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Scheduling theory and applications revisited

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Abstract

In this paper, the scheduling theory and its applications are considered. The problems with one machine and two machines, as well as the multi-machine problem, are discussed. The schedules are presented by Gantt charts, and the Johnson's theorem is stated. The calendar planning and the traveling salesman problem, as scheduling-type problems, are also considered.

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1. Introduction

Scheduling theory is part of operations research, connected with mathematical statement of problems of optimal ordering (sequencing) and coordination of certain operations during the time, methods for solving the obtained mathematical problems, and development of optimal (concerning some criteria) schedules. In other words, scheduling theory studies allocation problems, in which the limited resource is time. The methods and results of scheduling theory are applicable to production planning, management, transportation, coordination of various activities, computer systems, etc.

Combinatorial algorithms are a powerful tool for analysis of discrete optimization problems whose feasible region is not connected or whose set of objective function values is finite. Usually there is not a system of basic rules and theorems, which allow to obtain general results about these algorithms. However, numerous studies have shown that, in many cases, there are general principles and schemes of calculations, which take into account the main characteristics of the

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corresponding class of problems. Such methods are, for example, the branch and bound method, methods for solving optimization problems on graphs and networks, some methods of dynamic programming, etc.

Although most problems have a relatively simple statement, only some of these problems can be solved exactly. This is mainly due to the difficulties in studying the convergence of methods and algorithms, proposed for solving the considered problems.

Schedule is a document which contains information about:

- the quantity and the nomenclature of the performed activities, including their stages;
- the start time and the end time of each activity;
- the place and the technical tools for performing each activity;
- the costs of time and material resources for the performed activities.

This information is sufficient for presenting the schedules, although this information can be supplemented and specified for a better and more complete description of the reality.

The topics and problems, considered in this paper, and related to them, are discussed in references [1]-[28], etc.

Rest of the paper is organized as follows. In Section 2, the problems with one machine, two machines, and the multi-machine problem are discussed, the schedules are presented by Gantt charts, the Johnson's theorem is stated, and the calendar planning is analyzed. In Section 3, the traveling salesman problem, as scheduling-type problem, is considered. In Section 4, the contents of the paper is summarized.

2. Problems with one machine, two machines, and the multi-machine problem. Gantt chart

There are various ways of presenting schedules. The most visual is the geometric (graphical) way, using the so-called *Gantt chart*. In this approach,

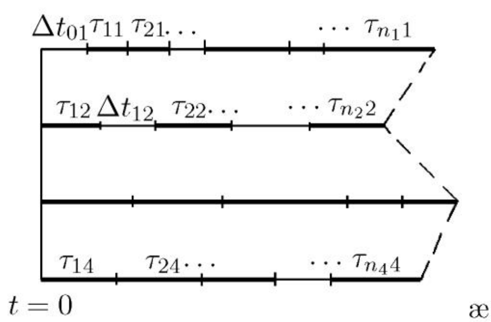

Figure 1
Example of Gantt chart.

each activity is assigned a line segment of a certain length, and each type of equipment is assigned a straight line (time axis), on which the line segments (that is, the activities), that are performed by this equipment, are located. For a known start time $t = 0$, the mutual location of the line segments gives all necessary information (Figure 1).

Problem 1: The simplest scheduling problem is that of choosing the sequence of performing n activities by one performer. This problem can be formulated as follows: n activities (tasks, jobs, operations) have to be done by one performer (machine, technology line, etc.). The duration t_j, $j = 1, \ldots, n$, of each activity is known, as well as the efficiency criterion K. Find such an order of performing the activities, such that K attains a minimum (or a maximum).

This problem can be solved, for example, by the method of item-by-item examination (explicit enumeration) of all possible variants for performing the activities.

This simplest example shows the main features of the problems of scheduling theory - a large number of possible variants, combinatorial schemes for finding a solution, various ways of choosing the criterion K.

Problem 2: Another problem, connected with the scheduling theory, is the problem of two machines, that perform sequentially n activities. At the initial moment of time $t = 0$ the first activity (according to the established order) goes to the first machine, and this requires time τ_{11}. After that it goes to the second available machine, which requires time τ_{21} for performing this activity. This cycle is repeated for other activities in the order, in which the machines become available, complying with the following rules:

- the sequence of activities for both machines is the same;
- at a certain moment of time, only one activity is performed by each machine;
- each activity is performed without interruption, and activity goes to a second machine only after this activity has entirely performed by the first machine.

Determine the order of performing the activities, such that the total time T_c of employment of the two-machine system be minimal.

The Gantt chart of this problem is presented in Figure 2.

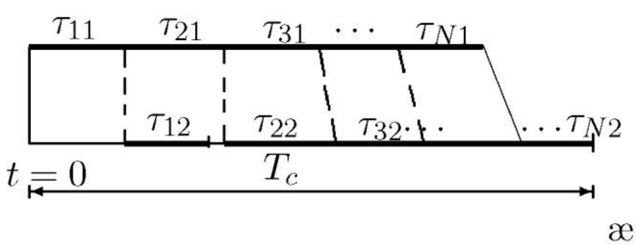

Figure 2
Gantt chart for Problem 2.

The following theorem holds true.

Theorem 1: (Johnson's theorem) *The order of performance of n activities, for which T_c attains minimum, satisfies the inequalities*

$$\min\{\tau_{1j}, \tau_{2,j+1}\} < \min\{\tau_{1,j+1}, \tau_{2j}\}, \quad j = 1, \dots, n-1.$$

One of the most complex problems of scheduling theory is the general problem of analyzing multi-stage technological processes.

Given a set of details (tasks, activities) $d_1, \dots, d_n$, which have to be processed by the machines $\{M_1, \dots, M_m\}$. The technological routes TR_j, $j = 1, \dots, n$, for processing the tasks are also given, which determine the order of using the machines. In general, the technological routes are different for the different tasks.

For the sake of simplicity, let all TR_j be the same and

$$TR_j = \{M_1 - M_2 - \dots - M_m\} = TR, \quad j = 1, \dots, n.$$

Since at a certain moment of time, a machine cannot process more than one detail, queues of details arise for some machines.

Denote by τ_{ij} the duration of processing the detail d_j by machine M_i; by T_j^d the necessary (directive) period for completing the processing of j −th detail by the last machine; by T_j^r the real time limit for completing the processing of j −th detail; by r_j the relative losses per unit time in case of delay of the processing of d_j with respect to the time T_j^d.

Determine the order $\sigma_1, \sigma_2, \dots, \sigma_m$ of processing the details by each machine, respectively, such that a certain criterion, for example, the total duration of processing, is *optimal*.

This problem is known as *the problem of calendar planning* or *the problem of composing a schedule*, and the choice of the order of processing the details is called *ordering*.

The composition of admissible calendar plan (schedule, timetable) is equivalent to coordination of many manufacturing operations during the time, which helps to improve (optimize) the functioning of complex systems.

In the calendar planning problems, the following criteria are most commonly used:

1. Minimizing the total duration of the processing, that is, of the interval between the beginning of the first operation and the end of the last operation for the last detail:
$$\min_{\sigma_1, \dots, \sigma_m} T_c(\sigma_1, \dots, \sigma_m).$$
2. Minimizing the total losses due to delays:
$$\min \sum_{j=1}^n r_j z_j(\sigma_1, \dots, \sigma_m) = \min \sum_{j=1}^n r_j z_j(\sigma),$$
where $z_j(\sigma) = \max[0, T_j^d(\sigma) - T_j^r]$ is the value of the delay for the j −th detail.

3. Minimizing the total delay:

$$\min_{\sigma} \sum_{j=1}^{n} z_j(\sigma).$$

4. Minimizing the maximal delay:

$$\min_{\sigma} \max_{j} z_j(\sigma).$$

5. Maximizing the load (employment) of the machines.

The problem of calendar planning is a combinatorial problem. The total number of possible variants in the general problem with m machines is

$$N_{\Sigma} = (n!)^m,$$

where $n! = 1.2.3 \dots n$. Since this number is enormous for large values of m and n, approximate heuristic models are used for solving combinatorial problems, with the exception of the special cases for $m = 1,2,3$, and the case when the criterion $\min T_c$ is used.

The calendar planning problems are classified according to the following items:

- according to the number of machines: $m = 1$, $m = 2$, $m = 3$, and the general case (multi-machine problem);
- according to the type of technological routes: with the same TR (of the conveyor type) and with different TR (general case);
- according to the optimality criterion used;
- according to the type of machines: with identical and with non-identical machines.

3. The traveling salesman problem

The *traveling salesman problem* can be stated as follows.

Given a list of $n+1$ cities $A_0, A_1, \dots, A_n$ and a matrix $C = (c_{ij})_{(n+1)\times(n+1)}$ of distances between any two cities. In the general case, matrix C may not be symmetric, that is, we may have $c_{ij} \neq c_{ji}$ for some pairs of indices (i, j), because the routes between two cities in one direction and in the opposite direction may be different. Starting from an initial city A_0, a traveling salesman must visit all other cities exactly once and return to the initial city A_0. Find the order, in which the traveling salesman must visit the cities under the above requirements, so that the total traveled distance be minimal, that is, find the shortest possible route which satisfies the above requirements.

Obviously this problem can be reformulated in the terms of machine-scheduling problem: “cities” are “machines” and “distances” are the “set-up times".

Introduce the variables as follows:

$$x_{ij} = \begin{cases} 1, & \text{if the traveling salesman goes from } A_i \text{ to } A_j \text{ directly,} \\ 0, & \text{otherwise,} \end{cases}$$

$$i, j = 0,1, \dots, n.$$

In the above definition of variables, “directly” is important because in any case, the traveling salesman goes from one city to another city after *some* number of traveling steps.

The mathematical model of the traveling salesman problem is:

$$\min\{L(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x_{ij}\} \quad (1)$$

subject to

$$\sum_{j=0}^{n} x_{ij} = 1, \quad i = 0,1,\dots,n, \quad (2)$$

$$\sum_{i=0}^{n} x_{ij} = 1, \quad j = 0,1,\dots,n, \quad (3)$$

$$u_i - u_j + (n+1)x_{ij} \le n, \quad i,j = 0,1,\dots,n, i \neq j, \quad (4)$$

$$x_{ij} \in \{0,1\}, \quad i,j = 0,1,\dots,n, \quad (5)$$

where u_i, u_j, $i,j = 0,1,\dots,n$, are nonnegative integers.

Constraints (2) and (3) mean that the traveling salesman enters each city exactly once and he leaves each city exactly once, respectively. If the constraint (4) is missing, the problem (1), (2), (3), (5) is the Assignment problem, whose solution, in the general case, is not cyclic, that is, it is possible the path, obtained as a solution of the Assignment problem, to be a sequence of unconnected sub-cycles. Namely constraint (4) guarantees that the traveling salesman route be connected, and hence cyclic.

The traveling salesman problem is an NP-hard problem of combinatorial optimization (see, for example, [Cook 3] and [Karp 9]), and this problem is important in both operations research and theoretical computer science.

4. Conclusions

This paper is devoted to the scheduling theory and its applications. The problems with one machine and two machines, as well as the multi-machine problem are discussed. The schedules are presented by Gantt charts, and the Johnson's theorem is stated. The calendar planning and the traveling salesman problem, as scheduling-type problems, are also considered.

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