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Traveling wave solutions to the class of nonlinear partial differential equations with a Riccati-Bernoulli sub-ODE method

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Abstract

This research investigates the application of the Riccati-Bernoulli (R-B) sub-ODE technique to analyze the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) and the three-dimensional Potential Yu-Toda-Sasa-Fukuyama (3D-PYTSF) equations. By integrating traveling wave transformations with the R-B framework, both equations are systematically reduced to algebraic systems. Various soliton solutions, including kink-type, singular, and periodic solutions, are derived. To visualize the physical significance of these solutions, 3D plots, density graphs, and contour diagrams are generated based on appropriate parameter

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selections. The R-B sub-ODE method proves to be an effective yet straightforward technique for addressing nonlinear partial differential equations (NLPDEs).

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1. Introduction

Solitary wave solutions have long been a subject of fascination and importance in the realm of nonlinear partial differential equations (PDEs). These intriguing solutions represent localized, self-sustaining waveforms that retain their shape and amplitude while propagating through a medium. The study of solitary waves is not only of theoretical interest but also holds significant practical implications, finding applications in diverse fields such as fluid dynamics, plasma physics, optics, and nonlinear optics, among others.

In recent years, considerable attention has been directed towards understanding and characterizing solitary waves in various classes of nonlinear PDEs. This stems from the inherent complexity and ubiquity of nonlinear phenomena observed in physical systems. Nonlinear PDEs, governing the evolution of complex systems, often exhibit solitary waves as solutions, providing valuable insights into the underlying dynamics.

The investigation of traveling wave solutions of partial differential equations with non-linear characteristics (NLPDEs) plays an important role in optical fibers, engineering, chemical reaction, plasma and particle physics [1, 2, 3, 4, 5]. In recent years, construction of solitary wave solutions to NLPDEs has become a part of people's consciousness. Some of the existing methods developed to solve NLPDEs includes tanh-sech method [6, 7], optical homotopy asymptotic method [8, 9, 10], Hirota's method [11], and so on. This paper aims to investigate the traveling wave solutions of the nonlinear (3+1)-dimensional BLMPE and 3D-PYTFE via **(**R-B) sub-ODE method.

The (3+1)-dimensional BLMPE is given by [12]

$$-3u_x\left(u_{xy}+u_{xz}\right)-3u_{xx}\left(u_y+u_z\right)+u_{xxxy}+u_{xxxz}+u_{yt}+u_{zt}=0, \tag{1}$$

Where u=u(x,y,z,t). On the other hand, the (3+1)-dimensional PYTFE is a NLPDE which is often used to solve problems in fluid mechanics [13]. Chen *et al.* [14] obtained rogue wave solution for the (3+1)-dimensional PYTSFE by using the extended homoclinic test method. The 3D-PYTSF equation is

$$4(-u_{xt}+4u_xu_{xz})+3u_{yy}+2u_zu_{xx}+u_{xxxz}=0, \tag{2}$$

In recent years, various methods have been employed to solve partial differential equations (PDEs) with increasing complexity. For instance, Mehdi *et al.* [15] utilized the SEJI integral transform to solve PDEs, demonstrating the potential for this approach in handling nonlinear problems effectively . Similarly, Mechee and Aidi [16] explored generalized numerical methods for higher-order fractional PDEs, contributing valuable insights into the application of these techniques in solving complex mathematical models. This research focuses specifically on exploring solitary wave solutions within a particular class of nonlinear partial differential equations. The equations under consideration represent a broad spectrum of physical and mathematical models, each posing unique challenges and opportunities for analytical and numerical investigations. By delving into the solitary wave solutions of these equations, we aim to unravel the rich dynamics and nonlinear behaviors inherent in the underlying systems. The investigation of solitary waves in nonlinear PDEs involves a multidisciplinary approach, drawing on techniques from mathematical analysis, numerical simulations, and physical interpretations. Through this interdisciplinary lens, we aim to not only unveil novel solitary wave solutions but also to deepen our understanding of the fundamental principles governing the dynamics of the studied class of equations.

2. The Riccati-Bernoulli Sub-ODE Method

The R-B sub-ODE method is essentially employed to construct solitary wave solutions of NLPDEs. Suppose we have a given NLPDE in three independent variables $x, y, and\ t$ of the form

$$Q(\varphi, \varphi_t, \varphi_x, \varphi_{tt}, \varphi_{xx}, \dots) = 0, \tag{3}$$

"In the general formulation, Q represents a polynomial expression incorporating the relevant variables, with subscripts indicating partial differentiation. The R-B sub-ODE method involves a series of steps, beginning with

Step 1: A transformation is applied to combine the independent variables *x* and *t* into a single composite variable

$$\xi = k(x + y + z \pm v\, t) \tag{4}$$

This assumes that the solution, $\varphi(x,t)$, can be expressed as a function of this composite variable only

$$\varphi(x,t) = \varphi(\xi) \tag{5}$$

Where $\varphi(\xi)$ represents a localized traveling wave solution propagating at a velocity V. By utilizing equations (4) and (5), the original governing equation, Eq. (3), is converted into an ODE.

$$Q(\varphi, \varphi', \varphi'', \dots) = 0, \tag{6}$$

Where φ' denotes $\frac{d\varphi}{d\xi}$

Step 2: Assume that the function satisfying Eq. (6) also satisfies the R-B equation.

$$\varphi' = a\varphi^{2-m} + b\varphi + c\varphi^m \tag{7}$$

Here, *a, b, c,* and *m* represent constants that will be specified subsequently. Utilizing Eq. (7) and performing straightforward computations, we derive the following

$$\varphi'' = \mathrm{ab}\,(3-\mathrm{m})\varphi^{2-\mathrm{m}} + \mathrm{a}^2\,(2-\mathrm{m})\varphi^{3-2\mathrm{m}} +$$
$$\mathrm{mc}^2\varphi^{2\mathrm{m}-1} + \mathrm{bc}\,(\mathrm{m}+1)\,\varphi^{\mathrm{m}} + (2\mathrm{ac}+\mathrm{b}^2)\varphi, \tag{8}$$

$$\varphi''' = \frac{d\varphi''}{d\mathrm{m}}, \dots \tag{9}$$

Comments: Special Cases and the Characteristics of Solutions of Eq. (7)

(i) Eq. (7) becomes Riccati equation for $ac \neq 0$ and $m = 0$,

(ii) Eq. (7) forms Bernoulli equation for $a \neq 0, c = 0$ and $m \neq 1$,

(iii) Eq. (7) has the following solutions:

$$\varphi(\xi) = C\,e^{(a+b+c)\xi}, m = 1 \tag{10}$$

$$\varphi(\xi) = (a\,(m-1)\,(\xi + C)^{\frac{1}{m-1}}, m \neq 1, b = 0, c = 0 \tag{11}$$

$$\varphi(\xi) = \left(-\frac{a}{b} + Ce^{b(m-1)\xi}\right)^{\frac{1}{m-1}}, m \neq 1, b \neq 0, c = 0. \tag{12}$$

For $m \neq 1, a \neq 0, D = b^2 - 4ac < 0$, we have

$$\varphi(\xi) = \left(-\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a} tan\left(\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi + C)\right)\right)^{\frac{1}{1-m}}. \tag{13}$$

and

$$\varphi(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a} cot\left(\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi + C)\right)\right)^{\frac{1}{1-m}}. \tag{14}$$

Also, for $m \neq 1, a \neq 0$, and $D > 0$, we obtain

$$\varphi(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a} coth\left(\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi + C)\right)\right)^{\frac{1}{1-m}}. \tag{15}$$

and

$$\varphi\,(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a} tanh\left(\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi + C)\right)\right)^{\frac{1}{1-m}}. \tag{16}$$

For $m \neq 1, a \neq 0$, and $D = 0$, we have

$$\varphi(\xi) = \left(\frac{1}{a(m-1)(\xi + C)} - \frac{b}{2a}\right)^{\frac{1}{1-m}}. \tag{17}$$

With an arbitrary constant, *C*.

In Step 3, by inserting the derivatives of φ into Eq. (6), an algebraic equation involving φ is obtained. By comparing the coefficients of φ^i, we obtain a system of algebraic equations for a, b, c, and V. Solving this system and substituting m, a, b, c, V, and $\xi = k\,(x + y + z \pm Vt)$ into Eqs. (10)–(17) leads to the traveling wave solutions of Eq. (3).

3. Backlund Transformation

Suppose Eq. (3). has the following solution namely, $\varphi_n(\xi)$ and $\varphi_{n-1}(\xi)$. Then

$$\frac{d\varphi_n}{d\xi} = \frac{d\varphi_n(\xi)}{d\,\varphi_{n-1}(\xi)}\frac{d\,\varphi_{n-1}(\xi)}{d\varphi_n(\xi)} = \frac{d\varphi_n(\xi)}{d\,\varphi_{n-1}(\xi)}(a{\varphi_{n-1}}^{2-m} + \,b\varphi_{n-1} \,+\, c\varphi_{n-1}^m),$$

Thus,

$$\frac{d\varphi_n(\xi)}{a{\varphi_n}^{2-m}+b\varphi_n+c\varphi_n^m} = \frac{d\,\varphi_{n-1}(\xi)}{a{\varphi_{n-1}}^{2-m}+b\varphi_{n-1}+c\varphi_{n-1}^m} \tag{18}$$

Solving (18) and simplifying further, yields

$$\varphi_n(\xi) = \left(\frac{-cA_1+aA_2\left(\varphi_{n-1}(\xi)\right)^{1-m}}{bA_1+aA_2+aA_1\left(\varphi_{n-1}(\xi)\right)^{1-m}}\right)^{1-m}. \tag{19}$$

4. Applications

4.1 *First application*

To obtain the solution of Eq. (1), we use the traveling wave transformation (twt) $\varphi(x,y,z,t) = \varphi(\xi)$ and $\xi = k(x+y+z-vt)$ (20)

Where $\xi = k(x + y + z - v\,t)$ reduces (1) to the following ordinary differential equations

$$k(-1+6kv^2u')u'' + k^3v^3u'''' = 0. \tag{21}$$

If the solution of Eq. (21) is also a solution of Eq. (7), then it satisfies both equations. Substituting Eqs. (7)-(9) into Eq. (21), one obtains:

$$-ku^{-1-2m}(au^2+cu^{2m}+bu^{1+m})(-a(-2+m)u^2+cmu^{2m}+bu^{1+m}) + 6k^2v^2u^{-1-2m}(bu+au^{2-m}+cu^m)(au^2+cu^{2m}+bu^{1+m})(-a(-2+m)u^2+cmu^{2m}+bu^{1+m}) + k^3v^3u^{-3(1+m)}(bu+au^{2-m}+cu^m)\Big(-a^3(-2+m)(-3+2m)(-4+3m)u^6 + c^3m\big(2+m(-7+6m)\big)u^{6m} - 3a^2b(-2+m)^2(-3+2m)u^{5+m} - a(-2+m)\Big(b^2(7+(-5+m)m)+ac\big(8+m(-7+2m)\big)\Big)u^{4+2m} + b\big(b^2+2ac(4+(-2+m)m)\big)u^{3+3m} + cm\Big(b^2(1+m+m^2)+ac\big(2+m(-1+2m)\big)\Big)u^{2+4m} + 3bc^2m^2(-1+2m)u^{1+5m}\Big) = 0 \tag{22}$$

Setting m= 0 and collecting the coefficients of $\varphi^i(\xi)$, for $i = 0,1,2,3,4,5$, we then equate each coefficient to zero, leading to the following system:

$\varphi^0(\xi) = bck(-1+b^2k^2v^3+2ckv^2(3+4akv))$=0,

$\varphi^1(\xi) = k(b^4k^{ip2}v^3+2ac(-1+2ckv^2(3+4akv))+b^2(-1+2ckv^2(6+11akv)))$ = 0,

$$\varphi^2(\xi) = 3bk\big(2b^2kv^2+20a^2ck^2v^3+a(-1+12ckv^2+5b^2k^2v^3)\big) = 0, \tag{23}$$

$$\varphi^3(\xi) = 2ak\big(12b^2kv^2+20a^2ck^2v^3+a(-1+12ckv^2+25b^2k^2v^3)\big) = 0,$$

$$\varphi^4(\xi) = 30a^2bk^2v^2(1+2akv) = 0,$$

$$\varphi^5(\xi) = 12a^3k^2v^2(1 + 2akv) = 0.$$

Solving equation (23) using Mathematica produces the following set of values.

Set 1:

$$kv \neq 0, a = -\frac{1}{2kv}, c = \frac{1}{2}k(4a^2 - b^2v).$$

Set 2:

$$kv \neq 0, a = -\frac{1}{2kv}, b = 0, c = 2a^2k.$$

Set 3:

$$kv \neq 0, a = -\frac{1}{2kv}, b = \pm 2\mathrm{i}\sqrt{2}a^{3/2}\sqrt{k}, c = 0.$$

Substituting the values of set 1 into Eqs. $(13) - (16)$, we obtain the following cases of solutions

CASE A: When $m \neq 0, a \neq 0, and\ b^2 - 4ac < 0$, the solutions are

$$\varphi_1(x,y,z,t) = kv\left(b - \sqrt{-\frac{1}{k^2v^3}}Tan\left[\frac{1}{2}\sqrt{-\frac{1}{k^2v^3}}\big(c + k(-tv + x + y + z)\big)\right]\right), \tag{24}$$

and

$$\varphi_2(x,y,z,t) = kv\left(b + \sqrt{-\frac{1}{k^2v^3}}\mathrm{Cot}\left[\frac{1}{2}\sqrt{-\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]\right). \tag{25}$$

CASE B: When $m \neq 0, a \neq 0, and\ b^2 - 4ac > 0$, the solutions are

$$\varphi_4(x,y,z,t) = kv\left(b + \sqrt{\frac{1}{k^2v^3}}\mathrm{Coth}\left[\frac{1}{2}\sqrt{\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]\right). \tag{26}$$

$$\varphi_3(x,y,z,t) = kv\left(b + \sqrt{\frac{1}{k^2v^3}}\mathrm{Tanh}\left[\frac{1}{2}\sqrt{\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]\right). \tag{27}$$

Substituting the values of set 2 into Eqs. $(13) - (16)$, we obtain the following cases of solution

CASE C: When $m \neq 0, a \neq 0, and\ b^2 - 4ac < 0$, the solutions are

$$\varphi_5(x,y,z,t) = -k\sqrt{-\frac{1}{k^2v^3}}v\mathrm{Tan}\left[\frac{1}{2}\sqrt{-\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]. \tag{28}$$

and

$$\varphi_6(x,y,z,t) = kv\left(b + \sqrt{-\frac{1}{k^2v^3}}\mathrm{Cot}\left[\frac{1}{2}\sqrt{-\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]\right). \tag{29}$$

CASE D: When $m \neq 0, a \neq 0, and\ b^2 - 4ac > 0$, the solutions are

$$\varphi_7(x,y,z,t) = k\sqrt{\frac{1}{k^2v^3}}v\mathrm{Coth}\left[\frac{1}{2}\sqrt{\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]. \tag{30}$$

and

$$\varphi_8(x,y,z,t) = k\sqrt{\frac{1}{k^2v^3}}v\mathrm{Tanh}\left[\frac{1}{2}\sqrt{\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]. \tag{31}$$

Substituting the values of set 2 into Eqs. $(13) - (16)$, we obtain the following cases of solution

CASE E: When $m \neq 0, a \neq 0, and\ b^2 - 4ac < 0$, the solutions are:

$$\varphi_9(x,y,z,t) = -k\sqrt{-\frac{1}{k^2v^3}}v\mathrm{Tan}\left[\frac{1}{2}\sqrt{-\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]. \tag{32}$$

and

$$\varphi_{10}(x,y,z,t) = k\sqrt{-\frac{1}{k^2v^3}}v\mathrm{Cot}\left[\frac{1}{2}\sqrt{-\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]. \tag{33}$$

CASE F: When $m \neq 0, a \neq 0, and\ b^2 - 4ac > 0$, the solutions are:

$$\varphi_{11}(x,y,z,t) = k\sqrt{\frac{1}{k^2v^3}}v\mathrm{Coth}\left[\frac{1}{2}\sqrt{\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]. \tag{34}$$

and

$$\varphi_{12}(x,y,z,t) = k\sqrt{\frac{1}{k^2v^3}}v\mathrm{Tanh}\left[\frac{1}{2}\sqrt{\frac{1}{k^2v^3}}\big(C + k(-tv + x + y + z)\big)\right]. \tag{35}$$

4.2 *Second Application*

To get the solution of the 3D Yu-Toda, we consider Eq. (20) and use it in Eq. (2), we get the following nonlinear ODE

$$k(3 + 4v + 6ku')u'' + k^3u'''' = 0. \tag{36}$$

Provided that the solution to Eq. (36) simultaneously satisfies Eq. (7), then it satisfies both equations. Substituting Eqs. (7)-(9) into Eq. (36), we have:

$3ku^{-1-2m}(au^2 + cu^{2m} + bu^{1+m})(-a(-2+m)u^2 + cmu^{2m} + bu^{1+m}) +$
$4kvu^{-1-2m}(au^2 + cu^{2m} + bu^{1+m})(-a(-2+m)u^2 + cmu^{2m} + bu^{1+m}) +$
$6k^2u^{-1-2m}(bu + au^{2-m} + cu^m)(au^2 + cu^{2m} + bu^{1+m})(-a(-2+m)u^2 +$
$cmu^{2m} + bu^{1+m}) + k^3u^{-3(1+m)}(bu + au^{2-m} + cu^m)\Big(-a^3(-2+m)(-3 +$
$2m)(-4 + 3m)u^6 + c^3m\big(2 + m(-7 + 6m)\big)u^{6m} - 3a^2b(-2+m)^2(-3 +$

$2m)u^{5+m} - a(-2+m)\left(b^2(7+(-5+m)m) + ac\big(8+m(-7+$
$2m)\big)\right)u^{4+2m} + b\big(b^2 + 2ac(4+(-2+m)m)\big)u^{3+3m} + cm\left(b^2(1+m+\right.$
$\left.m^2) + ac\big(2+m(-1+2m)\big)\right)u^{2+4m} + 3bc^2m^2(-1+2m)u^{1+5m}\Big) = 0.$ (37)

Setting m= 0 and collecting the coefficients of $\varphi^i(\xi)$, for $i = 0,1,2,3,4,5$, we then equate each coefficient to zero, leading to the following system:

$$\begin{gathered}
\varphi^0(\xi) = bck(3 + b^2k^2 + 2ck(3+4ak) + 4v) = 0,\\
\varphi^1(\xi) = k\big(b^4k^2 + 2ac(3 + 2ck(3+4ak) + 4v) \\
+ b^2(3 + 2ck(6+11ak) + 4v)\big) = 0,\\
\varphi^2(\xi) = 3bk\big(2b^2k + 20a^2ck^2 + a(3 + 12ck + 5b^2k^2 + 4v)\big) = 0, \quad (38)\\
\varphi^3(\xi) = 2ak\big(12b^2k + 20a^2ck^2 + a(3 + 12ck + 25b^2k^2 + 4v)\big) = 0,\\
\varphi^4(\xi) = 30a^2bk^2(1+2ak) = 0,\\
\varphi^5(\xi) = 12a^3k^2(1+2ak) = 0.
\end{gathered}$$

Solving Eq. (38) with Mathematica yields the following set of values.

Set 4:

$$c = a(3+4v), a = -\tfrac{1}{2k}, b = 0,$$

Set 5:

$$k \neq 0, a = -\tfrac{1}{2k}, c = \tfrac{1}{2}(6a - b^2k + 8av),$$

Set 6:

$$k \neq 0, a = -\tfrac{1}{2k}, b = \pm 2\sqrt{-3a^2 - 4a^2v}, c = 0.$$

Substituting the values of set 4 into Eqs. (13) – (16), we obtain the following cases of solutions

CASE G: If $m \neq 0, a \neq 0, and\ b^2 - 4ac < 0$, the solutions are:

$$\varphi_{13}(x, y\, z, t) = -k\sqrt{\tfrac{3+4v}{k^2}}\,\mathrm{Tan}\left[\tfrac{1}{2}\sqrt{\tfrac{3+4v}{k^2}}\big(C + k(-tv + x + y + z)\big)\right], \quad (39)$$

$$\varphi_{14}(x, y, z, t) = \sqrt{3+4v}\,\mathrm{Cot}\left[\tfrac{\sqrt{3+4v}(C+k(-tv+x+y+z))}{2k}\right]. \quad (40)$$

CASE H: If $m \neq 0, a \neq 0, and\ b^2 - 4ac > 0$, the solutions are

$$\varphi_{15}(x, y, z, t) = k\sqrt{-\tfrac{3+4v}{k^2}}\,\mathrm{Coth}\left[\tfrac{1}{2}\sqrt{-\tfrac{3+4v}{k^2}}\big(C + k(-tv + x + y + z)\big)\right], \quad (41)$$

$$\varphi_{16}(x, y, z, t) = k\sqrt{-\tfrac{3+4v}{k^2}}\,\mathrm{Tanh}\left[\tfrac{1}{2}\sqrt{-\tfrac{3+4v}{k^2}}\big(C + k(-tv + x + y + z)\big)\right]. \quad (42)$$

Substituting the values of set 5 into Eqs. (11) – (14), we obtain the following cases of solutions

CASE I: If $m \neq 0, a \neq 0,$ and $b^2 - 4ac < 0$, the solutions are:

$$\varphi_{17}(x,y,z,t) = -k\left(\sqrt{-\frac{3+4v}{k^2}} + \sqrt{\frac{3+4v}{k^2}}\,\mathrm{Tan}\left[\frac{1}{2}\sqrt{\frac{3+4v}{k^2}}\left(C + k(-tv + x + y + z)\right)\right]\right) \tag{43}$$

and

$$\varphi_{18}(x,y,z,t) = -k\sqrt{\frac{-3-4v}{k^2}} + \sqrt{3+4v}\,\mathrm{Cot}\left[\frac{\sqrt{3+4v}\left(C+k(-tv+x+y+z)\right)}{2k}\right]. \tag{44}$$

CASE J: If $m \neq 0, a \neq 0, and\ b^2 - 4ac > 0$, the solutions are

$$\varphi_{19}(x,y,z,t) = k\sqrt{-\frac{3+4v}{k^2}}\left(-1 + \mathrm{Coth}\left[\frac{1}{2}\sqrt{-\frac{3+4v}{k^2}}\left(C + k(-tv + x + y + z)\right)\right]\right). \tag{45}$$

and

$$\varphi_{20}(x,y,z,t) = k\sqrt{-\frac{3+4v}{k^2}}\left(-1 + \mathrm{Tanh}\left[\frac{1}{2}\sqrt{-\frac{3+4v}{k^2}}\left(C + k(-tv + x + y + z)\right)\right]\right). \tag{46}$$

Substituting the values of set 6 into Eqs. $(11) - (14)$, we obtain the following cases of solutions

CASE K: If $m \neq 0, a \neq 0,$ and $b^2 - 4ac < 0$, the solutions are:

$$\varphi_{21}(x,y,z,t) = -k\left(\sqrt{-\frac{3+4v}{k^2}} + \sqrt{\frac{3+4v}{k^2}}\,\mathrm{Tan}\left[\frac{1}{2}\sqrt{\frac{3+4v}{k^2}}\left(C + k(-tv + x + y + z)\right)\right]\right). \tag{47}$$

and

$$\varphi_{22}(x,y,z,t) = -k\sqrt{\frac{-3-4v}{k^2}} + \sqrt{3+4v}\,\mathrm{Cot}\left[\frac{\sqrt{3+4v}\left(C+k(-tv+x+y+z)\right)}{2k}\right]. \tag{48}$$

CASE L: If $m \neq 0, a \neq 0, and\ b^2 - 4ac > 0$, the solutions are

$$\varphi_{23}(x,y,z,t) = k\sqrt{-\frac{3+4v}{k^2}}\left(-1 + \mathrm{Coth}\left[\frac{1}{2}\sqrt{-\frac{3+4v}{k^2}}\left(C + k(-tv + x + y + z)\right)\right]\right). \tag{49}$$

and

$$\varphi_{24}(x,y,z,t) = k\sqrt{-\frac{3+4v}{k^2}}\left(-1 + \mathrm{Tanh}\left[\frac{1}{2}\sqrt{-\frac{3+4v}{k^2}}\left(C + k(-tv + x + y + z)\right)\right]\right). \tag{50}$$

5. Graphical explanation of the results obtained

This section presents the 3D, density and contour figures of the obtained results under a suitable selection of parameters. From the results obtained, Eqs. (24), (25), (26), (29), (30), and (44) are periodic soliton waveforms. Equation (27) represents a basic kink-type soliton solution, while Eq. (28) corresponds to a periodic singular soliton. Furthermore, Eqs. (31) and (46) yield ideal kink soliton profiles. (Refer to Figures 1–10 for illustration).

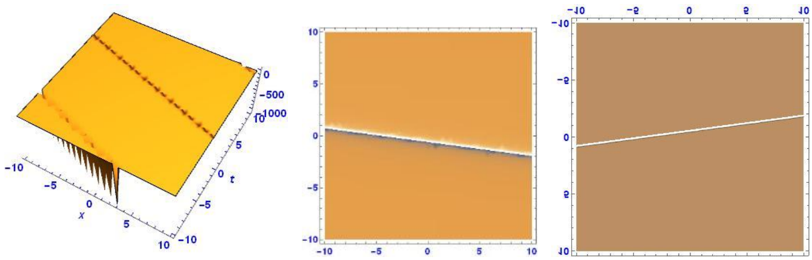

Figure 1

Density and contour of solution (24) when $m = 4, v = -7.5, k = 7, y = 0.5, z = 2.9,$ and $b = 8.2$.

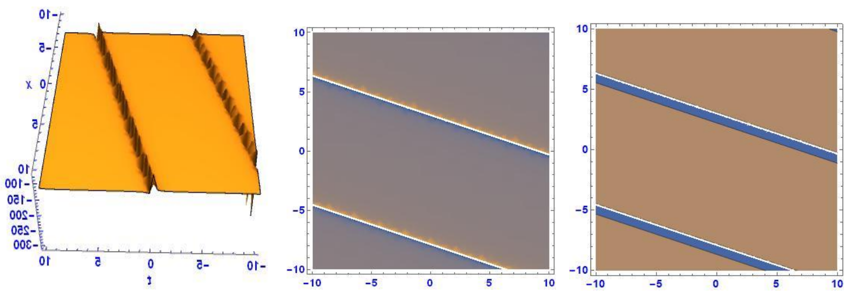

Figure 2

Density and contour of solution (25) when $m = 2.9, v = -3, k = 5, b = 8.5, y = 0.7, z = 6$.

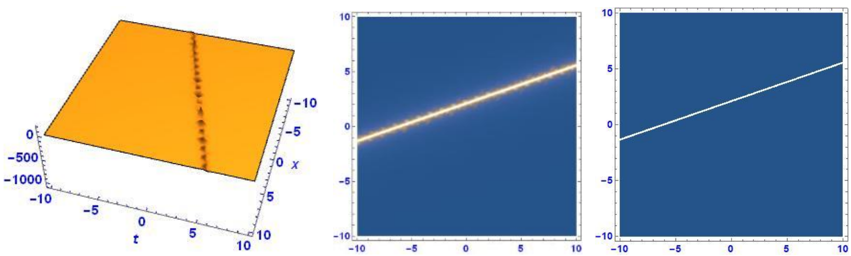

Figure 3

Density and contour of solution (26) when $m = 6.7, v = 2.9, k = 8, y = 0.3,$ and $z = 5$.

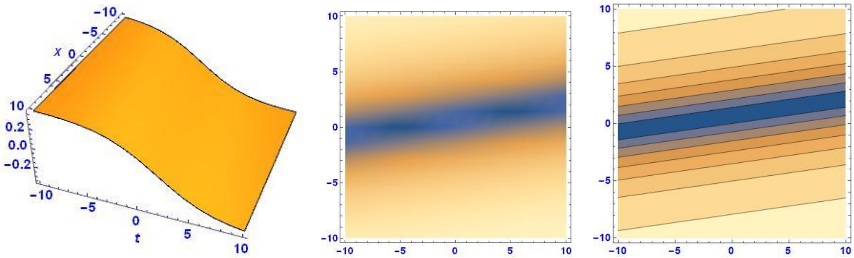

Figure 4

Density and contour of solution (27)
when $m = 5, v = 7, k = 12, y = 0.6,$ and $z = 4$.

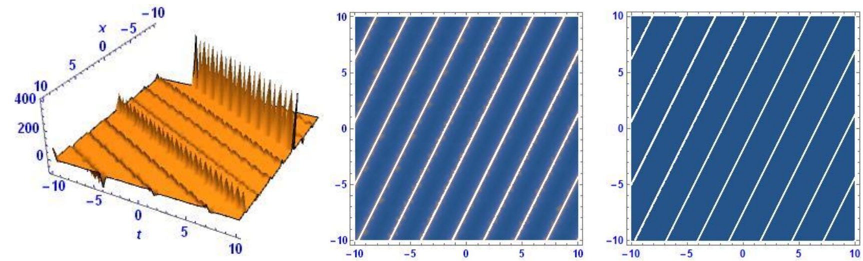

Figure 5

Density and contour of solution (28) when $m = 2.7, v = 0.5, k = 1, y = 1.8$ and $z = 1.6$.

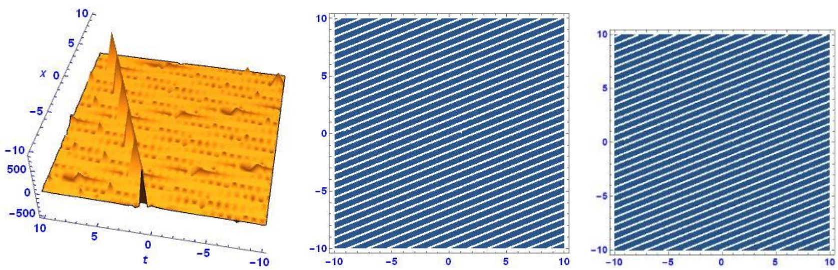

Figure 6

Density and contour of solution (29) when $m = 3.5, v = 2.5, k = 4, y = 1.2$ and $z = 1$.

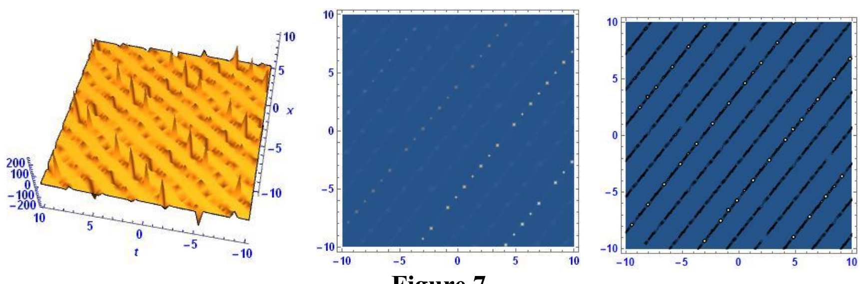

Figure 7

Density and contour of solution (30) when $m = 3, v = 0.8, k = 1.5, y = 1.7$ and $z = 2$.

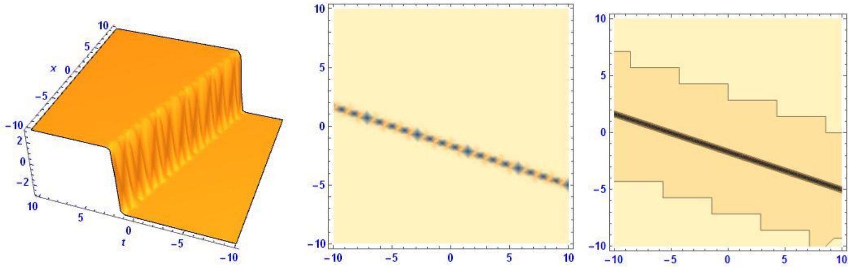

Figure 8

Density and contour of solution (31) when $m = 2, v = 2.3, k = 1.9, y = 1.7$ and $z = 2.3$.

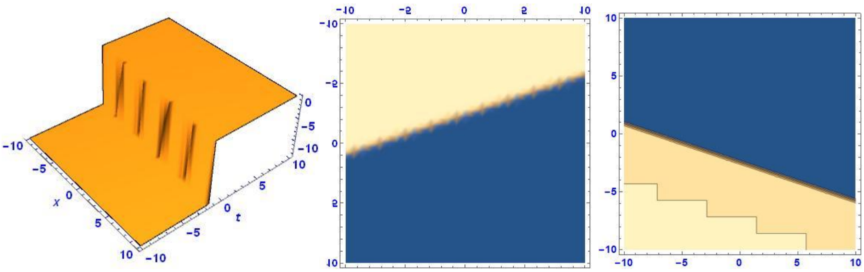

Figure 9

Density and contour of solution (46) when $m = 6, v = -12, k = 5, y = 1.3$ and $z = 4$

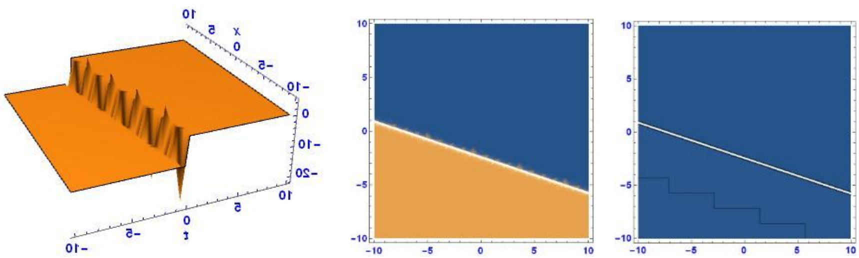

Figure 10

Density and contour of solution (44) when $m = 5, v = -6, k = 1.2, y = 2$ and $z = 3$

6. Concluding Remark

The theoretical solutions of (1) and (2) are investigated using the R-B sub-ODE method. As can be seen from the study, the obtained solution acquired some singular solitons, periodic solitons and optical solitons to the (3+1)-dimensional BLMPE and the 3D-PYTDFE. The obtained results are verified by Mathematica software and affirmed that the applied technique is optimal for obtaining analytical solutions to NLPDEs. In conclusion, this study contributes to the broader scientific discourse on nonlinear waves and their implications across disciplines. By identifying and characterizing solitary wave solutions, we provide valuable insights into the stability, existence, and properties of localized structures within the context of nonlinear PDEs. The outcomes of this study have the potential to inform and impact diverse scientific and engineering fields, paving the way for advancements in our comprehension of complex systems governed by nonlinear dynamics.

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