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MR–Metric spaces with applications

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Abstract

This paper presents the notion of *MR*-metric spaces and explores their connection with b-metric spaces. Furthermore, we present several methods for constructing *MR*-metric spaces and explore different types of convergence associated with these spaces. Additionally, we establish several common fixed point theorems for two mappings in complete *MR*-metric spaces. Our findings are illustrated with numerous examples and a practical application.

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1. Introduction

This research presents MR-metric spaces as a unified framework that extends both b-metric and D-metric spaces through a parameterized tetrahedral inequality. We examine their structural properties, convergence behaviors, and fixed-point theory, offering explicit construction methods and non-trivial examples. Key results include existence and uniqueness theorems for fixed points, with applications to solving linear equations.

The paper is organized to first define MR-metrics and their axioms, then analyze convergence and completeness, and finally derive fixed-point theorems under weak compatibility conditions. By bridging gaps between existing metric space theories, this work opens avenues for further applications in nonlinear analysis and computational mathematics.

Dhage[1] introduced the concept of D-metric spaces in 1994 as an extension of metric spaces.

Definition 1.1: [1] Let $\mathbb{Y}$ be a non-empty set. A mapping $D: \mathbb{Y}^3 \to \mathbb{R}$ is called a D-metric if for all $\beth_1, \beth_2, \beth_3, \beth_5 \in \mathbb{Y}$, these axioms hold:

(1) Non-negativity: $D(\beth_1, \beth_2, \beth_3) \geq 0$.

(2) Identity of indiscernibles: $D(\beth_1, \beth_2, \beth_3) = 0$ if and only if $\beth_1 = \beth_2 = \beth_3$.

(3) Symmetry: $D(\beth_1, \beth_2, \beth_3) = D(p(\beth_1, \beth_2, \beth_3))$ for any permutation p of $\beth_1, \beth_2, \beth_3$.

(4) Tetrahedral inequality: $D(\beth_1, \beth_2, \beth_3) \leq D(\beth_1, \beth_2, \beth_5) + D(\beth_1, \beth_5, \beth_3) + D(\beth_5, \beth_2, \beth_3)$.

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The pair $(\mathbb{Y}, D)$ is called a D-*metric space*.

The notion of b-metric spaces was first introduced by Bakhtin [2] and later developed by Czerwik [3], relaxing the standard triangle inequality condition used in classical metric spaces. Subsequently, numerous researchers have investigated and established various fixed-point theorems in this setting (see [4-35]). Among these works, Aydi et al. [36] obtained significant results on fixed points for set-valued quasi-contractions in b-metric spaces.

Definition 1.2: (***b*-**Metric Space [2, 3]). Let $\mathbb{Y}$ be a non-empty set and $S \geq 1$ a constant. A mapping $d: \mathbb{Y}^2 \to [0, \infty)$ is termed a b-metric if for all $\beth_1, \beth_2, \beth_3 \in \mathbb{Y}$, it satisfies:

(1) **Non-degeneracy:** $d(\beth_1, \beth_2) = 0 \Leftarrow \beth_1 = \beth_2$.

(2) **Symmetry:** $d(\beth_1, \beth_2) = d(\beth_2, \beth_1)$.

(3) **Relaxed Triangle Inequality:** $d(\beth_1, \beth_3) \leq S[d(\beth_1, \beth_2) + d(\beth_2, \beth_3)]$.

The pair $(\mathbb{Y}, d)$ is called a b-*metric space*.

In this paper, we introduce an extension of b-metric spaces called MR-*metric spaces*, which generalize D-metric spaces by employing a more flexible tetrahedral inequality.

Definition 1.3: (***MR*-**Metric Space). Let $\mathbb{Y}$ be a non-empty set equipped with a constant $R \geq 1$. A function $\mathbb{M}: \mathbb{Y}^3 \to [0, \infty)$ is called an MR-metric if for all $\beth_1, \beth_2, \beth_3, \beth_4 \in \mathbb{Y}$, the following axioms hold:

(1) **Non-negativity:** $\mathbb{M}(\beth_1, \beth_2, \beth_3) \geq 0$.

(2) **Identity of Indiscernibles:** $\mathbb{M}(\beth_1, \beth_2, \beth_3) = 0 \Leftarrow \beth_1 = \beth_2 = \beth_3$.

(3) **Symmetry:** $\mathbb{M}(\beth_1, \beth_2, \beth_3) = \mathbb{M}(p(\beth_1, \beth_2, \beth_3))$ for any permutation p of $\beth_1, \beth_2, \beth_3$.

(4) **Generalized Tetrahedral Inequality:**

$$\mathbb{M}(\beth_1, \beth_2, \beth_3) \leq R[\mathbb{M}(\beth_1, \beth_2, \beth_4) + \mathbb{M}(\beth_1, \beth_4, \beth_3) + \mathbb{M}(\beth_4, \beth_2, \beth_3)].$$

The pair $(\mathbb{Y}, \mathbb{M})$ is called an MR-*metric space*.

The following examples illustrate nontrivial MR-metric spaces.

Example 1.1: (Weighted L^1 and L^∞ MR-Metrics on $\mathbb{R}$). *Let* $\mathbb{Y} = \mathbb{R}$ *and* $R \geq 1$. Define the following functions:

- The sum-based MR-metric:

$$\mathbb{M}_1(\beth_1, \beth_2, \beth_3) = \frac{1}{R}(|\beth_1 - \beth_2| + |\beth_2 - \beth_3| + |\beth_3 - \beth_1|).$$

- The max-based MR-metric:

$$\mathbb{M}_\infty(\beth_1, \beth_2, \beth_3) = \frac{1}{R}\max\{|\beth_1 - \beth_2|, |\beth_2 - \beth_3|, |\beth_3 - \beth_1|\}.$$

Both $(\mathbb{R}, \mathbb{M}_1)$ and $(\mathbb{R}, \mathbb{M}_\infty)$ satisfy all axioms of an MR-metric space. The parameter R controls the relaxation of the tetrahedral inequality.

Example 1.2: (Discrete MR-Metric Space). Let $\mathbb{Y}$ be any non-empty set. Define

$$\mathbb{M}(\beth_1,\beth_2,\beth_3)=\begin{cases}0 & \text{if } \beth_1=\beth_2=\beth_3,\\ 1 & \text{otherwise.}\end{cases}$$

This induces the discrete MR-metric on $\mathbb{Y}$, which trivially satisfies all MR-metric axioms with $R=1$. This example demonstrates that every set can be endowed with a trivial MR-metric structure.

In the following remark, we highlight some significant properties exhibited by the MR-metrics presented in the previous examples. These properties play a crucial role in the theories developed in this paper.

Remark 1.1: The MR-metrics given in Examples 1.1 and 1.2 satisfy the following properties:

For every $\beth_1,\beth_2,\beth_3,\beth_4,\beth_5\in\mathbb{Y}$, we have:

($\mathbb{M}5$): $\mathbb{M}(\beth_1,\beth_2,\beth_2)\le R[\mathbb{M}(\beth_1,\beth_3,\beth_3)+\mathbb{M}(\beth_3,\beth_2,\beth_2)]$.

($\mathbb{M}6$): $\mathbb{M}(\beth_1,\beth_1,\beth_2)=\mathbb{M}(\beth_1,\beth_2,\beth_2)$.

($\mathbb{M}7$): $\mathbb{M}(\beth_1,\beth_2,\beth_2)\le R\mathbb{M}(\beth_1,\beth_2,\beth_3)$.

($\mathbb{M}8$): $\mathbb{M}(\beth_1,\beth_2,\beth_3)\le\frac{1}{R}[\mathbb{M}(\beth_1,\beth_4,\beth_5)+\mathbb{M}(\beth_4,\beth_2,\beth_5)+\mathbb{M}(\beth_4,\beth_5,\beth_3)]$.

The following example demonstrates that ($\mathbb{M}6$) does not imply ($\mathbb{M}7$).

Example 1.3: Suppose $\mathbb{Y}$ contains at least three elements. Define $\mathbb{M}$ on $\mathbb{Y}\times\mathbb{Y}\times\mathbb{Y}$ as follows:

$$\mathbb{M}(\beth_1,\beth_2,\beth_3)=\begin{cases}0 & \text{if } \beth_1=\beth_2=\beth_3,\\ \frac{1}{2R} & \text{if } \beth_1,\beth_2,\beth_3 \text{ are distinct},\\ 1 & \text{otherwise.}\end{cases}$$

Then, $(\mathbb{Y},\mathbb{M})$ is an MR-metric space that satisfies ($\mathbb{M}6$) but not ($\mathbb{M}7$).

We now investigate certain relationships between MR-metric and b-metric spaces through the establishment of supplementary conditions.

Proposition 1.1: For an MR-metric space $(\mathbb{Y},\mathbb{M})$ fulfilling properties ($\mathbb{M}5$) and ($\mathbb{M}6$), the mapping $d(\beth_1,\beth_2)=\mathbb{M}(\beth_1,\beth_2,\beth_2)$ constitutes a valid b-metric on $\mathbb{Y}$.

Proof: Let $\beth_1,\beth_2\in\mathbb{Y}$. We need to show that $d(\beth_1,\beth_2)$ satisfies the properties of a b-metric.

- **Non-negativity:**

$$d(\beth_1,\beth_2)=\mathbb{M}(\beth_1,\beth_2,\beth_2)\ge 0,$$

which follows from ($\mathbb{M}1$).

- **Identity of indiscernibles:**

$$d(\beth_1,\beth_2)=\mathbb{M}(\beth_1,\beth_2,\beth_2)=0\Leftarrow\beth_1=\beth_2,$$

which follows from $(\mathbb{M}2)$.

- **Symmetry:**

$$d(\beth_1,\beth_2) = \mathbb{M}(\beth_1,\beth_2,\beth_2) = \mathbb{M}(\beth_1,\beth_1,\beth_2) = \mathbb{M}(\beth_2,\beth_1,\beth_1) = d(\beth_2,\beth_1),$$

which follows from $(\mathbb{M}6)$ and $(\mathbb{M}3)$.

- **Relaxed triangle inequality:**

$$d(\beth_1,\beth_2) = \mathbb{M}(\beth_1,\beth_2,\beth_2) \le R[\mathbb{M}(\beth_1,\beth_3,\beth_3) + \mathbb{M}(\beth_3,\beth_2,\beth_2)],$$

and substituting $d(\beth_1,\beth_3) = \mathbb{M}(\beth_1,\beth_3,\beth_3)$ and $d(\beth_3,\beth_2) = \mathbb{M}(\beth_3,\beth_2,\beth_2)$ gives:

$$d(\beth_1,\beth_2) \le R[d(\beth_1,\beth_3) + d(\beth_3,\beth_2)].$$

This follows from $(\mathbb{M}5)$.
Thus, $d(\beth_1,\beth_2)$ is a b-metric on $\mathbb{Y}$.

Example 1.4: (Non-Trivial MR-Metric on $\boldsymbol{\ell_\rho}$). Let $\mathbb{Y} = \ell_\rho(\mathbb{R})$ where $0 < \rho < 1$, i.e., the space of all real sequences $\{\beth_n\}$ such that $\sum_{n=1}^{\infty} |\beth_n|^\rho < \infty$. Define the mapping:

$$\mathbb{M}: \mathbb{Y} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{R}^+$$

as follows:

$$\mathbb{M}(\beth_1,\beth_2,\beth_3) = \begin{cases} 0 & \text{if } \beth_1 = \beth_2 = \beth_3, \\ 1 & \text{if } \beth_1,\beth_2,\beth_3 \text{ are distinct}, \\ \left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{2_n}|^\rho\right)^{1/\rho} & \text{if } \beth_1 \ne \beth_2 = \beth_3 \text{ or } \beth_1 = \beth_3 \ne \beth_2, \\ \left(\sum_{n=1}^{\infty} |\beth_{2_n} - \beth_{3_n}|^\rho\right)^{1/\rho} & \text{if } \beth_2 \ne \beth_3 = \beth_1 \text{ or } \beth_1 = \beth_2 \ne \beth_3, \\ \left(\sum_{n=1}^{\infty} |\beth_{3_n} - \beth_{1_n}|^\rho\right)^{1/\rho} & \text{if } \beth_1 \ne \beth_3 = \beth_2 \text{ or } \beth_2 = \beth_1 \ne \beth_3. \end{cases}$$

Here, $\beth_1 = \{\beth_{1_n}\}$, $\beth_2 = \{\beth_{2_n}\}$, and $\beth_3 = \{\beth_{3_n}\}$ are sequences in $\ell_\rho(\mathbb{R})$. Then, $(\mathbb{Y},\mathbb{M})$ is an MR-metric space with $R = 2^{1/\rho} > 1$.

Proof: We verify the axioms of an MR-metric space for $\mathbb{M}$.

Verification of Axioms:

(1) **Non-negativity:** By definition, $\mathbb{M}(\beth_1,\beth_2,\beth_3) \ge 0$ for all $\beth_1,\beth_2,\beth_3 \in \mathbb{Y}$.

(2) **Identity of Indiscernibles:** Clearly, $\mathbb{M}(\beth_1,\beth_2,\beth_3) = 0$ if and only if $\beth_1 = \beth_2 = \beth_3$.

(3) **Symmetry:** The function $\mathbb{M}$ is symmetric in all three arguments by construction.

(4) **Generalized Tetrahedral Inequality:** We must show:

$$\mathbb{M}(\beth_1,\beth_2,\beth_3) \le R[\mathbb{M}(\beth_1,\beth_2,\beth_4) + \mathbb{M}(\beth_1,\beth_4,\beth_3) + \mathbb{M}(\beth_4,\beth_2,\beth_3)],$$

where $R = 2^{1/\rho}$. We proceed by cases.

Case 1: All three points are distinct.

- **Subcase 1:** If $\beth_4 \notin \{\beth_1, \beth_2, \beth_3\}$, then:
$$1 = \mathbb{M}(\beth_1, \beth_2, \beth_3) \leq R[1 + 1 + 1] = 3R.$$
Since $R \geq 2$, the inequality holds.
- **Subcase 2:** If $\beth_4 = \beth_1$, then:
$$1 \leq R\left[1 + \left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho} + \left(\sum_{n=1}^{\infty} |\beth_{2_n} - \beth_{1_n}|^{\rho}\right)^{1/\rho}\right].$$
The right-hand side is clearly larger than 1, so the inequality holds. Similar reasoning applies if $\beth_4 = \beth_2$ or $\beth_4 = \beth_3$.

Case 2: Two points coincide. Assume $\beth_1 = \beth_2 \neq \beth_3$. We consider three subcases:

- **Subcase 1:** If $\beth_4 \notin \{\beth_1, \beth_3\}$, then:
$$\mathbb{M}(\beth_1, \beth_2, \beth_3) = \left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho}.$$
The tetrahedral inequality becomes:
$$\left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho} \leq R\left[1 + 1 + \left(\sum_{n=1}^{\infty} |\beth_{1_n} - \ell_{1_n}|^{\rho}\right)^{1/\rho}\right].$$

This holds because $\left(\sum |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho} \leq 2^{1/\rho}\left(\sum |\beth_{1_n} - \ell_{1_n}|^{\rho} + |\ell_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho}$ by the quasi-triangle inequality in ℓ_{ρ}.
- **Subcase 2:** If $\beth_4 = \beth_1$, then:
$$\left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho} \leq R\left[\left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho} + \left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho} + 0\right].$$
Simplifying, this reduces to $1 \leq 2R$, which holds since $R \geq 2$.
- **Subcase 3:** If $\beth_4 = \beth_3$, then:
$$\left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho} \leq R\left[0 + 0 + \left(\sum_{n=1}^{\infty} |\beth_{1_n} - \beth_{3_n}|^{\rho}\right)^{1/\rho}\right].$$
This simplifies to $1 \leq R$, which is true.

Case 3: Violation of D-Metric Axiom. Consider the vectors:
$\beth_1 = (1,1, \ldots, 1,0,0, \ldots), \beth_2 = (-1,-1, \ldots, -1,0,0, \ldots), \beth_3 = (1,-1, \ldots, -1,0,0, \ldots)$, each with $2n$ non-zero elements. Then:
$$\mathbb{M}(\beth_1, \beth_1, \beth_2) = 2 \cdot (2n)^{1/\rho}, \mathbb{M}(\beth_1, \beth_1, \beth_3) = 2 \cdot n^{1/\rho}, \mathbb{M}(\beth_1, \beth_3, \beth_2) = 1.$$
For $n = 2$ and $\rho = 1/2$, we have:
$$32 = \mathbb{M}(\beth_1, \beth_1, \beth_2)\mathbb{M}(\beth_1, \beth_1, \beth_3) + \mathbb{M}(\beth_1, \beth_3, \beth_2) + \mathbb{M}(\beth_3, \beth_1, \beth_2) = 8 + 1 + 1 = 10.$$
Thus, $\mathbb{M}$ is not a D-metric.

Theorem 1.1: *Let* $(\mathbb{Y}, \mathbb{M})$ *be an* MR*-metric space satisfying property* $(\mathbb{M}5)$. *Then any function* $d: \mathbb{Y} \times \mathbb{Y} \to \mathbb{R}^+$ *defined as follows is a* b*-metric on* $\mathbb{Y}$*:*

(1) For $1 \leq q < \infty$,

$$d(\beth_1, \beth_2) = (\mathbb{M}^q(\beth_1, \beth_2, \beth_2) + \mathbb{M}^q(\beth_1, \beth_1, \beth_2))^{1/q}.$$

(2) For the limiting case ($q \to \infty$),

$$d(\beth_1, \beth_2) = \max\{\mathbb{M}(\beth_1, \beth_2, \beth_2), \mathbb{M}(\beth_1, \beth_1, \beth_2)\}, \forall \beth_1, \beth_2 \in \mathbb{Y}.$$

Proof: We provide a detailed proof for case (i). The proof for case (ii) follows similarly by replacing the q-norm with the maximum norm.

Part (i): Verification of b-Metric Axioms for $1 \leq q < \infty$

Axiom 1 (Non-negativity and Identity of Indiscernibles):

- By definition, $\mathbb{M}(\beth_1, \beth_2, \beth_2) \geq 0$ and $\mathbb{M}(\beth_1, \beth_1, \beth_2) \geq 0$ for all $\beth_1, \beth_2 \in \mathbb{Y}$. Thus, $d(\beth_1, \beth_2) \geq 0$.
- If $\beth_1 = \beth_2$, then $\mathbb{M}(\beth_1, \beth_2, \beth_2) = \mathbb{M}(\beth_1, \beth_1, \beth_2) = 0$, so $d(\beth_1, \beth_2) = 0$.
- Conversely, if $d(\beth_1, \beth_2) = 0$, then $\mathbb{M}^q(\beth_1, \beth_2, \beth_2) = \mathbb{M}^q(\beth_1, \beth_1, \beth_2) = 0$. By property $(\mathbb{M}2)$ of the MR-metric, this implies $\beth_1 = \beth_2$.

Axiom 2 (Symmetry): For all $\beth_1, \beth_2 \in \mathbb{Y}$, we have:

$$d(\beth_1, \beth_2) = (\mathbb{M}^q(\beth_1, \beth_2, \beth_2) + \mathbb{M}^q(\beth_1, \beth_1, \beth_2))^{1/q}.$$

By the symmetry property $(\mathbb{M}3)$ of the MR-metric, $\mathbb{M}(\beth_1, \beth_2, \beth_2) = \mathbb{M}(\beth_2, \beth_2, \beth_1)$ and $\mathbb{M}(\beth_1, \beth_1, \beth_2) = \mathbb{M}(\beth_2, \beth_1, \beth_1)$. Thus:

$$d(\beth_1, \beth_2) = (\mathbb{M}^q(\beth_2, \beth_2, \beth_1) + \mathbb{M}^q(\beth_2, \beth_1, \beth_1))^{1/q} = d(\beth_2, \beth_1).$$

Axiom 3 (Relaxed Triangle Inequality): Let $\beth_1, \beth_2, \beth_3 \in \mathbb{Y}$. We need to show:

$$d(\beth_1, \beth_2) \leq R[d(\beth_1, \beth_3) + d(\beth_3, \beth_2)],$$

for some $R \geq 1$.

Using the definition of d and property $(\mathbb{M}5)$ of the MR-metric, we have:

$$\mathbb{M}(\beth_1, \beth_2, \beth_2) \leq R[\mathbb{M}(\beth_1, \beth_3, \beth_3) + \mathbb{M}(\beth_3, \beth_2, \beth_2)],$$

and

$$\mathbb{M}(\beth_1, \beth_1, \beth_2) \leq R[\mathbb{M}(\beth_1, \beth_1, \beth_3) + \mathbb{M}(\beth_3, \beth_3, \beth_2)].$$

Raising both inequalities to the power q and summing them, we obtain:

$$\mathbb{M}^q(\beth_1, \beth_2, \beth_2) + \mathbb{M}^q(\beth_1, \beth_1, \beth_2) \leq R^q[(\mathbb{M}(\beth_1, \beth_3, \beth_3) + \mathbb{M}(\beth_3, \beth_2, \beth_2))^q + (\mathbb{M}(\beth_1, \beth_1, \beth_3) + \mathbb{M}(\beth_3, \beth_3, \beth_2))^q].$$

Applying the Minkowski inequality for sums (which holds for $q \geq 1$), we get:

$$(\mathbb{M}^q(\beth_1, \beth_2, \beth_2) + \mathbb{M}^q(\beth_1, \beth_1, \beth_2))^{1/q} \leq R[(\mathbb{M}^q(\beth_1, \beth_3, \beth_3) + \mathbb{M}^q(\beth_1, \beth_1, \beth_3))^{1/q} + (\mathbb{M}^q(\beth_3, \beth_2, \beth_2) + \mathbb{M}^q(\beth_3, \beth_3, \beth_2))^{1/q}].$$

This simplifies to:

$$d(\beth_1, \beth_2) \leq R[d(\beth_1, \beth_3) + d(\beth_3, \beth_2)],$$

which is the relaxed triangle inequality for a b-metric.

The function d satisfies all axioms of a b-metric with coefficient $R \geq 1$. The proof for case (ii) is analogous, replacing the q-norm with the maximum norm and using the same properties of $\mathbb{M}$.

Theorem 1.2: *Consider a function* $\mathbb{M}: \mathbb{Y} \times \mathbb{Y} \times \mathbb{Y} \to [0, \infty)$ *that verifies the following conditions:*

$(\mathbb{M}1)$ *through* $(\mathbb{M}3)$,

- *along with* $(\mathbb{M}7)$ *and* $(\mathbb{M}8)$.

Then $\mathbb{M}$ defines a valid *MR-metric* on the set $\mathbb{Y}$.

Proof: The proof reduces to establishing property $(\mathbb{M}4)$. For arbitrary elements $\beth_1, \beth_2, \beth_3 \in \mathbb{Y}$, we derive:

$$\begin{aligned}\mathbb{M}(\beth_1, \beth_2, \beth_3) &\leq \frac{1}{R}[\mathbb{M}(\beth_1, \beth_4, \beth_4) + \mathbb{M}(\beth_4, \beth_2, \beth_4) \\ &\quad +\mathbb{M}(\beth_4, \beth_4, \beth_3)] \text{ (using } (\mathbb{M}4)). \\ &\leq \frac{R}{R}[\mathbb{M}(\beth_1, \beth_4, \beth_3) + \mathbb{M}(\beth_2, \beth_4, \beth_1) \\ &\quad +\mathbb{M}(\beth_3, \beth_4, \beth_2)] \text{ for some } \beth_4 \in \mathbb{Y} (\text{by } (\mathbb{M}7)) \\ &\leq R[\mathbb{M}(\beth_1, \beth_4, \beth_3) + \mathbb{M}(\beth_2, \beth_4, \beth_1) + \mathbb{M}(\beth_3, \beth_4, \beth_2)].\end{aligned}$$

This establishes $(\mathbb{M}4)$, confirming that $\mathbb{M}$ indeed forms an MR-metric on $\mathbb{Y}$.

2. Construction Methods for MR-Metrics

This section presents several extension theorems in the context of MR-metric spaces. We define the set

$$\aleph = \left\{ (\eta_1, \eta_2, \eta_3) \in (\mathbb{R}^+)^3 : \quad \begin{array}{l} \eta_1 \leq \frac{1}{R}(\eta_2 + \eta_3), \\ \eta_2 \leq \frac{1}{R}(\eta_1 + \eta_3), \\ \eta_3 \leq \frac{1}{R}(\eta_1 + \eta_2) \end{array} \right\},$$

where $R > 1$ is a given constant.

Theorem 2.1: *Consider a mapping* $\pi: \aleph \to \mathbb{R}^+$ *that satisfies the following conditions:*

(1) $\pi(\eta_1, \eta_2, \eta_3) = \pi(p(\eta_1, \eta_2, \eta_3))$ for any permutation p of η_1, η_2, η_3.

(2) $\pi(\eta_1, \eta_2, \eta_3) = 0$ if and only if $\eta_1 = \eta_2 = \eta_3 = 0$.

(3) $\pi(t, t, 0) \leq \pi(\eta, \eta_1, \eta_2)$ for all $(\eta, \eta_1, \eta_2) \in \aleph$.

(4) $\pi(\eta_1, \eta_2, \eta_3) \leq \frac{1}{R}[\pi(\eta_1', \eta, \eta_1'') + \pi(\eta_2', \eta, \eta_2'') + \pi(\eta_3', \eta, \eta_3'')]$ for all $(\eta_1, \eta_2, \eta_3), (\eta_1', \eta, \eta_1''), (\eta_2', \eta, \eta_2''), (\eta_3', \eta, \eta_3'') \in \aleph$, where the triples $(\eta_1, \eta_1', \eta_2'), (\eta_1, \eta_1'', \eta_2''), (\eta_2, \eta_2', \eta_3'), (\eta_2, \eta_2'', \eta_3''), (\eta_3, \eta_3', \eta_1')$, $(\eta_3, \eta_3'', \eta_1'')$ are in $\aleph$.

Let $(\mathbb{Y}, d)$ be a b-metric space. Define $\mathbb{M}: \mathbb{Y} \times \mathbb{Y} \times \mathbb{Y} \to [0, \infty)$ by:

$$\mathbb{M}(\beth_1, \beth_2, \beth_3) = R\pi(d(\beth_1, \beth_2), d(\beth_2, \beth_3), d(\beth_3, \beth_1)).$$

Then, $\mathbb{M}$ defines an MR-metric on $\mathbb{Y}$.

Proof: We verify that $\mathbb{M}$ satisfies all axioms of an MR-metric. By construction, $\mathbb{M}$ inherits properties $(\mathbb{M}1)$, $(\mathbb{M}2)$, and $(\mathbb{M}3)$ directly from π and d. Thus, we focus on proving $(\mathbb{M}7)$ and $(\mathbb{M}8)$.

Part 1: Verification of Property $(\mathbb{M}7)$

For any $\beth_1, \beth_2 \in \mathbb{Y}$, we have:

$$\mathbb{M}(\beth_1, \beth_2, \beth_2) = R\pi(d(\beth_1, \beth_2), 0, d(\beth_2, \beth_1)).$$

By the symmetry of d and property (i) of π, this simplifies to:

$$\mathbb{M}(\beth_1, \beth_2, \beth_2) = R\pi(d(\beth_1, \beth_2), d(\beth_1, \beth_2), 0).$$

From property (iii) of π, we know:

$$\pi(t, t, 0) \leq \pi(\eta, \eta_1, \eta_2) \text{ for any} (\eta, \eta_1, \eta_2) \in \aleph.$$

Applying this with $t = d(\beth_1, \beth_2)$ and $(\eta, \eta_1, \eta_2) = (d(\beth_1, \beth_2), d(\beth_2, \beth_3), d(\beth_3, \beth_1))$, we obtain:

$$\pi(d(\beth_1, \beth_2), d(\beth_1, \beth_2), 0) \leq \pi(d(\beth_1, \beth_2), d(\beth_2, \beth_3), d(\beth_3, \beth_1)).$$

Multiplying both sides by R yields:

$$\mathbb{M}(\beth_1, \beth_2, \beth_2) \leq \mathbb{M}(\beth_1, \beth_2, \beth_3).$$

This establishes property $(\mathbb{M}7)$.

Part 2: Verification of Property $(\mathbb{M}8)$

Let $\beth_1, \beth_2, \beth_3, \beth_4, \beth_5 \in \mathbb{Y}$. We analyze $\mathbb{M}(\beth_1, \beth_2, \beth_3)$ using property (iv) of π. First, observe that:

$$\mathbb{M}(\beth_1, \beth_2, \beth_3) = R\pi(d(\beth_1, \beth_2), d(\beth_2, \beth_3), d(\beth_3, \beth_1)).$$

By property (iv) of π, we have:

$$\pi(d(\beth_1, \beth_2), d(\beth_2, \beth_3), d(\beth_3, \beth_1)) \leq \frac{1}{R}[\pi_1 + \pi_2 + \pi_3],$$

where:

$$\begin{aligned} \pi_1 &= \pi(d(\beth_1, \beth_4), d(\beth_4, \beth_5), d(\beth_5, \beth_1)), \\ \pi_2 &= \pi(d(\beth_2, \beth_4), d(\beth_4, \beth_5), d(\beth_5, \beth_2)), \\ \pi_3 &= \pi(d(\beth_3, \beth_4), d(\beth_4, \beth_5), d(\beth_5, \beth_3)). \end{aligned}$$

Multiplying through by R gives:

$$\mathbb{M}(\beth_1, \beth_2, \beth_3) \leq \pi_1 + \pi_2 + \pi_3.$$

Now, recognize that:

$$\pi_1 = \frac{1}{R}\mathbb{M}(\beth_1, \beth_4, \beth_5), \quad \pi_2 = \frac{1}{R}\mathbb{M}(\beth_4, \beth_2, \beth_5), \quad \pi_3 = \frac{1}{R}\mathbb{M}(\beth_4, \beth_5, \beth_3).$$

Substituting these back, we obtain:

$$\mathbb{M}(\beth_1, \beth_2, \beth_3) \leq \frac{1}{R}[\mathbb{M}(\beth_1, \beth_4, \beth_5) + \mathbb{M}(\beth_4, \beth_2, \beth_5) + \mathbb{M}(\beth_4, \beth_5, \beth_3)].$$

This verifies property $(\mathbb{M}8)$.

Since $\mathbb{M}$ satisfies all required properties $(\mathbb{M}1)$–$(\mathbb{M}8)$, it defines a valid MR-metric on $\mathbb{Y}$.

Theorem 2.2: *Consider a mapping* $F: \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying:*

(1) *Non-triviality:* $F(\eta) = 0 \Leftarrow \eta = 0$;

(2) *Monotonicity:* $\eta_1 \leq \eta_2 \Rightarrow F(\eta_1) \leq F(\eta_2)$;

(3) *Weighted subadditivity:* $F(s+t) \leq R^{-1}[F(s) + F(t)]$ *for all* $s, t \in \mathbb{R}^+$ *(*$R \geq 1$*).*

Then the function $\pi(\eta_1, \eta_2, \eta_3) = \sum_{i=1}^{3} F(\eta_i)$ satisfies all conditions of Theorem 2.1.

Proof: We systematically verify each required property for π.

1. **Verification of Property (i) (Symmetry)** For any permutation p of $\{\eta_1, \eta_2, \eta_3\}$:
$$\pi(\eta_1, \eta_2, \eta_3) = F(\eta_1) + F(\eta_2) + F(\eta_3) = \pi(p(\eta_1, \eta_2, \eta_3)).$$
Thus, π is fully symmetric.

2. **Verification of Property (ii) (Definiteness)**
$$\begin{aligned} \pi(\eta_1, \eta_2, \eta_3) = 0 &\Leftarrow F(\eta_1) + F(\eta_2) + F(\eta_3) = 0 \\ &\Leftarrow F(\eta_i) = 0 \text{ for } i = 1,2,3 \text{ (since } F \geq 0) \\ &\Leftarrow \eta_1 = \eta_2 = \eta_3 = 0 \text{ (by non} - \text{triviality of } F). \end{aligned}$$

3. **Verification of Property (iii) (Dominance of Diagonal)** For any $(\eta, \eta_1, \eta_2) \in \aleph$:
$$\pi(t, t, 0) = 2F(t) + F(0) = 2F(t).$$
By monotonicity of F:
$$2F(t) \leq F(\eta) + F(\eta_1) + F(\eta_2) = \pi(\eta, \eta_1, \eta_2).$$

4. **Verification of Property (iv) (Generalized Subadditivity)** Let $(\eta_1, \eta_2, \eta_3) \in \aleph$ and consider the auxiliary triples:
$$(\eta_1', \eta, \eta_1''), (\eta_2', \eta, \eta_2''), (\eta_3', \eta, \eta_3'') \in \aleph$$
$$\text{with } (\eta_1, \eta_1', \eta_2'), (\eta_1, \eta_1'', \eta_2''), \dots, (\eta_3, \eta_3', \eta_1'') \in \aleph.$$

We analyze $\pi(\eta_1, \eta_2, \eta_3)$ through multiple applications of the weighted subadditivity:
$$\begin{aligned} \pi(\eta_1, \eta_2, \eta_3) &= F(\eta_1) + F(\eta_2) + F(\eta_3) \\ &\leq F(\eta_1' + \eta + \eta_2'') + F(\eta_2' + \eta + \eta_3'') + F(\eta_3' + \eta + \eta_1'') \\ &\leq R^{-1}[F(\eta_1') + F(\eta) + F(\eta_2'')] \\ &\quad + R^{-1}[F(\eta_2') + F(\eta) + F(\eta_3'')] \\ &\quad + R^{-1}[F(\eta_3') + F(\eta) + F(\eta_1'')] \text{ (by sub additivity)} \\ &= R^{-1}[(F(\eta_1') + F(\eta) + F(\eta_1'')) \end{aligned}$$

$$+(F(\eta_2') + F(\eta) + F(\eta_2''))$$
$$+(F(\eta_3') + F(\eta) + F(\eta_3''))]$$
$$= R^{-1}[\pi(\eta_1', \eta, \eta_1'') + \pi(\eta_2', \eta, \eta_2'') + \pi(\eta_3', \eta, \eta_3'')].$$

Conclusion

The function π satisfies all four required properties:

(1) Complete symmetry under permutations

(2) Definiteness (vanishing only when all inputs are zero)

(3) Dominance of the diagonal term $\pi(t, t, 0)$

(4) Generalized subadditivity condition

Thus, π fulfills all conditions specified in Theorem 2.1.

To clarify the difference between requirements (2) and (3) in Theorem 2.1, we construct an explicit example.

Example 2.1: (Function Verification and Independence of Conditions). Consider the linear function $F: \mathbb{R}^+ \to \mathbb{R}^+$ defined by $F(\eta) = 2\eta$ for all $\eta \in \mathbb{R}^+$. We verify it satisfies all conditions of Theorem 2.2:

(i) **Non-triviality:** $F(\eta) = 0 \Leftrightarrow 2\eta = 0 \Leftrightarrow \eta = 0$.

(ii) **Monotonicity:** For $\eta_1 \leq \eta_2$, we have $2\eta_1 \leq 2\eta_2$, so $F(\eta_1) \leq F(\eta_2)$.

(iii) **Weighted subadditivity:** $F(s + t) = 2(s + t) = 2s + 2t = \frac{1}{1}[F(s) + F(t)]$ (with $R = 1$).

To demonstrate the independence of conditions (ii) and (iii):

- **Example satisfying (ii) but not (iii):** Take $\pi(\eta) = 2\eta^2$. This is strictly increasing (satisfies (ii)), but fails subadditivity since:
$$\pi(1 + 1) = 8 > 4 = \pi(1) + \pi(1).$$
- **Example satisfying (iii) but not (ii):** Consider the piecewise function:
$$F(\eta) = \begin{cases} 0 & \eta = 0, \\ \eta + \frac{1}{\eta} & \eta > 0. \end{cases}$$
This satisfies (iii) with $R = 1$ since for $s, t > 0$:
$$F(s + t) = s + t + \frac{1}{s+t} \leq s + \frac{1}{s} + t + \frac{1}{t} = F(s) + F(t).$$
However, it's not monotonic: $F(0.5) = 2.5 > 2.1 \approx F(1)$.

Theorem 2.3: (Construction of MR-Metrics from b-Metrics). *Consider a* b *-metric space* $(\mathbb{Y}, d)$ *with constant* $R \geq 1$. *Define* $\mathbb{M}_i: \mathbb{Y}^3 \to \mathbb{R}^+$ *(* $i = 1, \infty, 3, 4$*) by:*

$$\mathbb{M}_1(\beth_1, \beth_2, \beth_3) = d(\beth_1, \beth_2) + d(\beth_2, \beth_3) + d(\beth_3, \beth_1),$$
$$\mathbb{M}_\infty(\beth_1, \beth_2, \beth_3) = \max\{d(\beth_1, \beth_2), d(\beth_2, \beth_3), d(\beth_3, \beth_1)\},$$

$$\mathbb{M}_3(\beth_1,\beth_2,\beth_3) = \begin{cases} \mathbb{M}_1(\beth_1,\beth_2,\beth_3) & \text{if distinct,} \\ \mathbb{M}_\infty(\beth_1,\beth_2,\beth_3) & \text{otherwise,} \end{cases}$$
$$\mathbb{M}_4(\beth_1,\beth_2,\beth_3) = \begin{cases} \mathbb{M}_\infty(\beth_1,\beth_2,\beth_3) & \text{if distinct,} \\ \mathbb{M}_1(\beth_1,\beth_2,\beth_3) & \text{otherwise.} \end{cases}$$

Then all four functions are MR-metrics on $\mathbb{Y}$ with the same constant R.

Proof. We verify all MR-metric axioms for $\mathbb{M}_4$, focusing on the most intricate case - the tetrahedral inequality.

Case 1: Distinct Points ($\beth_1,\beth_2,\beth_3$ pairwise distinct). Assume without loss of generality that:

$$d(\beth_1,\beth_2) \le d(\beth_2,\beth_3) \le d(\beth_3,\beth_1) = \mathbb{M}_4(\beth_1,\beth_2,\beth_3).$$

Subcase 1.1: $\beth_4 \notin \{\beth_1,\beth_2,\beth_3\}$

$$\begin{aligned} \mathbb{M}_4(\beth_1,\beth_2,\beth_3) &= d(\beth_3,\beth_1) \\ &\le R[d(\beth_3,\beth_4) + d(\beth_4,\beth_1)] \text{ (b} - \text{metric inequality)} \\ &\le R[\mathbb{M}_4(\beth_4,\beth_2,\beth_3) + \mathbb{M}_4(\beth_1,\beth_4,\beth_3)] \\ &\le R\textstyle\sum_{i=1}^{3} \mathbb{M}_4(\text{appropriate terms}). \end{aligned}$$

Subcase 1.2: $\beth_4 = \beth_1$

$$\begin{aligned} \mathbb{M}_4(\beth_1,\beth_2,\beth_3) &= d(\beth_3,\beth_1) \\ &= \mathbb{M}_4(\beth_4,\beth_2,\beth_3) \\ &\le R[\mathbb{M}_4(\beth_1,\beth_2,\beth_4) + \mathbb{M}_4(\beth_1,\beth_4,\beth_3) + \mathbb{M}_4(\beth_4,\beth_2,\beth_3)]. \end{aligned}$$

Case 2: Two Points Equal ($\beth_1 = \beth_2 \ne \beth_3$) Here $\mathbb{M}_4(\beth_1,\beth_2,\beth_3) = 2d(\beth_1,\beth_3)$.

Subcase 2.1: $\beth_4 \notin \{\beth_1,\beth_3\}$

$$\begin{aligned} 2d(\beth_1,\beth_3) &\le R[2d(\beth_1,\beth_4) + 2d(\beth_4,\beth_3)] \\ &= R[2\mathbb{M}_4(\beth_1,\beth_1,\beth_4) + 2\mathbb{M}_4(\beth_4,\beth_3,\beth_3)] \\ &\le R\textstyle\sum \mathbb{M}_4(\text{appropriate terms}). \end{aligned}$$

Subcase 2.2: $\beth_4 = \beth_1$

$$\begin{aligned} 2d(\beth_1,\beth_3) &= \mathbb{M}_4(\beth_1,\beth_1,\beth_3) \\ &\le R[\mathbb{M}_4(\beth_1,\beth_1,\beth_4) + \mathbb{M}_4(\beth_1,\beth_4,\beth_3) + \mathbb{M}_4(\beth_4,\beth_1,\beth_3)]. \end{aligned}$$

Verification of Other Axioms:

- **Symmetry:** Immediate from the symmetry of d and the definitions.
- **Definiteness:** $\mathbb{M}_4(\beth_1,\beth_2,\beth_3) = 0 \Leftrightarrow \beth_1 = \beth_2 = \beth_3$ follows from the definiteness of d.
- **Non-negativity:** Clear since $d \ge 0$.

Thus, $\mathbb{M}_4$ satisfies all axioms of an MR-metric. The proofs for $\mathbb{M}_1$, $\mathbb{M}_\infty$, and $\mathbb{M}_3$ follow similar patterns with appropriate modifications for their respective definitions.

3. Variants of Convergence in MR-Metric Spaces

Building upon previous work in generalized metric spaces, we develop new convergence criteria specifically for MR-metric spaces. These concepts provide a framework for analyzing sequence behavior in this extended context.

Definition 3.1: (MR-Limit of a Sequence). In an MR-metric space $(\mathbb{Y},\mathbb{M})$, a sequence $(\beth_n)_{n\in\mathbb{N}}$ is said to converge to a limit $\beth_* \in \mathbb{Y}$ with respect to the MR-metric if the following condition holds:

$$(\forall\varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)\ \mathbb{M}(\beth_n, \beth_m, \beth_*) < \varepsilon$$

When this condition holds, we denote $\beth_n \overset{MR}{\longrightarrow} \beth_*$.

Definition 3.2: (MR-Cauchy Sequence). A sequence $(\beth_n)$ in $(\mathbb{Y},\mathbb{M})$ is MR-Cauchy if:

$$(\forall\varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n, p \geq N)\quad \mathbb{M}(\beth_n, \beth_m, \beth_p) < \varepsilon$$

We introduce two strengthened convergence notions:

Definition 3.3: (Strong MR-Convergence). A sequence $(\beth_n)$ converges strongly to $\beth_*$ in $(\mathbb{Y},\mathbb{M})$ if:

(1) $\lim\limits_{\min(m,n)\to\infty} \mathbb{M}(\beth_n, \beth_m, \beth_*) = 0$

(2) For all anchor points $\beta \in \mathbb{Y}$, $\lim\limits_{n\to\infty} \mathbb{M}(\beta, \beta, \beth_n) = \mathbb{M}(\beta, \beta, \beth_*)$

Definition 3.4: (Very Strong MR-Convergence). A sequence $(\beth_n)$ converges very strongly to $\beth_*$ when:

(1) $\lim\limits_{\min(m,n)\to\infty} \mathbb{M}(\beth_n, \beth_m, \beth_*) = 0$

(2) For all pairs $\beth_2, \beth_3 \in \mathbb{Y}$, $\lim\limits_{n\to\infty} \mathbb{M}(\beth_2, \beth_3, \beth_n) = \mathbb{M}(\beth_2, \beth_3, \beth_*)$

Our investigation focuses on four fundamental convergence types in MR-metric spaces:

- Standard MR-convergence
- MR-Cauchy sequences
- Strong MR-convergence
- Very strong MR-convergence

The subsequent analysis develops the interrelationships between these concepts, supported by foundational results from preliminary observations.

Theorem 3.1: *Let* $(\mathbb{Y},\mathbb{M})$ *be an* MR*-metric space that satisfies condition* $(\mathbb{M}5)$. *A sequence* $\{\beth_n\}$ *in* $(\mathbb{Y},\mathbb{M})$ *is* MR*-strongly convergent to* $\beth_*$ *if and only if:*

(1) It is MR-convergent to $\beth_*$;

(2) $\lim\limits_{n\to\infty} \mathbb{M}(\beth_*, \beth_*, \beth_n) = 0.$

Proof: We prove both directions of the equivalence with careful attention to the underlying metric space properties.

Necessity ($\Rightarrow$): Assume $\{ב_n\}$ is MR-strongly convergent to $ב_*$. By definition, this immediately gives us two crucial properties. First, the sequence must be MR-convergent to $ב_*$ since strong convergence implies standard convergence. Second, taking the fixed point $\beta = ב_*$ in the definition of strong convergence yields:

$$\lim_{n\to\infty} \mathbb{M}(ב_*, ב_*, ב_n) = \mathbb{M}(ב_*, ב_*, ב_*) = 0$$

where the final equality follows from the definiteness property of the MR-metric.

Sufficiency ($\Leftarrow$): Now assume $\{ב_n\}$ is MR-convergent to $ב_*$ and satisfies $\lim_{n\to\infty} \mathbb{M}(ב_*, ב_*, ב_n) = 0$. To establish strong convergence, we need to verify both conditions in its definition.

For the joint limit condition, observe that for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$:

$$\mathbb{M}(ב_*, ב_*, ב_n) < \frac{\varepsilon}{3R}$$

and by MR-convergence, there exists $N_2 \in \mathbb{N}$ such that for all $m, n \geq N_2$:

$$\mathbb{M}(ב_n, ב_m, ב_*) < \frac{\varepsilon}{3}$$

Taking $N = \max\{N_1, N_2\}$, we can bound the metric for all $m, n \geq N$ using the tetrahedral inequality:

$$\begin{aligned} \mathbb{M}(ב_n, ב_m, ב_*) &\leq R[\mathbb{M}(ב_n, ב_*, ב_*) + \mathbb{M}(ב_*, ב_m, ב_*) + \mathbb{M}(ב_*, ב_*, ב_*)] \\ &< R\left[\frac{\varepsilon}{3R} + \frac{\varepsilon}{3R} + 0\right] = \frac{2\varepsilon}{3} < \varepsilon \end{aligned}$$

For the fixed-point condition, let $\beta \in \mathbb{Y}$ be arbitrary. Using condition $(\mathbb{M}5)$, we have for each n:

$$\mathbb{M}(\beta, \beta, ב_n) \leq R[\mathbb{M}(ב_n, ב_*, ב_*) + \mathbb{M}(ב_*, \beta, \beta)]$$

which implies:

$$|\mathbb{M}(\beta, \beta, ב_n) - \mathbb{M}(\beta, \beta, ב_*)| \leq R\mathbb{M}(ב_*, ב_*, ב_n)$$

Since the right side tends to 0 as $n \to \infty$, we conclude:

$$\lim_{n\to\infty} \mathbb{M}(\beta, \beta, ב_n) = \mathbb{M}(\beta, \beta, ב_*)$$

This completes the verification of both conditions for strong convergence.

The proof establishes not only the equivalence but also reveals the fundamental relationship between the diagonal behavior of the metric $(\mathbb{M}(ב_*, ב_*, \cdot))$ and the strong convergence property. The argument carefully employs all relevant metric space axioms, particularly condition $(\mathbb{M}5)$, to maintain strict mathematical rigor throughout.

Theorem 3.2: *Consider an MR-metric space $(\mathbb{Y}, \mathbb{M})$, where $\{ב_n\}$ is a sequence of elements in $\mathbb{Y}$ and $ב \in \mathbb{Y}$ is a given point. Assume the following conditions hold:*

(1) $\mathbb{M}(\beth_1, \beth_1, \beth_{1_l}) \to 0 \;\; as \;\; l \to \infty$,
(2) $\mathbb{M}(\beth_1, \beth_{1_l}, \beth_{1_l}) \to 0 \;\; as \;\; l \to \infty$,
(3) $\mathbb{M}(\beth_1, \beth_{1_l}, \beth_{1_m}) \to 0 \;\; as \;\; l, m \to \infty$,
(4) $\mathbb{M}(\beth_2, \beth_2, \beth_{1_l}) \to \mathbb{M}(\beth_2, \beth_2, \beth_1) \;\; as \;\; l \to \infty \;\; for\ all \;\; \beth_2 \in \mathbb{Y}$,
(5) $\mathbb{M}(\beth_2, \beth_{1_l}, \beth_{1_m}) \to \mathbb{M}(\beth_2, \beth_1, \beth_1) \;\; as \;\; l, m \to \infty \;\; for\ all \;\; \beth_2 \in \mathbb{Y}$,
(6) $\mathbb{M}(\beth_2, \beth_1, \beth_{1_l}) \to \mathbb{M}(\beth_2, \beth_1, \beth_1) \;\; as \;\; l \to \infty \;\; for\ all \;\; \beth_2 \in \mathbb{Y}$,
(7) $\mathbb{M}(\beth_2, \beth_3, \beth_{1_l}) \to \mathbb{M}(\beth_2, \beth_3, \beth_1) \;\; as \;\; l \to \infty \;\; for\ all \;\; \beth_2, \beth_3 \in \mathbb{Y}$.

The following implications hold:

- $(7) \Rightarrow (6) \Rightarrow (1)$,
- $(7) \Rightarrow (4) \Rightarrow (1)$,
- $(5) \Rightarrow (3) \Rightarrow (2)$.

In the following example, we show that the converse of some properties in Theorem 3.2 need not be true.

Example 3.1: Neither (3) nor (4) necessarily imply (5), (6), or (7).

Consider the space $\mathbb{Y} = \mathbb{R}$ equipped with an MR-metric. Let the function $\mathbb{M}_3$ be defined on $\mathbb{Y} \times \mathbb{Y} \times \mathbb{Y}$ as follows:

$$\mathbb{M}_3(\beth_1, \beth_2, \beth_3) = \begin{cases} 0 & \text{if } \beth_1 = \beth_2 = \beth_3, \\ |\beth_1 - \beth_2| + |\beth_2 - \beth_3| + |\beth_3 - \beth_1| & \text{if } \beth_1, \beth_2, \beth_3 \text{ are distinct,} \\ \max\{|\beth_1 - \beth_2|, |\beth_2 - \beth_3|, |\beth_3 - \beth_1|\} & \text{otherwise.} \end{cases}$$

The structure $(\mathbb{Y}, \mathbb{M}_3)$ constitutes an MR-metric space that verifies axioms (3) and (4), while failing to satisfy (5), (6), and (7).

Consider the sequence $\{\alpha_n\}$ defined by $\alpha_n = 2^{1/n}$ for $n \in \mathbb{N}$. This sequence -converges to 1 in the given metric space.

For indices $m > n$, we compute:

$$\mathbb{M}_3(1, \alpha_n, \alpha_m) = |1 - \alpha_n| + |\alpha_m - \alpha_n| + |\alpha_m - 1| \to 0 \text{ as } n, m \to \infty,$$

demonstrating the convergence of $\{\alpha_n\}$ to 1 under $\mathbb{M}_3$.

For arbitrary $\beta \in \mathbb{Y}$, we observe:

$$\mathbb{M}_3(\beta, \beta, \alpha_n) = \max\{|\beta - \alpha_n|, 0, |\beta - \alpha_n|\} = |\beta - \alpha_n|,$$

which establishes that $\lim_{n\to\infty} \mathbb{M}_3(\beta, \beta, \alpha_n) = \mathbb{M}_3(\beta, \beta, 1)$, confirming axioms (3) and (4).

However, taking $\beta = 3$ and $\gamma = 1$ reveals:

$$\lim_{n\to\infty} \mathbb{M}_3(3,1,\alpha_n) \neq \mathbb{M}_3(3,1,1),$$

showing the failure of (6) and (7). Moreover:

$$\lim_{n,m\to\infty} \mathbb{M}_3(3, \alpha_n, \alpha_m) \neq \mathbb{M}_3(3,1,1),$$

which violates axiom (5).

Theorem 3.3: *Consider an MR-metric space* $(\mathbb{Y}, \mathbb{M})$ *that satisfies axioms* $\mathbb{M}6$*,* $\mathbb{M}7$*, and* $\mathbb{M}8$*. The function* $d: \mathbb{Y} \times \mathbb{Y} \to [0, \infty)$ *given by:*

$$d(\beth, \beta) = \mathbb{M}(\beth, \beta, \beta)$$

forms a b-metric on $\mathbb{Y}$*. Under these conditions, the following statements are equivalent:*

(1) *The sequence* $\{\beth_n\}$ *converges to* $\beth$ *in the metric space* $(\mathbb{Y}, d)$*;*

(2) The sequence $\{\beth_n\}$ converges to $\beth$ in the MR-metric space $(\mathbb{Y}, \mathbb{M})$;

(3) The sequence $\{\beth_n\}$ exhibits strong convergence to $\beth$ in $(\mathbb{Y}, \mathbb{M})$.

Proof. First, Proposition 1.1 immediately establishes that $(\mathbb{Y}, d)$ forms a b-metric space.

Implication (1) ⇒ (2): Suppose $\{\beth_n\}$ converges to $\beth$ in $(\mathbb{Y}, d)$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$d(\beth, \beth_n) < \frac{\varepsilon}{2}$$

Applying axiom $\mathbb{M}8$ for $n, m \geq N$:

$$\begin{aligned}\mathbb{M}(\beth, \beth_n, \beth_m) &\leq \frac{1}{R}[\mathbb{M}(\beth, \beth, \beth_m) + \mathbb{M}(\beth, \beth, \beth_n)] \\ &= \frac{1}{R}[d(\beth, \beth_m) + d(\beth, \beth_n)] < \varepsilon\end{aligned}$$

This proves convergence in $(\mathbb{Y}, \mathbb{M})$.

Implication (2) ⇒ (3): Assume $\{\beth_n\}$ converges to $\beth$ in $(\mathbb{Y}, \mathbb{M})$. For any $\varepsilon > 0$, there exists N such that for $n, m \geq N$:

$$\mathbb{M}(\beth, \beth_n, \beth_m) < \varepsilon$$

For any $\beta \in \mathbb{Y}$ and $n \geq N$:

$$\begin{aligned}\mathbb{M}(\beta, \beta, \beth_n) &\leq R\mathbb{M}(\beth, \beta, \beth_n) \text{ (by } \mathbb{M}7) \\ &\leq \mathbb{M}(\beth, \beth, \beth_n) + \mathbb{M}(\beta, \beth, \beth) \text{ (by } \mathbb{M}8) \\ &\leq \mathbb{M}(\beth, \beth_n, \beth_n) + \mathbb{M}(\beta, \beta, \beth) \text{ (by } \mathbb{M}6)\end{aligned}$$

This yields:

$$|\mathbb{M}(\beta, \beta, \beth_n) - \mathbb{M}(\beta, \beta, \beth)| < \frac{\varepsilon}{2}$$

establishing strong convergence.

Implication (3) ⇒ (2): This direction is immediate.

Implication (2) ⇒ (1): Given convergence in $(\mathbb{Y}, \mathbb{M})$, for any $\varepsilon > 0$ there exists N such that for $n \geq N$:

$$\mathbb{M}(\beth, \beth_n, \beth_n) < \varepsilon$$

By $\mathbb{M}7$:

$$d(\beth, \beth_n) = \mathbb{M}(\beth, \beth, \beth_n) \leq R\mathbb{M}(\beth, \beth_m, \beth_n) < R\varepsilon$$

proving convergence in $(\mathbb{Y}, d)$.

4. On the Existence of Unique Simultaneous Fixed Points in Complete MR-Metric Spaces

This section develops new fixed point theory for pairs of mappings in MR-metric spaces. We introduce and examine three fundamental concepts:

- Open set structures in MR-metric spaces
- MR-boundedness of functions
- MR-continuity conditions

Additionally, we establish a crucial inequality for MR-metric spaces through the following lemma.

Lemma 4.1: *Let* $(\mathbb{Y}, \mathbb{M})$ *be an* MR*-metric space. Then the following inequality holds:*

$$\mathbb{M}(\beth_1, \beth_1, \beth_2) \leq \tfrac{2R}{1-R}\mathbb{M}(\beth_1, \beth_2, \beth_2).$$

Proof: Using condition $(\mathbb{M}4)$ and setting $u = \beth_2$, we obtain:

$$\mathbb{M}(\beth_1, \beth_1, \beth_2) \leq R\mathbb{M}(\beth_1, \beth_1, \beth_2) + R\mathbb{M}(\beth_1, \beth_2, \beth_2) + R\mathbb{M}(\beth_2, \beth_1, \beth_2).$$

Applying condition $(\mathbb{M}3)$, this becomes:

$$\mathbb{M}(\beth_1, \beth_1, \beth_2) \leq 2R\mathbb{M}(\beth_1, \beth_2, \beth_2) + R\mathbb{M}(\beth_1, \beth_1, \beth_2).$$

Rearranging terms, we get:

$$\mathbb{M}(\beth_1, \beth_1, \beth_2) \leq \tfrac{2R}{1-R}\mathbb{M}(\beth_1, \beth_2, \beth_2).$$

Next, we introduce key definitions, including the MR-metric ball, open sets, MR-boundedness, and MR-continuity.

Definition 4.1: Given an MR-metric space $(\mathbb{Y}, \mathbb{M})$, we define the open ball centered at $\beth \in \mathbb{Y}$ with radius $r > 0$ as:

$$\mathcal{B}(\beth, r) = \{\beta \in \mathbb{Y}: \mathbb{M}(\beth, \beta, \beta) < r\}.$$

Example 4.1: Consider $\mathbb{Y} = \mathbb{R}$ equipped with the MR-metric:

$$\mathbb{M}(x, y, z) = \tfrac{1}{3}(|x-y| + |y-z| + |z-x|).$$

The open ball centered at $a \in \mathbb{R}$ with radius r corresponds to the interval:

$$\mathcal{B}(a, r) = \{x \in \mathbb{R}: |x-a| < r\}.$$

Definition 4.2: For a subset A of an MR-metric space $(\mathbb{Y}, \mathbb{M})$:

(1) A is open if for each $\beth \in A$, there exists $\varepsilon > 0$ such that $\mathcal{B}(\beth, \varepsilon) \subseteq A$.

(2) A is MR-bounded if $\sup\{\mathbb{M}(\beth, \beta, \beta): \beth, \beta \in A\} < \infty$.

Definition 4.3: An MR-metric $\mathbb{M}$ on $\mathbb{Y}$ is continuous if for any convergent sequences:

$$\beth_n \to \beth, \beta_n \to \beta, \gamma_n \to \gamma \Rightarrow \mathbb{M}(\beth_n, \beta_n, \gamma_n) \to \mathbb{M}(\beth, \beta, \gamma).$$

Lemma 4.2*:* (Uniqueness of MR-Metric Limits). *In any MR-metric space* $(\mathbb{Y}, \mathbb{M})$*, if a sequence* $\{\gamma_n\}$ *converges to two points* γ *and* γ' *under the MR-metric, then* $\gamma = \gamma'$.

Proof: Assume for contradiction that $\{\gamma_n\}$ MR-converges to distinct points γ and γ'. By the convergence definition, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$:

$$\mathbb{M}(\gamma, \gamma_m, \gamma_n) < \frac{\varepsilon}{3R},$$
$$\mathbb{M}(\gamma_n, \gamma_m, \gamma') < \frac{\varepsilon}{3R}.$$

Applying the MR-metric axioms, we derive for $m \geq N$:

$$\begin{aligned}\mathbb{M}(\gamma_m, \gamma, \gamma') &\leq R[\mathbb{M}(\gamma_m, \gamma, \gamma_N) + \mathbb{M}(\gamma_m, \gamma_N, \gamma') + \mathbb{M}(\gamma_N, \gamma, \gamma')] \\ &< R\left(\frac{\varepsilon}{3R} + \frac{\varepsilon}{3R} + \frac{\varepsilon}{3R}\right) = \varepsilon.\end{aligned}$$

Taking $m \to \infty$ yields $\mathbb{M}(\gamma, \gamma, \gamma') \leq \varepsilon$ for arbitrary $\varepsilon > 0$. This implies $\mathbb{M}(\gamma, \gamma, \gamma') = 0$, contradicting $\gamma \neq \gamma'$. Therefore, limits must be unique.

Lemma 4.3: (Convergence implies Cauchy Property in MR-spaces). *In any MR-metric space* $(\mathcal{X}, \mathcal{M})$*, every* $\mathcal{M}$*-convergent sequence* $\{y_n\}$ *is necessarily an* $\mathcal{M}$*-Cauchy sequence.*

Proof: Assume $\{y_n\}$ $\mathcal{M}$-converges to $y \in \mathcal{X}$. For arbitrary $\delta > 0$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$:

$$\mathcal{M}(y_k, y_k, y) < \frac{\delta}{3R^2},$$
$$\mathcal{M}(y, y_k, y_l) < \frac{\delta}{3R^2} \text{ for all } l \geq K.$$

Selecting $N = K$, we apply the MR-metric properties for any $m, n \geq N$:

$$\begin{aligned}\mathcal{M}(y_n, y_n, y_m) &\leq R[\mathcal{M}(y_n, y_n, y) + \mathcal{M}(y_n, y, y_m) + \mathcal{M}(y, y_n, y_m)] \\ &< R\left(\frac{\delta}{3R^2} + \frac{\delta}{3R^2} + \frac{\delta}{3R^2}\right) = \frac{\delta}{R}.\end{aligned}$$

The final inequality holds because:

$$\begin{aligned}\mathcal{M}(y_n, y, y_m) &\leq R\mathcal{M}(y_n, y, y) + R\mathcal{M}(y_n, y, y_m) \\ &\leq R \cdot 0 + R \cdot \frac{\delta}{3R^2} = \frac{\delta}{3R}.\end{aligned}$$

Therefore, $\mathcal{M}(y_n, y_n, y_m) < \delta$ for all $m, n \geq N$, establishing the Cauchy criterion.

Definition 4.4: (Compatible Mappings in MR-Spaces). For self-mappings S and R on an MR-metric space $(X, \mathcal{M})$, we say S and R are weakly compatible in the MR-sense when they commute at any coincidence point. That is, whenever $Sx = Rx$ for some $x \in X$, it follows that $SRx = RSx$.

Given an MR-metric space $(X, \mathcal{M})$, we define the set diameter function for subsets $B_1, B_2, B_3 \subseteq X$ as:

$$\Delta_{\mathcal{M}}(B_1, B_2, B_3) = \inf\{K > 0 : \mathcal{M}(b_1, b_2, b_3) \leq K \ \forall b_i \in B_i\}.$$

When dealing with singleton sets, we adopt the simplified notation:

- $\Delta_{\mathcal{M}}(\{x\}, B_2, B_3) = \Delta_{\mathcal{M}}(x, B_2, B_3)$
- $\Delta_{\mathcal{M}}(\{x\}, \{y\}, \{z\}) = \mathcal{M}(x, y, z)$

This diameter function satisfies the following fundamental properties:

- $\Delta_{\mathcal{M}}(B_1, B_2, B_3) = 0$ iff $B_1 = B_2 = B_3$ are singleton sets containing a common zero-distance point
- The value remains unchanged under any permutation of the sets B_1, B_2, B_3 and is always non-negative

For any non-empty subset $B \subseteq X$, we define its MR-diameter as:

$$\Delta_{\mathcal{M}}(B) = \sup\{\mathcal{M}(b_1, b_2, b_3) : b_1, b_2, b_3 \in B\}.$$

Remark 4.1: (Properties of Set Diameters in MR-Spaces). *In any MR-metric space* $(X, \mathcal{M})$*, the diameter function* $\Delta_{\mathcal{M}}$ *satisfies:*

(1) Monotonicity: For nested subsets $B \subseteq C \subseteq X$, we have $\Delta_{\mathcal{M}}(B) \leq \Delta_{\mathcal{M}}(C)$

(2) For the tail sequence $T_n = \{x_k\}_{k=n}^{\infty}$ with diameters $d_n = \Delta_{\mathcal{M}}(T_n)$, the following hold:
   - $d_n \geq d_{n+1}$ (as $T_n \supseteq T_{n+1}$)
   - $\mathcal{M}(x_p, x_q, x_r) \leq d_n$ for all $p, q, r \geq n$
   - $0 \leq d_{n+1} \leq d_n$ for all $n \in \mathbb{N}$

This establishes that $\{d_n\}$ is a non-increasing, bounded below sequence that converges to some limit $d \geq 0$.

Lemma 4.4: (Characterization of Cauchy Sequences in MR-Metric Spaces). *Let* $(\mathbb{Y}, \mathbb{M})$ *be an MR-metric space and* $\{\beth_n\}$ *a sequence of points in* $\mathbb{Y}$*. If the sequence of diameters* $\delta_n = \sup_{k,l \geq n} \mathbb{M}(\beth_k, \beth_l, \beth_p)$ *for any* $p \geq n$ *satisfies* $\lim_{n\to\infty} \delta_n = 0$*, then* $\{\beth_n\}$ *must be an MR-Cauchy sequence.*

Proof: Assume $\lim_{n\to\infty} d_n = 0$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

1. $d_n = \sup\{\mathcal{M}(x_i, x_j, x_k) : i, j, k \geq n\} < \varepsilon$

For any three indices $p, q, r \geq N$, we have:

$$\begin{aligned} \mathcal{M}(x_p, x_q, x_r) &\leq \sup\{\mathcal{M}(x_i, x_j, x_k) : i, j, k \geq N\} \\ &= d_N \\ &< \varepsilon \end{aligned}$$

This satisfies the MR-Cauchy condition, completing the proof.

Theorem 4.1: (Generalized Fixed Point Theorem for Weakly Compatible Mappings in Complete MR-Spaces). *Let* $(\mathbb{Y}, \mathbb{M})$ *be a complete MR-metric space and consider two self-mappings* $S, T : \mathbb{Y} \to \mathbb{Y}$ *satisfying the following conditions:*

(i) ***Range Inclusion****: The range of* T *is contained in the range of* S*, and the image* $S(T(\mathbb{Y}))$ *forms a closed subset of* $\mathbb{Y}$*.*

(ii) ***Compatibility Condition****: The pair* (S, T) *is weakly compatible, meaning they commute at their coincidence points (i.e., whenever* $Sx = Tx$*, then* $STx = TSx$*).*

(ii) ***Contraction Property****: There exists a control function $\phi: [0, \infty) \to [0, \infty)$ which is:*

(1) *Non-decreasing:* $t_1 \leq t_2 \Rightarrow \phi(t_1) \leq \phi(t_2)$.

(2) *Upper semicontinuous.*

(3) *Strictly contractive:* $\phi(t) < t$ *for all* $t > 0$*such that for all triples* $x, y, z \in \mathbb{Y}$*:*

$$\mathbb{M}(Tx, Ty, Tz) \leq \phi(\mathbb{M}(Sx, Sy, Sz))$$

Then there exists a unique point $\xi \in \mathbb{Y}$ satisfying $S\xi = T\xi = \xi$.

Proof: The proof proceeds through several carefully constructed steps:

1. **Sequence Construction and Diameter Analysis**

- Select an arbitrary initial point $x_0 \in \mathbb{Y}$.
- Using condition (i), inductively construct sequences $\{x_n\}$ and $\{y_n\}$ with:

$$y_n = Tx_n = Sx_{n+1} \; \forall n \geq 0$$

- Define the diameter sequence $D_n = \sup_{k,l \geq n} \mathbb{M}(y_k, y_l, y_p)$ for any $p \geq n$.
- By the contraction property (iii), for any indices n, m, l:

$$\begin{aligned} \mathbb{M}(y_{n+1}, y_{m+1}, y_{l+1}) &= \mathbb{M}(Tx_n, Tx_m, Tx_l) \\ &\leq \phi(\mathbb{M}(Sx_n, Sx_m, Sx_l)) \\ &= \phi(\mathbb{M}(y_n, y_m, y_l)) \end{aligned}$$

- This establishes the crucial inequality:

$$D_{n+1} \leq \phi(D_n)$$

- Through iterative application and the properties of ϕ, we conclude:

$$\lim_{n \to \infty} D_n = 0$$

proving $\{y_n\}$ is MR-Cauchy.

2. Limit Point Identification

- By completeness, $\exists z \in \mathbb{Y}$ with $\lim y_n = z$.
- Since $S(T(\mathbb{Y}))$ is closed, $\exists w \in \mathbb{Y}$ such that $Sw = z$.
- Applying the contraction to (x_n, x_n, w):

$$\mathbb{M}(Tx_n, Tx_n, Tw) \leq \phi(\mathbb{M}(Sx_n, Sx_n, Sw)) \to \phi(0) = 0$$

Hence $Tw = z$.

3. Fixed Point Verification

- From weak compatibility (ii), since $Tw = Sw = z$, we have $STw = TSw \; Sz = Tz$.
- Consider the sequence $\mathbb{M}(Tz, Tz, y_n)$:

$$\begin{aligned} \mathbb{M}(Tz, Tz, y_n) &= \mathbb{M}(Tz, Tz, Tx_n) \\ &\leq \phi(\mathbb{M}(Sz, Sz, Sx_n)) \\ &= \phi(\mathbb{M}(Tz, Tz, y_{n-1})) \end{aligned}$$

- Taking limits yields:
$$\mathbb{M}(Tz, Tz, z) \leq \phi(\mathbb{M}(Tz, Tz, z))$$
which implies $Tz = z$ (and consequently $Sz = z$).

4. **Uniqueness Argument**
   - Suppose u, v are distinct fixed points.
   - Then:
   $\mathbb{M}(u, u, v) = \mathbb{M}(Tu, Tu, Tv) \leq \phi(\mathbb{M}(Su, Su, Sv)) = \phi(\mathbb{M}(u, u, v))$
   - This leads to the contradiction:
   $$\mathbb{M}(u, u, v) \leq \phi(\mathbb{M}(u, u, v)) < \mathbb{M}(u, u, v)$$
   - Hence, $u = v$.

The proof establishes not only existence but also the essential uniqueness of the common fixed point through careful application of the MR-metric properties and the control function ϕ.

Notation: From this point onward, we will denote ψ as previously defined, unless stated otherwise.

Finally, we present some corollaries and an example related to Theorem 4.1.

Corollary 4.1: (Generalized Triple Mapping Fixed Point Theorem in Complete MR-Spaces). *Let* $(\mathbb{Y}, \mathbb{M})$ *be a complete MR-metric space and consider three self-mappings* $f, g, p: \mathbb{Y} \to \mathbb{Y}$ *satisfying the following conditions:*

(1) **Range Inclusion Condition:**
   - $g(\mathbb{Y}) \subseteq (f \circ p)(\mathbb{Y})$
   - $(f \circ p)(\mathbb{Y})$ forms a closed subspace of $\mathbb{Y}$

(2) **Commutativity and Compatibility:**
   - The mappings commute: $f \circ p = p \circ f$ and $g \circ p = p \circ g$
   - The pair $(f \circ p, g)$ is weakly compatible, meaning they commute at coincidence points

(3) **Generalized Contraction Principle**: There exists a control function $\phi: [0, \infty) \to [0, \infty)$ satisfying:
   - Monotonicity: $t_1 \leq t_2 \Rightarrow \phi(t_1) \leq \phi(t_2)$
   - Strict contractivity: $\phi(t) < t$ for all $t > 0$

   such that for all triples $x, y, z \in \mathbb{Y}$:
$$\mathbb{M}(gx, gy, gz) \leq \phi(\mathbb{M}((f \circ p)x, (f \circ p)y, (f \circ p)z))$$

Then there exists a unique element $\xi \in \mathbb{Y}$ that serves as a common fixed point for all three mappings, i.e., $f\xi = g\xi = p\xi = \xi$.

Proof: We establish this result through several carefully constructed steps:

Part 1: Existence of Common Fixed Point for $f \circ p$ and g

- Apply Theorem 4.1 to the mappings $f \circ p$ and g:
  - Condition (i) is satisfied by hypothesis

- Condition (ii) holds due to weak compatibility
- Condition (iii) follows from the given contraction

- This guarantees existence of $u \in \mathbb{Y}$ with:

$$(f \circ p)u = gu = u$$

Part 2: Verification of Fixed Point Property for p

- Consider the MR-metric at the point pu:

$$\begin{aligned}\mathbb{M}(pu,u,u) &= \mathbb{M}(p(gu),gu,gu) \text{ (since } gu = u) \\ &= \mathbb{M}(g(pu),gu,gu) \text{ (by commutativity } gp = pg) \\ &\leq \phi(\mathbb{M}((f \circ p)(pu),(f \circ p)u,(f \circ p)u)) \\ &= \phi\big(\mathbb{M}(p(f \circ p)u,u,u)\big) \text{ (by commutativity } fp = pf) \\ &= \phi(\mathbb{M}(pu,u,u))\end{aligned}$$

- This inequality $\mathbb{M}(pu,u,u) \leq \phi(\mathbb{M}(pu,u,u))$ implies:
  - If $\mathbb{M}(pu,u,u) > 0$, we get the contradiction $\mathbb{M}(pu,u,u) < \mathbb{M}(pu,u,u)$
  - Therefore $\mathbb{M}(pu,u,u) = 0$, proving $pu = u$

Part 3: Conclusion of Fixed Points

- From $pu = u$ and $(f \circ p)u = u$, we get $fu = u$
- Thus u satisfies:

$$fu = gu = pu = u$$

Part 4: Uniqueness Argument

- Suppose $v \neq u$ is another common fixed point
- Then:

$$\begin{aligned}\mathbb{M}(u,u,v) &= \mathbb{M}(gu,gu,gv) \\ &\leq \phi(\mathbb{M}((f \circ p)u,(f \circ p)u,(f \circ p)v)) \\ &= \phi(\mathbb{M}(u,u,v)) < \mathbb{M}(u,u,v)\end{aligned}$$

- This contradiction establishes uniqueness

The proof is complete, showing all three mappings share exactly one common fixed point.

Corollary 4.2 (Iterated Mapping Fixed Point Theorem). *Let* $(\mathbb{Y},\mathbb{M})$ *be a complete MR-metric space and* $T\colon \mathbb{Y} \to \mathbb{Y}$ *a self-mapping satisfying for some* $n \in \mathbb{N}$*:*

$$\mathbb{M}(T^n x, T^n y, T^n z) \leq \varphi(\mathbb{M}(x,y,z)) \; \forall x,y,z \in \mathbb{Y}$$

where $\varphi\colon [0,\infty) \to [0,\infty)$ *is a comparison function satisfying:*

- *Monotonicity:* $t_1 \leq t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$
- *Strict contractivity:* $\varphi(t) < t$ *for all* $t > 0$

Then T *possesses exactly one fixed point in* $\mathbb{Y}$.

Proof: We establish this through two main steps:

Part 1: Fixed Point for T^n

- Apply Theorem 4.1 with:
  - $S = \mathrm{id}_{\mathbb{Y}}$ (identity mapping)
  - $R = T^n$
- The contraction condition becomes:

$$\mathbb{M}(T^n x, T^n y, T^n z) \leq \varphi(\mathbb{M}(x, y, z))$$

- This guarantees a unique $p \in \mathbb{Y}$ with $T^n p = p$

Part 2: Fixed Point for T

- Observe that:

$$T^n(Tp) = T(T^n p) = Tp$$

- Thus Tp is also a fixed point of T^n
- By uniqueness, $Tp = p$
- The fixed point is unique because:
  - Any fixed point of T is clearly a fixed point of T^n
  - T^n has only one fixed point

Corollary 4.3: (Common Fixed Point Theorem for Power Mappings). *Let* $(\mathbb{Y}, \mathbb{M})$ *be a complete MR-metric space and* $S, T: \mathbb{Y} \to \mathbb{Y}$ *self-mappings satisfying for some* $m, n \in \mathbb{N}$*:*

(1) **Range Condition**: $T^n(\mathbb{Y}) \subseteq S^m(\mathbb{Y})$ with $S^m(\mathbb{Y})$ closed

(2) **Commutativity**: $S^m \circ T = T \circ S^m$ and $T^n \circ S = S \circ T^n$

(3) **Power Contraction**:

$$\mathbb{M}(T^n x, T^n y, T^n z) \leq \phi(\mathbb{M}(S^m x, S^m y, S^m z)) \ \forall x, y, z \in \mathbb{Y}$$

where ϕ *is a comparison function as above*

Then S and T share a unique common fixed point.

Proof: The proof develops through several logical stages:

Stage 1: Application of Main Theorem

- Apply Theorem 4.1 to S^m and T^n to obtain unique $p \in \mathbb{Y}$ with:

$$S^m p = T^n p = p$$

Stage 2: Verification for T

- Compute:

$$Tp = T(T^n p) = T^n(Tp)$$
$$Tp = T(S^m p) = S^m(Tp) \text{ (by commutativity)}$$

- Thus Tp is fixed under both S^m and T^n

- By uniqueness, $Tp = p$

Stage 3: Verification for S

- Similarly:

$$Sp = S(S^m p) = S^m(Sp)$$
$$Sp = S(T^n p) = T^n(Sp) \text{ (by commutativity)}$$

- Hence Sp is also a common fixed point of S^m and T^n
- By uniqueness, $Sp = p$

Stage 4: Uniqueness Argument

- Any common fixed point of S and T would:
    - Be fixed under S^m and T^n
    - Must therefore coincide with p

This completes the proof of the corollary.

Example 4.2: (Concrete Realization in Function Space). Consider the closed interval $\mathbb{Y} = [0,4]$ equipped with the canonical MR-metric:

$$\mathbb{M}(x, y, z) = \frac{1}{3}(|x - y| + |y - z| + |z - x|)$$

which forms a complete MR-metric space.

Define the affine mappings $f, g: \mathbb{Y} \to \mathbb{Y}$ by:

$$f(x) = \frac{x+4}{3}, g(x) = \frac{x+16}{8}$$

and take the contraction function $\phi(t) = \frac{2}{5}t$.

Verification of Conditions:

- **Contraction Condition**: For any $x, y, z \in \mathbb{Y}$:

$$\begin{aligned} \mathbb{M}(gx, gy, gz) &= \frac{1}{3}\left(\left|\frac{x-y}{8}\right| + \left|\frac{y-z}{8}\right| + \left|\frac{z-x}{8}\right|\right) \\ &= \frac{1}{24}(|x - y| + |y - z| + |z - x|) \\ &\leq \frac{2}{15}(|x - y| + |y - z| + |z - x|) \\ &= \phi(\mathbb{M}(fx, fy, fz)) \end{aligned}$$

- **Fixed Point Calculation**: Solving $f(x) = x$ and $g(x) = x$ simultaneously:

$$\frac{x+4}{3} = x \Rightarrow x = 2 \text{ and } \frac{x+16}{8} = x \Rightarrow x = 2$$

confirming 2 is the unique common fixed point.

This example illustrates all conditions of Theorem 4.1 in a concrete setting.

5. MR-Contraction and Its Use in Solving Systems of Linear Equations

This section develops a novel approach for solving systems of linear equations within the MR-metric framework. We establish fundamental theoretical results to support our methodology.

Definition 5.1: (Contractive Mapping in MR-Spaces). Let $(\mathcal{X}, \mathbb{M}_R)$ be an MR-metric space with parameter $R \geq 1$, where $\mathcal{X}$ is non-empty. A mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ is called an MR-contraction if there exists a contraction constant $\kappa \in [0,1/R)$ such that for all $x, y \in \mathcal{X}$:

$$\mathbb{M}_R(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) \leq \kappa \cdot \mathbb{M}_R(x, x, y)$$

Remark 5.1: The condition $\kappa < 1/R$ is crucial for ensuring the contractive property in the MR-metric framework, as it compensates for the relaxation parameter R in the metric axioms.

Theorem 5.1: (Existence and Uniqueness of Fixed Points). *Let $(\mathcal{X}, \mathbb{M}_R)$ be a complete MR-metric space with $R \geq 1$, and let $\mathcal{F}: \mathcal{X} \to \mathcal{X}$ be an MR-contraction with constant $\kappa \in [0,1/R)$. If there exists an initial point $x_0 \in \mathcal{X}$ with $\mathbb{M}_R(x_0, x_0, \mathcal{F}x_0) < \infty$ and the space satisfies axioms (M5)-(M6), then $\mathcal{F}$ admits exactly one fixed point in $\mathcal{X}$.*

Proof: We establish this through several carefully constructed steps:

1. **Iterative Sequence Construction**
   - Define the Picard sequence $(x_n)_{n=0}^{\infty}$ by $x_{n+1} = \mathcal{F}x_n$
   - From the contraction property:

   $$\mathbb{M}_R(x_{n+1}, x_{n+1}, x_{n+2}) \leq \kappa \mathbb{M}_R(x_n, x_n, x_{n+1}) \leq \kappa^n \mathbb{M}_R(x_0, x_0, x_1)$$

2. **Cauchy Sequence Verification** For $m > n \geq 1$, the polygonal inequality yields:

   $$\begin{aligned} \mathbb{M}_R(x_n, x_n, x_m) &\leq R \textstyle\sum_{i=0}^{m-n-1} R^i \mathbb{M}_R(x_{n+i}, x_{n+i}, x_{n+i+1}) \\ &\leq R\kappa^n \textstyle\sum_{i=0}^{m-n-1} (R\kappa)^i \mathbb{M}_R(x_0, x_0, x_1) \\ &< \tfrac{R\kappa^n}{1-R\kappa} \mathbb{M}_R(x_0, x_0, x_1) \to 0 \text{as} n \to \infty \end{aligned}$$

3. **Limit Point Identification**
   - By completeness, $\exists p \in \mathcal{X}$ with $\lim_{n\to\infty} x_n = p$
   - The mapping continuity gives:

   $$\begin{aligned} \mathbb{M}_R(\mathcal{F}p, \mathcal{F}p, p) &\leq R[\mathbb{M}_R(\mathcal{F}p, \mathcal{F}p, x_n) + \mathbb{M}_R(x_n, x_n, p)] \\ &\leq R[\kappa \mathbb{M}_R(p, p, x_{n-1}) + \mathbb{M}_R(x_n, x_n, p)] \to 0 \end{aligned}$$

4. **Uniqueness Demonstration** For any two fixed points p, q:

   $$\mathbb{M}_R(p, p, q) = \mathbb{M}_R(\mathcal{F}p, \mathcal{F}p, \mathcal{F}q) \leq \kappa \mathbb{M}_R(p, p, q) \Longrightarrow p = q$$

Theorem 5.2: (Solvability of Linear Systems in MR-Spaces). *Consider the Euclidean space $\mathbb{R}^n$ endowed with the MR-metric:*

$$\mathbb{M}_R(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \textstyle\sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i| + |z_i - x_i|)$$

where $\mathbf{x} = (x_i), \mathbf{y} = (y_i), \mathbf{z} = (z_i) \in \mathbb{R}^n$.

For any matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ satisfying the column sum condition:

$$\max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \le \lambda < 1$$

the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution in $\mathbb{R}^n$.

Proof:

1. **Metric Space Properties**
   - $(\mathbb{R}^n, \mathbb{M}_R)$ is complete as it's equivalent to the standard ℓ^1 space
   - The MR-metric satisfies all axioms (M1)-(M6) with $R = 1$
2. **Operator Analysis** Define the affine operator $\mathcal{T}_A(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$. For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$:

$$\begin{aligned} \mathbb{M}_1(\mathcal{T}_A\mathbf{x}, \mathcal{T}_A\mathbf{y}, \mathcal{T}_A\mathbf{z}) &= \sum_{i=1}^{n} \left(|(A(\mathbf{x}-\mathbf{y}))_i| + |(A(\mathbf{y}-\mathbf{z}))_i| + |(A(\mathbf{z}-\mathbf{x}))_i|\right) \\ &\le \sum_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|\right)\left(|x_j - y_j| + |y_j - z_j| + |z_j - x_j|\right) \\ &\le \lambda \mathbb{M}_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) \end{aligned}$$

3. **Conclusion**
   - $\mathcal{T}_A$ is an MR-contraction with constant $\lambda < 1$
   - By the fixed point theorem, $\exists!\, \mathbf{x}^* \in \mathbb{R}^n$ with $\mathcal{T}_A\mathbf{x}^* = \mathbf{x}^*$
   - This $\mathbf{x}^*$ solves $A\mathbf{x}^* = \mathbf{b}$ uniquely

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