations, Preethi and Revathi [10] develop a novel concept for Pythagorean fuzzy digital feeble topological ($\mathcal{PFDFTS}$) spaces, nets, and filters. In accordance with that, the Pythagorean fuzzy digital feeble-centered structure (PFDFCS) system is presented in this study. This article's primary goal is to investigate a novel concept known as PFDFCS compactification by utilizing PFDFCS filters and PFDFCS nets. Several findings about compactifications and their characteristics are investigated in this idea.

1.1 Research gap and Motivation

Building on the foundational work of Dhavaseelan, Roja, and Uma [5] who introduced the fuzzy-centered net and intuitionistic fuzzy-centered open filter this study addresses the growing interest in compactification structures motivated by applications in robotics, physics, and Lie group theory. The increasing depth of theoretical contributions in intuitionistic fuzzy topology has been evident through works such as those by Atanassov and Tsvetkov [4], who proposed the intuitionistic fuzzy operations # and *, generalizing classical union and intersection. Recognizing the potential of these operations, Preethi and Revathi [10] developed a novel framework for Pythagorean fuzzy digital feeble topology. Extending this foundation, the present work introduces and examines the structure of Pythagorean fuzzy digital feeble centered spaces through the formulation of new operators, contributing further to the field's development.
ment.

1.2 Novelty

This paper dissertation the concept of Pythagorean fuzzy digital feeble centered structure space along with the introduction of novel operators, denoted by # and *. These operators facilitate the construction of a weak topological structure within this framework. Despite the inherent limitations of such a weak structure, an attempt is made to develop a form of compactification in this study. The novelty of this work lies in the formulation and analysis of various compactification structures within the context of Pythagorean fuzzy digital feeble topology.

2. Preliminaries

Definition 2.1. [1,7,8,14] A non-empty set with PFS $\beth$. A pair $(\mu_\beth, \nu_\beth)$ is defined for $\mathbf{E}^\sim$, where the M and NM functions are, respectively, $\mu_\beth : \mathbf{E}^\sim \to [0,1]$ and $\nu_\beth : \mathbf{E}^\sim \to [0,1]$. These functions satisfy the equation $\mu^2_\beth(y) + \nu^2_\beth(y) = r^2_\beth(y)$ for all $y \in \mathbf{E}^\sim$, where $r_\beth : \mathbf{E}^\sim \to [0,1]$ denotes the strength of commitment at point y.

Definition 2.2 (1,7,8)**.** Let $\beth = (\mu_\beth, \nu_\beth)$ and $\aleph = (\mu_\aleph, \nu_\aleph)$ can be any two PFS of $\mathbf{E}^\sim$. Then

I. $\beth^c = (\nu_\beth, \mu_\beth)$ defines the complement of $\beth$

II. $\beth \cap \aleph = \{\min(\mu_\beth, \mu_\aleph), \max(\nu_\beth, \nu_\aleph)\}$ determines the intersection of $\beth$ and $\aleph$.

III. $\beth \cup \aleph = \{\max(\mu_\beth, \mu_\aleph), \min(\nu_\beth, \nu_\aleph)\}$ is the definition of the union of $\beth$ and $\aleph$.

IV. $\beth$ is a subset of $\aleph$ or $\aleph$ contains $\beth$. If $\mu_\beth \leq \mu_\aleph$ and $\nu_\beth \geq \nu_\aleph$ then we write $\beth \subseteq \aleph$ or $\aleph \supseteq \beth$

Definition 2.3 (7, 8)**.** Let τ be a family of PFS of $\mathbf{E}^\sim$ and let $\mathbf{E}^\sim$ be a set if

(i) $1_\sim, 0_\sim \in \tau$,

(ii) For any $\{\beth_i / i \in \mathfrak{J}, \beth_i \in \tau\}, \cap_{i=1}^{n} \beth_i \in \tau$

(iii) $\cup_{i=1}\beth_i \in \tau$ for any $\{\beth_i/_{i\in\mathfrak{J}}, \beth_i \in \tau\}$, since $\mathfrak{J}$ represents any arbitrary index set, τ is known to as a PFT in $\mathbf{E}^{\sim}$.

In such instances, the pair $(\mathbf{E}^{\sim}, \tau)$ is termed a PFTS. Each element in τ is an open PFS, and a closed PFS is the complement connected to an open PFS. The discrete PFTS is the topology that includes all PFS.

Definition 2.4. [7, 8]. Consider any two PFS in a PFTSs, denoted as $\beth$ and U. U is defined as a neighborhood of $\beth$ if an open PFS $\aleph$ exist, such that $\beth \subseteq \aleph \subseteq U$.

Definition 2.5. [6,11,12]. Consider two points η and $\bar{\omega}$ in $\mathbf{E}^{\sim}$, consisting of points with integer coordinates. A path ρ connecting η to $\bar{\omega}$ is defined as a finite sequence $\eta = \eta_0, \eta_1, \eta_2, ..., \eta_n = \bar{\omega}$, where each successive point η_i is either 4-connected or 8-connected to η_{i-1}, for $1 \leq i \leq n$.

Definition 2.6. [6,11,12]. If everything on a path from η to $\bar{\omega}$ is points from η^*, where η^* is any subset of E, then points η and $\bar{\omega}$ are connected in S.

Definition 2.7. [9]. Consider $\mathbf{E}^{\sim}$ as a rectangular array (RA) of integer-coordinate points in the Euclidean plane $\sum$ and let $\beth = (\mu_\beth, \nu_\beth)$ be a PFS of $\sum$. Then, the degrees of M and NM of the Pythagorean fuzzy digital (PFD) subset of $\mathbf{E}^{\sim}$, depicted as $\beth\sim$ are defined as follows: $\mu_{\beth\sim}(\eta) = \max\{\mu_\beth(\mathbf{r}^*)|\mathbf{r}^* \in \eta^*\}$ and $\nu_{\beth\sim}(\mathbf{r}^*)|\mathbf{r}^* \in \eta^*$. Here $\mathbf{r}^*$ is the subset of the plane and $\eta*$ is the open unit square has the center η.

Definition 2.8. [5]. Let $(\mathbf{E}^{\sim}, \tau)$ be an intuitionistic fuzzy Hausdorff space. If there are finite collections of intuitionistic fuzzy sets $\{\mathfrak{I}_\mathfrak{i}\}_{i=1}^n$ such that $\cap_{i=1}^n \mathfrak{I}_\mathfrak{i} \neq 0$, for all intuitionistic open sets of $(\mathbf{E}^{\sim}, \tau)$ the system $\Gamma = \{\beth_i\}_{i\in\Delta}$ is $\mathcal{IFC}$. Suppose the system Γ is not able to be incorporated into any larger intuitionistic fuzzy-centered system, it is referred to as a maximum intuitionistic fuzzy-centered system or intuitionistic fuzzy end.

Definition 2.9. [5]. Define $\Gamma_{\mathbf{E}^{\sim}} = \{\Gamma_i : i \in \Delta\}$, where Γ_i's represent intuitionistic fuzzy-centered systems in $(\mathbf{E}^{\sim}, \tau)$. Another term for them is intuitionistic fuzzy-centered points. If the family τ_Γ has been determined to satisfy the following axioms, it is considered an intuitionistic fuzzy-centered structure:

- $\emptyset, \Gamma_{\mathbf{E}^{\sim}} \in \tau_\Gamma$.

- For any $\{\beth_i / i \in \mathfrak{J}, \beth_i \in \tau_\Gamma\}, \cap_{i=1}^{n} \beth_i \in \tau_\Gamma$
- $\cup_{i=1} \beth_i \in \tau_\Gamma$ for any $\{\beth_i /_{i \in \mathfrak{J}}, \beth_i \in \tau_\Gamma\}$

Here $(\Gamma_{\mathbf{E}^\sim}, \tau_\Gamma)$ is called intuitionistic fuzzy centred $(\mathcal{IFC})$ structure space. They are known as $\mathcal{IFC}$ open members of $(\Gamma_{\mathbf{E}^\sim}, \tau_\Gamma)$. An open set with $\mathcal{IFC}$ is complemented by an $\mathcal{IFC}$ closed set.

Definition 2.10. [5]. An $\mathcal{IFC}$ open cover of $\mathcal{IFC}$ subset $\aleph$ of $\Gamma_{\mathbf{E}^\sim}$ is defined as a collection $\{\Gamma_i : i \in \Delta\}$ of $\mathcal{IFC}$ open sets in an $\mathcal{IFC}$ structure space $(\Gamma_{\mathbf{E}^\sim}, \tau_\Gamma)$ if $\aleph \subset \cup\{\Gamma_i : i \in \Delta\}$.

Definition 2.11. [5]. An $\mathcal{IFC}$ structure space $(\Gamma_{\mathbf{E}^\sim}, \tau_\Gamma)$ is said to be $\mathcal{IFC}$ compact if every $\mathcal{IFC}$ open cover of $\Gamma_{\mathbf{E}^\sim}$ has a finite subcover.

Definition 2.12. [5]. Assume that $(\Gamma_{\mathbf{E}^\sim}, \tau_\Gamma)$ and $(\Gamma_{\mathfrak{F}}, \upsilon_\Gamma)$ are any two intuitionistic fuzzy-centered structure spaces. Given that $\mathcal{F}^{-1}(L)$ is an $\mathcal{IFC}$ open in (Γ_E, τ_Γ) for any intuitionistic fuzzy centered open set L in $(\Gamma_F, \upsilon_\Gamma)$, then $\mathcal{F} : (\Gamma_E, \tau_\Gamma) \to (\Gamma_F, \upsilon_\Gamma)$ is an intuitionistic fuzzy centered continuous function.

Definition 2.13. [5]. Let $\Gamma \in \Gamma_E$, and let (Γ_E, τ_Γ) be an intuitionistic fuzzy-centered structure space. If there is an $\mathcal{IFC}$ open set δ such that $\Gamma \in \delta \subseteq \rho$, then an $\mathcal{IFC}$ subset $\rho \subseteq \Gamma_E$ is considered an $\mathcal{IFC}$ neighbourhood.

Definition 2.14. [5]. $\mathrm{IFC}_\Gamma\, \mathrm{cl}(\mathfrak{J}) = \Gamma_E$ indicates that a subset $\mathfrak{J}$ of $\Gamma_{\mathfrak{J}}$ is an $\mathcal{IFC}$ dense

Definition 2.15. [10]. For any two PFD sets $\beth, \aleph$ the operators # and $*$ be are defined and denoted as

- $\beth \# \aleph = \{y / \mu_\beth(y)\mu_\aleph(y), \max(\upsilon_\beth(y), \upsilon_\aleph(y))\}$
- $\beth * \aleph = \{y / \max(\mu_\beth(y), \mu_\aleph(y)), \upsilon_\beth(y)\upsilon_\aleph(y)\}$

Definition 2.16. [10]. Let $\mathbf{E}^\sim$ be the RA of integer coordinates points and τ be the family of PFDS then $(\mathbf{E}^\sim, \tau)$ is known as $\mathcal{PFDFTS}$ if the following requirements satisfy

(i) $O^*, \mathbf{E}^{\sim *} \in \tau$

(ii) For any $\{\sqsupset_i/i = 1, 2, \cdots n, \sqsupset_i \in \tau\}$, $\#\sqsupset_i \in \tau$

(iii) For any arbitrary family $*\sqsupset_i \in \tau$.

Every element of $(\mathbf{E}^{\sim}, \tau)$ is said to be a PFD feeble open sets and its complement is said to be a PFD feeble closed sets, where $O^* = \{< y, 0, 1 > /y \in \mathbf{E}^{\sim}\}$ and $\mathbf{E}^{\sim *} = \{< y, 1, 0 > /y \in \mathbf{E}^{\sim}\}$.

Definition 2.17. [10]. Consider $\sqsupset = \{X, \mathfrak{G}_1, \mathfrak{G}_2\}$ as a PFD set in $\mathbf{E}^{\sim}$ and $(\mathbf{E}^{\sim}, \tau)$ as a $\mathcal{PFDFTS}$. Then, $\sqsupset$'s closure and interior are determined by

- PFDF $(\mathrm{cl}(\sqsupset)) = \#\{\mathbf{k}^{\sim}\colon \mathbf{k}^{\sim}$ is a PFDF closed set in $\mathbf{E}^{\sim}$ and $\sqsupset \subseteq \mathbf{k}^{\sim}\}$
- PFDF$(\mathrm{int}(\sqsupset)) = *\{\mathbf{G}^{\sim} : \mathbf{G}^{\sim}$ is a PFDF open set in $\mathbf{E}^{\sim}$ and $\mathbf{G}^{\sim} \subseteq \sqsupset\}$

3. Compactification in PFDF centred structure space

An introduction to PDFDCS nets and PDFDCS filters is given in this section. Throughout the present study, we refer lim (limit) as lt and ϵ as ι.

Definition 3.1. Let $(\mathbf{E}^{\sim}, \tau)$ be a PFDF topological space. Then $(\mathbf{E}^{\sim}, \tau)$ is said to be a PFDF Hausdorff space if and only is for every $e_1, e_2 \in \mathbf{E}^{\sim}$, and $e_1 \neq e_2$ implies that $\mathscr{U} =< e, \mu_{\mathscr{U}}, \nu_{\mathscr{U}} >, \Psi =< e, \mu_{\Psi}, \nu_{\Psi} >\in \tau$ with $\mu_{\mathscr{U}}(e_1) = \mathbf{E}^{\sim *}, \nu_{\mathscr{U}}(e_1) = O^*, \mu_{\Psi}(e_2) = \mathbf{E}^{\sim *}, \nu_{\Psi}(e_2) = O^*$ and $\mathscr{U}\#\Psi = O^*$.

Definition 3.2. Assume that the PFDF hausdorff space is $(\mathbf{E}^{\sim}, \tau)$. If for each finite collection of elements in PFDF structure space $\#_{i=1}^{n}\sqsupset_i \neq O^*$, then the collection $\Omega = \{\sqsupset_i\}_{i\in\iota}$ of all PFDF structure open sets of $(\mathbf{E}^{\sim}, \tau)$ is referred to as a PFDF structure centred system.

Definition 3.3. Let $\Omega_{\mathbf{E}^{\sim}} = \{\Omega_i : i \in \iota\}$ where Ω_i's are PGDFCS systems in $(\mathbf{E}^{\sim}, \tau)$ which are also known as PGDFCS points. Then the family τ_{Ω} is known as PFDFCS, if it satisfies the following conditions,

(i) $\emptyset, \Omega_{\mathbf{E}^{\sim}} \in \tau_{\Omega}$

(ii) For any $\{\sqsupset_i/i = 1, 2, \cdots n, \sqsupset_i \in \tau_{\Omega}\}$, $\#\sqsupset_i \in \tau_{\Omega}$

(iii) For any arbitrary family $*\sqsupset_i \in \tau_{\Omega}$.

The pair $(\Omega_{\mathbf{E}^\sim}, \tau_\Omega)$ is known as PFDFCS space. Each members of are said to be PFDFCS open set. The complement of a PFDFCS open set is a PFDFCS closed set.

Definition 3.4. Let $(\Omega_{\mathbf{E}^\sim}, \tau_\Omega)$ be a PFDFCS space and $\daleth \subseteq \Omega_{\mathbf{E}^\sim}$. Then PFDFCS closure and PFDFCS interior of $\daleth$ is defined as

- PFDFCS $\mathfrak{C}_\Omega\,\mathrm{cl}(\daleth) = \#\,\{\mathbf{k}^\sim : \mathbf{k}^\sim$ is a PFDFCS closed set and $\daleth \subseteq \mathbf{k}^\sim\}$
- PFDFCS $\mathfrak{C}_\Omega\,\mathrm{int}(\daleth) = *\{\mathbf{G}^\sim : \mathbf{G}^\sim$ is a PFDFCS open set in $\mathbf{E}^\sim$ and $\mathbf{G}^\sim \subseteq \daleth\}$

Definition 3.5. An collection $\{\Omega_i : i \in \iota\}$ of PFDFCS open sets in a PFDFCS space $(\Omega_{\mathbf{E}^\sim}, \tau_\Omega)$ is said to be PFDFCS open cover of PFDF centred subset $\aleph$ of $\Omega_{\mathbf{E}^\sim}$ if $\aleph \subseteq *\{\Omega_i : i \in \iota\}$ holds

Definition 3.6. The PFDFCS space $(\Omega_{\mathbf{E}^\sim}, \tau_\Omega)$ is known as PFDFCS compact if every PFDF centred open cover of $\Omega_{\mathbf{E}^\sim}$ has a finite subcover.

Definition 3.7. Let $(\Omega_{\mathbf{E}^\sim}, \tau_\Omega)$ and $(\Omega_{\mathfrak{F}}, \varphi_\Omega)$ be any PFDFCS space. Then $g : (\Omega_{\mathbf{E}^\sim}, \tau_\Omega) \to (\Omega_{\mathfrak{F}}, \varphi_\Omega)$ is an PFDFCS continuous function if $g^{-1}(\lambda)$ is a PFDFCS open in $(\Omega_{\mathbf{E}^\sim}, \tau_\Omega)$ for each PFDFCS open set λ in $(\Omega_{\mathfrak{F}}, \varphi_\Omega)$

Definition 3.8. Let $(\Omega_{\mathbf{E}^\sim}, \tau_\Omega)$ be a PFDFCS space and let $\Omega \in \Omega_{\mathbf{E}^\sim}$. Then a PFDFCS $\chi \subseteq \Omega_{\mathbf{E}^\sim}$ is called as PFDFCS neighbourhood if $\exists$ an PFDFCS open set $\mathscr{U}$ such that $\Omega \in \mathscr{U} \subseteq \chi$.

Definition 3.9. A PFDFCS subset $\daleth$ of $\Omega_{\mathbf{E}^\sim}$ is known as PFDFCS dense if PFDFCS $\mathfrak{C}_\Omega\,\mathrm{cl}(\daleth) = \Omega_{\mathbf{E}^\sim}$.

Definition 3.10. Let $\Omega_{\mathbf{E}^\sim}$ be a PFDFCS. Then a nonempty family Y of subsets of $\Omega_{\mathbf{E}^\sim}$ is called PFDFCS filter on $\Omega_{\mathbf{E}^\sim}$ iff the following axioms are satisfied:

(i) $\emptyset \in Y$

(ii) if $\mathbf{F}^\sim \in Y$ and $\mathbf{F}^\sim \subseteq \dot{H}$, then $\dot{H} \in Y$

(iii) if $\mathbf{F}^\sim{}_1, \mathbf{F}^\sim{}_2 \in Y$, then $\mathbf{F}^\sim{}_1 \# \mathbf{F}^\sim{}_2 \in Y$

Definition 3.11. A PFDFCS net in a PFDFCS space $(\Omega_{\mathbf{E}^{\sim}}, \tau_{\Omega}) : \Delta \to \Omega_{\mathbf{E}^{\sim}}$ is symbolized by $\{\Omega_\delta\}_{\delta \in \Delta}$, where Δ is a directed set.

Definition 3.12. Let $\{\Omega_\delta\}_{\delta \in \Delta}$ be a PFDFCS net in a PFDFCS $\Omega_{\mathbf{E}^{\sim}}$ and let $\mathscr{U}$ be a PFDFCS subset of $\Omega_{\mathbf{E}^{\sim}}$. Then the PFDFCS net is said to be (i)in $\mathscr{U}$ iff $\{\Omega_\delta\} \in \mathscr{U}$, $\forall \delta \in \Delta$.(ii) eventually in $\mathscr{U} \iff \exists\, \delta \in \Delta$ for all $\delta \in \Delta$, $\delta \geq \alpha$, such that $\{\Omega_\delta\}$ in $\mathscr{U}$.(iii)often in $\mathscr{U} \iff \forall\, \alpha \in \Delta, \exists\, \delta \in \Delta$, $\delta \geq \alpha$, and $\{\Omega_\delta\}$ in $\mathscr{U}$.

Definition 3.13. If $\{\Omega_\delta\}$ is a PFDFCS net in the PFDFCS $\Omega_{\mathbf{E}^{\sim}}$ and Ω is an PFDFCS element of $\Omega_{\mathbf{E}^{\sim}}$, the PFDFCS net is said to converge into $\Omega \iff$ for $\mathbf{r}^*$ every PFDFCS neighbourhood $\mathbf{r}^*$ of $\Omega, \{\Omega_\delta\}$ is eventually in $\mathbf{r}^*$.

Definition 3.14. A PFDFCS point Ω_1 in $\Omega_{\mathbf{E}^{\sim}}$ is said to be a PFDFCS accumulation point or cluster point of a PFDFCS net iff for every PFDFCS neighbourhood $\mathbf{r}^*$ of Ω_1, the PFDFCS net is frequently in $\mathbf{r}^*$.

Definition 3.15. A PFDFCS net $\{\Omega\}$ in a set $\Omega_{\mathbf{E}^{\sim}}$ is called a PFDFCS universal or a PFDFCS ultranet if for every PFDFCS subset $\beth$ of $\Omega_{\mathbf{E}^{\sim}}$, either $\{\Omega\}$ is essentially in $\beth$ or $\{\Omega\}$ is essentially in $\Omega_{\mathbf{E}^{\sim}} - \beth$.

Definition 3.16. Consider ρ as a family of PFDFCS continuous functions on a PFDFCS space $\Omega_{\mathbf{E}^{\sim}}$. A PFDFCS net $\{\Omega_i\}$ in $\Omega_{\mathbf{E}^{\sim}}$ will be called a PFDFCS ρ net if $\{g(\Omega_i)\}$ converges for each $g \in \rho$.

Definition 3.17. Consider ρ as a set of continuous bounded real valued functions $(\mathcal{BRVF})$ on $\Omega_{\mathbf{E}^{\sim}}$. Let $\Omega_{\mathbf{E}^{\sim}}$ is PFDFCS compact if every ρ net has a PFDFCS cluster point in $\Omega_{\mathbf{E}^{\sim}}$.

Definition 3.18. Let $\Omega_{\mathbf{E}^{\sim}}$ be any arbitrary PFDFCS space, $\mathfrak{C}(\Omega_{\mathbf{E}^{\sim}}) = \{g_\delta \mid \delta \in \iota\}$ the set of continuous, $\mathcal{BRVF}$ on $\Omega_{\mathbf{E}^{\sim}}$. For a $\mathfrak{C} - (\Omega_{\mathbf{E}^{\sim}})$ net $\{\Omega_i\}$, consider $G_{\{\Omega_i\}} = \{\mathbf{r}^* \mid \mathbf{r}^*$ is PFDFCS open in $\Omega_{\mathbf{E}^{\sim}}$ and $\{\Omega_i\}$ is eventually in $\mathbf{r}^*\}$. It is clear that $G_{\{\Omega_i\}}$ is a PFDFCS open filter, and for any $g_\delta \in \mathfrak{C} - (\Omega_{\mathbf{E}^{\sim}})$, any $\iota > 0$, $g_\delta^{-1}((r_\delta - \iota, r_\delta + \iota)) \in G_{\{\Omega_i\}}$, where $r_\delta = lt\{g_\delta(\Omega_i)\}$. $G_{\{\Omega_i\}}$ is the PFDFCS open filter on $\Omega_{\mathbf{E}^{\sim}}$ induced by $\{\Omega_i\}$.

Definition 3.19. If G is a PFDFCS filter on $\Omega_{\mathbf{E}^{\sim}}$, let $\Delta_G = \{(\Omega, \mathfrak{F}) \mid \Omega \in \mathfrak{F} \in G\}$. Then Δ_G is directed by the relation $(\Omega_1, \mathfrak{F}_1) \leq (\Omega_2, \mathfrak{F}_2) iff \mathfrak{F}_2 \subset \mathfrak{F}_1$., so the map $N : \Delta_G \to \Omega_{\mathbf{E}^{\sim}}$ defined by $N(\Omega, \mathfrak{F}_1) = \Omega$ is a PFDFCS net in $\Omega_{\mathbf{E}^{\sim}}$. This is known as the PFDFCS net with respect to G.

Definition 3.20. A PFDFCS filter G_Ω converges to Ω in $\Omega_{\mathbf{E}^\sim}$ if the PFDFCS net based on G_Ω converges to Ω.

Definition 3.21. Consider Ω be a PFDFCS open filter on $\Omega_{\mathbf{E}^\sim}$, $\{\Omega_i\}$ the PFDFCS net respect to Ω, and $\Gamma = \{\mathbf{r}^* \mid \mathbf{r}^*$ is PFDFCS open in $\Omega_{\mathbf{E}^\sim}$ and $\{\Omega_i\}$ is eventually in $\mathbf{r}^*\}$. Then $\Gamma = \Omega$.

Definition 3.22. For each $\mathfrak{C} - (\Omega_{\mathbf{E}^\sim})$ net $\{\Omega_i\}$ in $\Omega_{\mathbf{E}^\sim}$, let $\{\sigma_K^{\Omega_i}\}$ be the PFDFCS net respect to the PFDFCS open filter $G_{\{\Omega_i\}}$ caused by $\{\Omega_i\}$. *(a)* $\{\sigma_k^{\Omega_i}\}$ is uniquely determined by $G_{\{\Omega_i\}}$, and $G_{\{\Omega_i\}} = G_{\{\Omega_j\}}$ iff $\{\sigma_k^{\Omega_i}\} = \{\sigma_k^{(\Omega_j)}\}$.

(b) $G_{\{\Omega_i\}} = G_{\sigma_k^{\Omega_i}} = \{\mathscr{U} \mid \mathscr{U}$ is PFDFCS open in $\Omega_{\mathbf{E}^\sim}$ and $\{\sigma_k^{\Omega_i}\}$ is essentially in $\mathscr{U}$ $\}$. (c) $\{\sigma_k^{\Omega_i}\}$ is a $\mathfrak{C} - (\Omega_{\mathbf{E}^\sim})$-net, and $lt\{g_\delta(\sigma_k^{\Omega_i})\} = \{g_\delta(\Omega_i)\}$ for all $g_\delta in \mathfrak{C} - (\Omega_{\mathbf{E}^\sim})$.

The following are equivalent

(i)$\{\sigma_k^{\Omega_i}\}$ converges to Ω, (ii) $\{\Omega_i\}$ converges to Ω and (iii) $G_{\{\Omega_i\}}$ converges to Ω.

Let $\mathfrak{F}_\Omega = \{\{\sigma_k^{\Omega_i}\} | G_{\{\Omega_i\}}$ is a $\mathfrak{C} - (\Omega_{\mathbf{E}^\sim})$ net that does not converge in $\Omega_{\mathbf{E}^\sim}, \{\sigma_k^{\Omega_i}\}$ is the PFDFCS net based on PFDFCS filter $G_{\{\Omega_i\}}\}$, $\Omega_{\mathbf{E}^\sim}\hat{} = \Omega_{\mathbf{E}^\sim} * \mathfrak{F}_\Omega$, the disjoint union of $\Omega_{\mathbf{E}^\sim}$ and $\mathfrak{F}_\Omega$. For each PFDFCS open set $\mathbf{r}^* \subset \Omega_{\mathbf{E}^\sim}$, define $\mathbf{r}^{*\hat{}} \subset \Omega_{\mathbf{E}^\sim}\hat{}$ to be the set $\mathbf{r}^{*\hat{}} = \mathbf{r}^* * \{\{\sigma_k^{\Omega_i}\} : \{\sigma_k^{\Omega_i}\} \in \mathfrak{F}_\Omega$ and $\{\sigma_k^{\Omega_i}\}$ is eventually in $\mathbf{r}^*\}$. It is clear that if $\mathbf{r}^* \subset \mathbf{Q}^\sim$ then $\mathbf{r}^{*\hat{}} \subset \mathbf{Q}^{\sim\hat{}}$.

Proposition 3.23. *For any two PFDFCS open sets* $\mathbf{r}^*$ *and* $\mathbf{Q}^\sim$ *in* $\Omega_{\mathbf{E}^\sim}$*,* $(\mathbf{r}^* \# \mathbf{Q}^\sim)\hat{} = \mathbf{r}^{*\hat{}} \# \mathbf{Q}^{\sim\hat{}}$.

Proof. Let $\Omega_1 \in (\mathbf{r}^* \# \mathbf{Q}^\sim)\hat{} \# \mathfrak{F}_\Omega$, then $\Omega_1 = \{\sigma_k^{\Omega_i}\}\hat{}$ and $\{\sigma_k^{\Omega_i}\}$ is eventually in $\mathbf{r}^* \# \mathbf{Q}^\sim$. This implies that $\{\sigma_k^{\Omega_i}\}$ is eventually in $\mathbf{r}^*$ and in $\mathbf{Q}^\sim$, thus $\{\sigma_k^{\Omega_i}\}\hat{}$ is in $\mathbf{r}^{*\hat{}} \# \mathbf{Q}^{\sim\hat{}}$. For the other direction, if $\Omega_1 \in (\mathbf{r}^{*\hat{}} \# \mathbf{Q}^{\sim\hat{}}) \# \mathfrak{F}_\Omega$, then $\Omega_1 = \{\sigma_k^{\Omega_i}\}\hat{}$ and $\{\sigma_k^{\Omega_i}\}$ is eventually in $\mathbf{r}^*$ and in $\mathbf{Q}^\sim$, so $\{\sigma_k^{\Omega_i}\}$ is eventually in $\mathbf{r}^* \# \mathbf{Q}^\sim$. Thus, Ω_1 is in $(\mathbf{r}^* \# \mathbf{Q}^\sim)\hat{}$. □

Proposition 3.24. *Let* $\mathscr{B}^* = \{\mathbf{r}^{*\hat{}} | \mathbf{r}^*$ *be a PFDFCS open in* $\Omega_{\mathbf{E}^\sim}\}$*. Then* $\mathscr{B}^*$ *is a PFDFCS base for a PFDFCS structure on* $\hat{\Omega}_{\mathbf{E}^\sim}$ *if*

(a) $\hat{\Omega}_{\mathbf{E}^\sim} = \{\mathbf{r}^{*\hat{}} | \mathbf{r}^{*\hat{}} \in \mathscr{B}^*\}$

(b) whenever $\mathbf{r}^{*\hat{}}, \mathbf{Q}^{\sim\hat{}} \in \mathscr{B}^*$ *with* $\Omega_1 \in \mathbf{r}^{*\hat{}} \# \mathbf{Q}^{\sim\hat{}}$*, there is some* $\hat{\nu} = (\mathbf{r}^* \# \mathbf{Q}^\sim)\hat{} \in \mathscr{B}^*$*, such that* $\Omega_1 \in \hat{\nu} \subset \mathbf{r}^{*\hat{}} \# \mathbf{Q}^{\sim\hat{}}$.

Proof. (i) $\hat{\Omega_{\mathbf{E}^\sim}} = \{\hat{\mathbf{r}^*}|\hat{\mathbf{r}^*} \in \mathscr{B}^*\}$. Let $\Omega_1 \in \mathfrak{F}_\Omega$, then $\Omega_1 = \hat{\{\sigma_k^{\Omega_i}\}}$. For any $g_\delta \in \mathfrak{C} - (\Omega_{\mathbf{E}^\sim})$, let $r_\delta = lt\{g_\delta(\sigma_k^{\Omega_i})\}$. Then $\hat{\{\sigma_k^{\Omega_i}\}}$ is eventually in $g_\delta^{-1}((r_\delta - \iota, r_\delta + \iota))$ for any $\iota > 0$, that is, $\hat{\{\sigma_k^{\Omega_i}\}}$ is in $g_\delta^{-1}((r_\delta - \iota, r_\delta + \iota))$, for any $\iota > 0$. Thus, $\mathfrak{F}_\Omega \subset \#\{\hat{\mathbf{r}^*}|\hat{\mathbf{r}^*} \in \mathscr{B}^*\}$. Therefore, $\hat{\Omega_{\mathbf{E}^\sim}} \subset \#\{\hat{\mathbf{r}^*}|\hat{\mathbf{r}^*} \in \mathscr{B}^*\}$. For $\{\hat{\mathbf{r}^*}|\hat{\mathbf{r}^*} \in \mathscr{B}^*\} \subset \hat{\Omega_{\mathbf{E}^\sim}}$, it is clear.

(ii) If $\Omega_1 \in \hat{\mathbf{r}^*}\#\hat{\mathbf{Q}^\sim}$ for any $\hat{\mathbf{r}^*}$ and $\hat{\mathbf{Q}^\sim}$ in $\mathscr{B}^*$, since $\hat{(\mathbf{r}^*\#\mathbf{Q}^\sim)}$ is in $\mathscr{B}^*$ and $\hat{(\mathbf{r}^*\#\mathbf{Q}^\sim)} = \hat{\mathbf{r}^*}\#\hat{\mathbf{Q}^\sim}$, thus $\Omega_1 \in \hat{(\mathbf{r}^*\#\mathbf{Q}^\sim)} \subset \hat{\mathbf{r}^*}\#\hat{\mathbf{Q}^\sim}$.

□

Note 3.25. *Provide $\hat{\Omega_{\mathbf{E}^\sim}}$ with PFDFCS induced by PFDFCS base $\mathscr{B}^*$. For each $g_\delta \in \mathfrak{C} - (\Omega_{\mathbf{E}^\sim})$, define $\hat{g_\delta} : \hat{\Omega_{\mathbf{E}^\sim}} \to \mathbb{R}$ by setting $\hat{g_\delta}(\Omega_2) = g_\delta(\Omega_2)$ if $\Omega_2 \in \Omega_{\mathbf{E}^\sim}$; $\hat{g_\delta}(\hat{\sigma_k^{\Omega_i}}) = lt\{g_\delta(\sigma_k^{\Omega_i})\}$ for $\hat{\{\sigma_k^{\Omega_i}\}} \in \mathfrak{F}_\Omega$. Evidently $\hat{g_\delta}$ is precisely defined and is a $\mathcal{BRVF}$ on $\hat{\Omega_{\mathbf{E}^\sim}}$.*

Proposition 3.26. *For any $g_\delta \in \mathfrak{C} - (\Omega_{\mathbf{E}^\sim})$, $\hat{g_\delta}$ is a continuous $\mathcal{BRVF}$ on $\hat{\Omega_{\mathbf{E}^\sim}}$.*

Proof. To demonstrate $\hat{g_\delta}$'s continuation anywhere $\Omega_3 \in \hat{\Omega_{\mathbf{E}^\sim}}$, consider $y_\delta = \hat{g_\delta}(\Omega_3)$. It will be shown that for any $\iota > 0$, a PFDFCS open set $\hat{\mathbf{r}^*} \in \mathscr{B}^*$ exist to the extent that $\Omega_3 \in \hat{\mathbf{r}^*} \subset (\hat{g_\delta})^{-1}((y_\delta - \iota, y_\delta + \iota))$. Let $\mathbf{r}^* = (g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2))$. If $\Omega_3 \in \Omega_{\mathbf{E}^\sim}$, since $g_\delta(\Omega_3) = \hat{g_\delta}(\Omega_3) = y_\delta$, thus $\Omega_3 \in (g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2)) \subset \hat{((g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2)))}$. If $\Omega_3 \in \mathfrak{F}_\Omega$, then $\Omega_3 = \hat{\{\sigma_k^{\Omega_i}\}}$. Since $y_\delta = \hat{g_\delta}(\Omega_3) = lt\{g_\delta\{\sigma_k^{\Omega_i}\}\}$, so $\{\sigma_k^{\Omega_i}\}$ is eventually in $(g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2))$; that is, $\Omega_3 = \hat{\{\sigma_k^{\Omega_i}\}} \in \hat{((g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2)))}$.

Finally, we show that $\hat{((g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2)))} \subset (\hat{g_\delta})^{-1}((y_\delta - \iota, y_\delta + \iota))$. If $\Omega_1 \in \Omega_{\mathbf{E}^\sim}\#\hat{((g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2)))}$, then $\Omega_1 \in (g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2))$, that is, $\hat{g_\delta}(\Omega_1) = g_\delta(\Omega_1) \in (y_\delta - \iota, y_\delta + \iota)$. So $\Omega_1 \in (\hat{g_\delta})^{-1}((y_\delta - \iota, y_\delta + \iota))$. If $\Omega_2 \in \hat{((g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2)))}\#\mathfrak{F}_\Omega$, then $\Omega_2 = \hat{\{\sigma_k^{\Omega_i}\}}$, and $\{\sigma_k^{\Omega_i}\}$ is eventually in $(g_\delta)^{-1}((y_\delta - \iota/2, y_\delta + \iota/2))$, thus $\hat{g_\delta}(\Omega_2) = lt\{g_\delta\{\sigma_k^{\Omega_i}\}\} \in (y_\delta - \iota/2, y_\delta + \iota/2) \subset (y_\delta - \iota, y_\delta + \iota)$. That is, $\Omega_2 \in (\hat{g_\delta})^{-1}((y_\delta - \iota, y_\delta + \iota))$. □

Proposition 3.27. *Let $\mathbf{k}^\sim : \Omega_{\mathbf{E}^\sim} \to \hat{\Omega_{\mathbf{E}^\sim}}$ be the mapping such thatt $\mathbf{k}^\sim(\Omega_1) = \Omega_1$. Then $\mathbf{k}^\sim$ is a PFDFCS continuous function from $\Omega_{\mathbf{E}^\sim}$ into $\hat{\Omega_{\mathbf{E}^\sim}}$.*

Proof. For any PFDFCS open set $\mathbf{r}^{*\hat{}}$ in $\mathscr{B}^*$, $\mathbf{k}^{\sim -1}(\mathbf{r}^{*)} = \mathbf{r}^*$ is PFDFCS open in $\Omega_{\mathbf{E}^\sim}$, so K is PFDFCS continuous on $\Omega_{\mathbf{E}^\sim}$. □

Proposition 3.28. *For any Ω_2 in $\Omega_{\mathbf{E}^\sim}\hat{}-\Omega_{\mathbf{E}^\sim}$ with $\Omega_2 = \{\sigma_k^{\Omega_i}\}\hat{}$, $\{\mathbf{k}^\sim(\sigma_k^{\Omega_i})\}$ converges to $\Omega_2 = \{\sigma_k^{\Omega_i}\}\hat{}$.*

Proof. Let $\mathbf{r}^{*\hat{}}$ be any PFDFCS open set within $\mathscr{B}^*$ that contains Ω_2. Then the $\{\sigma_k^{\Omega_i}\}$ eventually belongs to $\mathbf{r}^*$ in the space $\Omega_{\mathbf{E}^\sim}$. It follows that $\{\mathbf{k}^\sim(\sigma_k^{\Omega_i})\}$ also eventually lies in $\mathbf{r}^{*\hat{}}$, and heance $\{\mathbf{k}^\sim(\sigma_k^{\Omega_i})\}$ converges to $\Omega_2 = \{\sigma_k^{\hat{\Omega}_i}\}$. □

Proposition 3.29. *$K(\Omega_{\mathbf{E}^\sim})$ is PFDFCS dense in $\Omega_{\mathbf{E}^\sim}\hat{}$.*

Proof. In the case of Ω_2 in $\Omega_{\mathbf{E}^\sim}\hat{} - \Omega_{\mathbf{E}^\sim}$, $\Omega_2 = \{\sigma_k^{\Omega_i}\}\hat{}$. Proposition 4.28 implies that $\{k(\sigma_k^{\Omega_i})\}$ converges to $\Omega_2 = \{\sigma_k^{\Omega_i}\}\hat{}$. Thus, PFDFCS $\mathfrak{C}\Omega\,\mathrm{cl}(k(\Omega_{\mathbf{E}^\sim})) = \Omega_{\mathbf{E}^\sim}\hat{}$. □

Note 3.30. *For convenience, consider $\mathfrak{C} = \{g_{\delta}\hat{} \mid \delta \in \iota\}$ represent $\{g_{\delta}\hat{} \mid g_\delta \in \mathfrak{C} - (\Omega_{\mathbf{E}^\sim})\}$. Consider any $\mathfrak{C}$ net $\{\Omega_i\}$ in $\Omega_{\mathbf{E}^\sim}\hat{}$, consider $A = \{\mathscr{U} \mid \mathscr{U}$ is PFDFCS open in $\Omega_{\mathbf{E}^\sim}\hat{}$ and $\{\Omega_i\}$ is eventually in $\mathscr{U}$ $\}$ and $\Lambda = \{\mathbf{r}^* \mid \mathbf{r}^*$ is PFDFCS open in $\Omega_{\mathbf{E}^\sim}$ and $\mathbf{r}^{*\in} A\}$.*

Proposition 3.31. *In the case of $\mathfrak{C}$ net $\{\Omega_i\}$ in $\Omega_{\mathbf{E}^\sim}\hat{}$, let $r_\delta = lt\{g_{\delta}\hat{}(\Omega_i)\}$ for each $g_{\delta}\hat{} \in \mathfrak{C}$. Then, for any $\iota > 0$, $(g_{\delta}\hat{})^{-1}((r_\delta - \iota, r_\delta + \iota)) \subset (g_\delta^{-1}((r_\delta - \iota, r_\delta + \iota)))\hat{}$.*

Proof. Let $\Omega_3 \in (g_{\delta}\hat{})^{-1}((r_\delta - \iota, r_\delta + \iota))$, then $g_{\delta}\hat{}(\Omega_3) \in (r_\delta - \iota, r_\delta + \iota)$. If $\Omega_3 = k(\Omega_1) = \Omega_1$ for all $\Omega_1 \in \Omega_{\mathbf{E}^\sim}$, since $g_\delta(\Omega_1) = g_{\delta}\hat{}(\Omega_3)$, so Ω_1 is in $(g_\delta^{-1}((r_\delta - \iota, r_\delta + \iota)))\hat{}$. If $\Omega_3 = \{\sigma_k^{\Omega_i}\hat{}\} \in \mathfrak{F}_\Omega$, then $lt\{g_\delta\{\sigma_k^{\Omega_i}\}\} = g_{\delta}\hat{}(\Omega_3) \in (r_\delta - \iota, r_\delta + \iota)$. This implies that $\{\sigma_k^{\Omega_i}\}$ is eventually in $(g_\delta)^{-1}((r_\delta - \iota, r_\delta + \iota))$, thus $\{\sigma_k^{\hat{\Omega}_i}\}$ is in $(g_\delta^{-1}((r_\delta - \iota, r_\delta + \iota)))\hat{}$. □

Proposition 3.32. *In the case of $\mathfrak{C}$ net $\{\Omega_i\}$ in $\Omega_{\mathbf{E}^\sim}\hat{}$, let $r_\delta = lt\{\hat{g_\delta}(\Omega_i)\}$ for each $\hat{g_\delta} \in \mathfrak{C}$. Then for any $\iota > 0$, $(\hat{g_\delta})^{-1}((r_\delta - \iota, r_\delta + \iota)) \in A$ and $(g_\delta)^{-1}((r_\delta - \iota, r_\delta + \iota)) \in \Lambda$.*

Proof. It is clear that $(\hat{g_\delta})^{-1}((r_\delta - \iota, r_\delta + \iota)) \in A$. By Proposition 4.31, $\{\Omega_i\}$ is eventually in $(g_\delta^{-1}((r_\delta - \iota, r_\delta + \iota)))\hat{}$, thus, $g_\delta^{-1}((r_\delta - \iota, r_\delta + \iota)) \in \Lambda$. □

Proposition 3.33. *A and Λ are PFDFCS open filters on $\Omega_{\mathbf{E}^\sim}\hat{}$ and $\Omega_{\mathbf{E}^\sim}$, respectively.*

Proof. Based on Proposition 4.23 and Corollary 4.32, it can be inferred that A is a PFDFCS filter on $\Omega_{\mathbf{E}^{\sim}}\hat{}$. By Corollary 4.32, $\Lambda \neq \emptyset$. If $\mathbf{r}^*$ and $\mathbf{Q}^{\sim}$ are PFDFCS open sets in Λ, then $\hat{\mathbf{r}^*}$ and $\hat{\mathbf{Q}^{\sim}}$ are in A. Since $(\mathbf{r}^*\#\mathbf{Q}^{\sim})\hat{} = \hat{\mathbf{r}^*}\#\hat{\mathbf{Q}^{\sim}}$ and $\hat{\mathbf{r}^*}\#\hat{\mathbf{Q}^{\sim}} \in A$, thus $\mathbf{r}^*\#\mathbf{Q}^{\sim} \in \Lambda$. If ν is PFDFCS open and $\nu \supset \mathscr{U}$, then $\hat{\nu} \supset \hat{\mathscr{U}}$. It indicates that $\hat{\nu} \in A$ and so, $\nu \in \Lambda$. □

Proposition 3.34. *The $\mathfrak{C}$ net $\{\Omega_i\}$ is convergent $\Omega_{\mathbf{E}^{\sim}}\hat{}$.*

Proof. Consider $\{\sigma_k\}$ is the PFDFCS net based on Λ. Since for any $\delta \in \iota$ and $\iota > 0$, $(g_\delta)^{-1}((r_\delta - \iota, r_\delta + \iota))$ is in Λ, where $r_\delta = lt\{\hat{g_\delta}(\Omega_i)\}$. Thus, $\{g_\delta(\sigma_k)\}$ dense towards r_δ $\forall$ $\delta \in \iota$; thus, $\{\sigma_k\}$ can be identify as $\hat{\mathfrak{C}}(\Omega_{\mathbf{E}^{\sim}})$ net. As the $\hat{\mathfrak{C}}(\Omega_{\mathbf{E}^{\sim}})$ net $\{\sigma_k\}$ induces the PFDFCS open filter $G_{\{\sigma_k\}}$ which is contained in Λ, hence if $\{\sigma_k^{\Omega_k}\}$ is the PFDFCS net based on $G\{\sigma_k\}$, then $\{\sigma_k\} = \{\sigma_k^{\Omega_k}\}$.

Case 1: If $\{\sigma_k\}$ converges to a PFDFCS point Ω in $\Omega_{\mathbf{E}^{\sim}}$. Let $\hat{\mathbf{r}^*}$ be a PFDFCS open set in $\mathscr{B}^*$ containing $\mathbf{k}^{\sim}(\Omega)$, following that Ω lies in $\mathbf{r}^*$ and $\mathbf{r}^*$ is PFDFCS open in $\Omega_{\mathbf{E}^{\sim}}$. Since $\{\sigma_k\}$ converges to Ω, by Definition 3.20, $\mathbf{r}^*$ lies in Λ, and therefore, $\hat{\mathbf{r}^*}$ lies in A. It suggests that $\{\Omega_i\}$ dense toward $\mathbf{k}^{\sim}(\Omega)$ in $\Omega_{\mathbf{E}^{\sim}}\hat{}$.

Case 2: If there is no convergence of $\{\sigma_k\}$ in $\Omega_{\mathbf{E}^{\sim}}$, following that $\{\hat{\sigma_k}\} = \{\sigma_k^{(\hat{\Omega}_k)}\}$ is in $\mathfrak{F}_\Omega$. For any $\hat{\mathbf{r}^*}$ in $\mathscr{B}^* \supset \{\sigma_k^{(\Omega_k)}\}$, it is eventually in $\mathbf{r}^*$ in $\Omega_{\mathbf{E}^{\sim}}$. Definition 3.21 implies that $\mathbf{r}^*$ is in Λ, and so, $\hat{\mathbf{r}^*}$ is in A. Thus, $\{\Omega_i\}$ converges to $\{\sigma_k^{(\hat{\Omega}_k)}\} = \{\hat{\sigma_k}\}$ in $\Omega_{\mathbf{E}^{\sim}}\hat{}$. □

Proposition 3.35. $(\Omega_{\mathbf{E}^{\sim}}\hat{}, \mathfrak{K})$ *is a PFDFCS compactification of* $\Omega_{\mathbf{E}^{\sim}}$.

Proof. given that $\mathfrak{C}$ is a collection of continuous $\mathcal{BRVF}$ on $\Omega_{\mathbf{E}^{\sim}}\hat{}$ and every $\mathfrak{C}$ net $\{\Omega_i\}$ dense towards $\Omega_{\mathbf{E}^{\sim}}\hat{}$, Definition 3.17 implies that $\Omega_{\mathbf{E}^{\sim}}\hat{}$ is PFDFCS compact. Hence, by Proposition 3.29, it follows that $(\Omega_{\mathbf{E}^{\sim}}\hat{}, \mathbf{k}^{\sim})$ is a PFDFCS compactification of $\Omega_{\mathbf{E}^{\sim}}$. □

Proposition 3.36. *Let $\mathfrak{C}(\Omega_{\mathbf{E}^{\sim}}\hat{})$ denote the collection of all continuous $\mathcal{BRVF}$ defined on $\Omega_{\mathbf{E}^{\sim}}\hat{}$. Then $\mathfrak{C}(\Omega_{\mathbf{E}^{\sim}}\hat{}) = \mathfrak{C} = \{\hat{g}_\delta \mid g_\delta \in \mathfrak{C}(\Omega_{\mathbf{E}^{\sim}}\hat{})\}$.*

Proof. Let $p \in \mathfrak{C}(\Omega_{\mathbf{E}^{\sim}}\hat{})$. Since $\Omega_{\mathbf{E}^{\sim}}\hat{}$ is PFDFCS compact, $p \circ \mathbf{k}^{\sim} \in \mathfrak{C}(\Omega_{\mathbf{E}^{\sim}}\hat{})$. By Proposition 3.27, 3.28, and the continuity of p, we have $(p \circ \hat{}\mathbf{k}^{\sim})(\sigma_k^{\Omega_i})\hat{} = lt(p \circ \mathbf{k}^{\sim})(\sigma_k^{\Omega_i}) = ltp(\mathbf{k}^{\sim}(\sigma_k^{\Omega_i})) = p\left(lt\{\mathbf{k}^{\sim}(\sigma_k^{\Omega_i})\}\right) = p(\sigma_k^{\Omega_i})\hat{}$ for all $\sigma_k^{\Omega_i}\hat{} \in \mathfrak{F}_\Omega$ and $(p \circ \hat{}\mathbf{k}^{\sim})(\mathbf{k}^{\sim}(\Omega)) = (p \circ \hat{}\mathbf{k}^{\sim})(\Omega) = p(\mathbf{k}^{\sim}(\Omega))$ for all $\Omega \in \Omega_{\mathbf{E}^{\sim}}\hat{}$. Hence, $\mathfrak{C}(\Omega_{\mathbf{E}^{\sim}}\hat{}) \subset \mathfrak{C} = \{\hat{g}_\delta \mid g_\delta \in \mathfrak{C}(\Omega_{\mathbf{E}^{\sim}}\hat{})\}$. □

Conclusion

In both physics and mathematics, compactification is an effective technique. It aids in filling in and extending gaps in mathematical analysis. In physics, it enables the explanation of our observed 4D universe by higher-dimensional theories.A finer understanding of PFDFCS space and its characteristics will result from this research study. It improves the theoretical methods applied to different compactification circumstances. A solid theoretical foundation for comprehending PFDFCS compactness in many kinds of environments is offered by the interactions between PFDFCS nets and PFDFCS filters. Similarly, convergence and compactness can be framed using PFDFCS nets and filters. Compactifications will be used in a variety of industries as a result.

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