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Khalouta reduced differential transform method and its convergence for nonlinear fractional biological population equation under Caputo's fractional operator

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Abstract

The current paper focuses on finding the exact solution of the nonlinear fractional biological population equation under Caputo's fractional operator by involving a new hybrid technique known as the Khalouta reduced differential transform method (KHRDTM). The exact solution is described in terms of the Mittag-Leffler functions. Moreover, the presented method is a powerful tool for solving a huge number of problems. The accuracy of the proposed technique is evaluated with the help of some examples and the results obtained are compared with the previous results of the techniques in the literature.

Subject Classification (2010): *26A33, 35R11, 35A22, 35C05.*

Keywords: *Khalouta transform (KHT), Reduced differential transform method (RDTM), Caputo's fractional derivative operator (CFDO), Riemann–Liouville fractional integral operator (RLFIO), Nonlinear fractional biological population equation (NFBPE).*

1. Introduction

Recently, nonlinear fractional partial differential equations (NFPDEs) and fractional calculus (FC) have attracted great attention due to their profound effectiveness in representing and examining real-world phenomena and distinct areas of natural and life sciences, including medicine, biological phenomena,

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population growth, control theory, physics, chemistry, viscoelasticity, medical sciences and signal processing [4, 5, 10, 11, 12, 13].

In most scientific fields, including applied mathematics, it is important to find exact solutions to NFPDEs. In general, these types of equations do not have exact closed-form solutions for most problems, and it is very difficult to obtain exact solutions. For this reason, techniques have been developed to solve NFPDEs.

In the past decade, several researchers have proposed new techniques that are a combination of integral transformations and different semi-analytical methods to find approximate analytical solutions to various kinds of NFPDEs [1, 2, 3, 14].

Our aim in this work is to drive a hybrid method using the KHT and the RDTM in order to solve the generalized form of the nonlinear fractional biological population equation involving Caputo's fractional operator given as follows

$$D_{\rho}^{\varepsilon}\psi = \frac{\partial^2\psi^2}{\partial\vartheta^2} + \frac{\partial^2\psi^2}{\partial\varphi^2} + F(\psi), \tag{1.1}$$

under the initial condition

$$\psi(\vartheta,\varphi,0) = \psi_0(\vartheta,\varphi). \tag{1.2}$$

Here D_{ρ}^{ε} represents the CFDO of order ε with $0 < \varepsilon \le 1, \psi = \{\psi(\vartheta,\varphi,\rho), (\vartheta,\varphi,\rho) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\}$ where ϑ and φ are space variables and ρ is a time variable, denotes the population density and $F(\psi) = h\psi^{\mu}(1 - q\psi^{\delta})$ is the supply of population due to births and deaths where h, q, μ and δ are real numbers.

This research paper is structured in the following way: In Sec. 2, we provide the essential definitions and important theorems. In Sec. 3, we describe the KHRDTM for solving the general NFPDEs with a source term. In Sec. 4, we give new results concerning the convergence analysis of the present technique. In Sec. 5, we explain our numerical results and demonstrate the efficiency and applicability of the proposed technique by considering three numerical examples of some special cases of the nonlinear fractional biological population equations. In Sec. 6, we present the conclusion of this work.

2. Fundamental definitions and results

This section provides fundamental definitions and results related to FC, KHT and RDTM to complement the current research work.

Definition 2.1: [10] The RLFIO of a continuous function $\psi(\vartheta,\varphi,\rho)$ is described by

$$I_{\rho}^{\varepsilon}\psi(\vartheta,\varphi,\rho) = \frac{1}{\Gamma(\varepsilon)}\int_0^{\rho} (\rho-\tau)^{\varepsilon-1}\psi(\vartheta,\varphi,\tau)d\tau, \text{ if } \varepsilon > 0,$$

and

$$I_{\rho}^{\varepsilon}\psi(\vartheta,\varphi,\rho) = \psi(\vartheta,\varphi,\rho), \text{ if } \varepsilon = 0,$$

Definition 2.2: [10] The CFDO of a continuous function $\psi(\vartheta,\varphi,\rho)$ is described by

$$D_\rho^\varepsilon\psi(\vartheta,\varphi,\rho)=\frac{1}{\Gamma(n-\varepsilon)}\int_0^\rho(\rho-\tau)^{n-\varepsilon-1}\psi^{(n)}(\vartheta,\varphi,\tau)d\tau,\ \text{if } n-1<\varepsilon<n,$$

and

$$D_\rho^\varepsilon\psi(\vartheta,\varphi,\rho)=\psi^{(n)}(\vartheta,\varphi,\rho),\ \text{if } \varepsilon=n,$$

Definition 2.3: [10] The Mittag-Leffler function is expressed as

$$E_\varepsilon(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\varepsilon+1)},z\in\mathbb{C},Re(\varepsilon)\geq 0. \tag{2.1}$$

Definition 2.4: [7] The KHT of a continuous function $\psi(\vartheta,\varphi,\rho)$ is described by

$$\mathbb{KH}[\psi(\vartheta,\varphi,\rho)]=\mathcal{K}(\vartheta,\varphi,s,\gamma,\eta)=\frac{s}{\gamma\eta}\int_0^\infty\exp\left(-\frac{s\rho}{\gamma\eta}\right)\psi(\vartheta,\varphi,\rho)d\rho, \tag{2.2}$$

where $s,\gamma,\eta>0$ are the Khalouta transform variables.

Theorem 2.1: [7] *The fundamental characteristics of KHT are:*

1) **Linearity property.**

$$\mathbb{KH}[a\psi(\vartheta,\varphi,\rho)+b\phi(\vartheta,\varphi,\rho)]=a\mathbb{KH}[\psi(\vartheta,\varphi,\rho)]+b\mathbb{KH}[\phi(\vartheta,\varphi,\rho)],a,b\in\mathbb{R}$$

2) **Differentiation property.**

$$\mathbb{KH}\left[\psi^{(n)}(\vartheta,\varphi,\rho)\right]=\frac{s^n}{\gamma^n\eta^n}\mathbb{KH}[\psi(\vartheta,\varphi,\rho)]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}\psi^{(k)}(\vartheta,\varphi,0),n\geq 1.$$

where $\psi^{(n)}(\vartheta,\varphi,\rho)$ is the n^{th} derivative of $\psi(\vartheta,\varphi,\rho)$.

3) **Convolution property.**

$$\mathbb{KH}[(\psi*\phi)(\vartheta,\varphi,\rho)]=\frac{\gamma\eta}{s}\mathbb{KH}[\psi(\vartheta,\varphi,\rho)]\mathbb{KH}[\phi(\vartheta,\varphi,\rho)],$$

where $\mathbb{KH}[(\psi*\phi)(\vartheta,\varphi,\rho)]$ is the Khalouta convolution of the functions $\psi(\vartheta,\varphi,\rho)$ and $\phi(\vartheta,\varphi,\rho)$.

4) **The KHTs of the usual functions.**

$$\begin{aligned}\mathbb{KH}[1]&=1,\\ \mathbb{KH}[\rho]&=\frac{\gamma\eta}{s},\\ \mathbb{KH}\left[\frac{\rho^n}{n!}\right]&=\frac{\gamma^n\eta^n}{s^n},n\geq 0,\\ \mathbb{KH}\left[\frac{\rho^\varepsilon}{\Gamma(\varepsilon+1)}\right]&=\frac{\gamma^\varepsilon\eta^\varepsilon}{s^\varepsilon},\varepsilon>-1,\end{aligned}$$

Proof: The proof of this theorem is found in [7].

Theorem 2.2: [8] *The KHT of the CFD of order* $n-1<\varepsilon\leq n$ *with* $n\geq 1$, *is described by*

$$\mathbb{KH}\left[D_\rho^\varepsilon\psi(\vartheta,\varphi,\rho)\right]=\frac{s^\varepsilon}{\gamma^\varepsilon\eta^\varepsilon}\mathbb{KH}[\psi(\vartheta,\varphi,\rho)]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\varepsilon-k}\psi^{(k)}(\vartheta,\varphi,0).$$

Proof: The proof of this theorem is found in [8].

Definition 2.5: [6] The RDT of a function $\psi(\vartheta,\varphi,\rho)$ can be given as

$$\Psi_k(\vartheta,\varphi)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial\rho^k}\psi(\vartheta,\varphi,\rho)\right]_{\rho=0}. \tag{2.3}$$

Here, $\psi(\vartheta,\varphi,\rho)$is the original function and $\Psi_k(\vartheta,\varphi)$ is the transformed function

Definition 2.6: The RDT inverse of a function $\Psi_k(\vartheta,\varphi)$ can be given as

$$\psi(\vartheta,\varphi,\rho)=\sum_{k=0}^{\infty}\Psi_k(\vartheta,\varphi)\rho^k. \tag{2.4}$$

Replacing (2.3) into (2.4), we find

$$\psi(\vartheta,\varphi,\rho)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial\rho^k}\psi(\vartheta,\varphi,\rho)\right]_{\rho=0}\rho^k.$$

Theorem 2.3: *Let $\Psi_k(\vartheta,\varphi),\Phi_k(\vartheta,\varphi)$ and $\Theta_k(\vartheta,\varphi)$ be the RDTs of the functions $\psi(\vartheta,\varphi,\rho),\phi(\vartheta,\varphi,\rho)$ and $\theta(\vartheta,\varphi,\rho)$ respectively, then*

(1) if

$$\theta(\vartheta,\varphi,\rho)=\lambda\psi(\vartheta,\varphi,\rho)+\mu\phi(\vartheta,\varphi,\rho),$$

then

$$\Theta_k(\vartheta,\varphi)=\lambda\Psi_k(\vartheta,\varphi)+\mu\Phi_k(\vartheta,\varphi),\lambda,\mu\in\mathbb{R}.$$

(2) if

$$\theta(\vartheta,\varphi,\rho)=\psi(\vartheta,\varphi,\rho)\phi(\vartheta,\varphi,\rho),$$

then

$$\Theta_k(\vartheta,\varphi)=\sum_{r=0}^{k}\Psi_r(\vartheta,\varphi)\Phi_{k-r}(\vartheta,\varphi).$$

(3) if

$$\theta(\vartheta,\varphi,\rho)=\psi^1(\vartheta,\varphi,\rho)\psi^2(\vartheta,\varphi,\rho)\times\ldots\times\psi^{n-1}(\vartheta,\varphi,\rho)\psi^n(\vartheta,\varphi,\rho),$$

then

$$\Theta_k(\vartheta,\varphi)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\Psi_{k_1}^1(\vartheta,\varphi)\Psi_{k_2-k_1}^2(\vartheta,\varphi)\times\ldots\times\Psi_{k_{n-1}-k_{n-2}}^{n-1}(\vartheta,\varphi)\Psi_{k-k_{n-1}}^n(\vartheta,\varphi).$$

3. Methodology of the KHRDTM

This section explains the main principle of the KHRDTM to create the approximate analytical solution for the general NFPDEs with a source term.

Theorem 3.1: *Assuming the general NFPDEs with a source term*

$$D_\rho^\varepsilon\psi(\vartheta,\varphi,\rho)+L\psi(\vartheta,\varphi,\rho)+N\psi(\vartheta,\varphi,\rho)=\theta(\vartheta,\varphi,\rho), \tag{3.1}$$

under the initial conditions

$$\psi^{(k)}(\vartheta,\varphi,0)=\psi_k(\vartheta,\varphi),k=0,1,2,\ldots,n-1. \tag{3.2}$$

Here D_{ρ}^{ε} represents the CFDO of order ε with $n-1<\varepsilon\leq n, n\geq 1$, L and N represent the linear and nonlinear operator respectively, and $\theta(\vartheta,\varphi,\rho)$ represents the source term.

Implementing the KHRDTM, the solution of equations (3.1) and (3.2) is described as

$$\psi(\vartheta,\varphi,\rho)=\sum_{r=0}^{\infty}\Psi_r(\vartheta,\varphi),$$

where $\Psi_r(\vartheta,\varphi)$ is the RDT function of $\psi(\vartheta,\varphi,\rho)$.

Proof: To begin the proof of this theorem, taking the KHT to equation (3.1) and using the linearity property of the KHT (See. Theorem 2.1), we have

$$\mathbb{KH}\left[D_{\rho}^{\varepsilon}\psi(\vartheta,\varphi,\rho)\right]+\mathbb{KH}[L\psi(\vartheta,\varphi,\rho)+N\psi(\vartheta,\varphi,\rho)]=\mathbb{KH}[\theta(\vartheta,\varphi,\rho)].$$

Utilizing Theorem 2.2, we obtain

$$\begin{aligned}\frac{s^{\varepsilon}}{\gamma^{\varepsilon}\eta^{\varepsilon}}\mathbb{KH}[\psi(\vartheta,\varphi,\rho)]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\varepsilon-k}\psi^{(k)}(\vartheta,\varphi,0)\\ =\mathbb{KH}[\theta(\vartheta,\varphi,\rho)]-\mathbb{KH}\begin{bmatrix}L\psi(\vartheta,\varphi,\rho)\\+N\psi(\vartheta,\varphi,\rho)\end{bmatrix}.\end{aligned}\tag{3.3}$$

Substituting the initial conditions (3.2) and simple calculate, then equation (3.3) becomes

$$\begin{aligned}\mathbb{KH}[\psi(\vartheta,\varphi,\rho)]\ &=\sum_{k=0}^{n-1}\left(\frac{\gamma\eta}{s}\right)^{k}\psi_k(\vartheta,\varphi)+\frac{\gamma^{\varepsilon}\eta^{\varepsilon}}{s^{\varepsilon}}\mathbb{KH}[\theta(\vartheta,\varphi,\rho)]\\ &-\frac{\gamma^{\varepsilon}\eta^{\varepsilon}}{s^{\varepsilon}}\mathbb{KH}[L\psi(\vartheta,\varphi,\rho)+N\psi(\vartheta,\varphi,\rho)].\end{aligned}\tag{3.4}$$

Application of the inverse KHT to equation (3.4), gives

$$\psi(\vartheta,\varphi,\rho)=\Phi(\vartheta,\varphi,\rho)-\mathbb{KH}^{-1}\left[\frac{\gamma^{\varepsilon}\eta^{\varepsilon}}{s^{\varepsilon}}\mathbb{KH}[L\psi(\vartheta,\varphi,\rho)+N\psi(\vartheta,\varphi,\rho)]\right].\tag{3.5}$$

Here $\Phi(\vartheta,\varphi,\rho)$ is the term resulting from the initial conditions and the source term.

Employing the RDTM to equation (3.5), we get

$$\begin{aligned}&\Psi_0(\vartheta,\varphi)=\Phi(\vartheta,\varphi,\rho),\\ &\Psi_{k+1}(\vartheta,\varphi)=-\mathbb{KH}^{-1}\left[\frac{\gamma^{\varepsilon}\eta^{\varepsilon}}{s^{\varepsilon}}\mathbb{KH}[L\Psi_k(\vartheta,\varphi)+A_k(\vartheta,\varphi)]\right],k\geq 0,\end{aligned}\tag{3.6}$$

where $A_k(\vartheta,\varphi)$ is the RDT of the nonlinear term $N\psi(\vartheta,\varphi,\rho)$.

Note that the recurrence formula (3.6) to the iterative terms of equations (3.1) and (3.2) is denoted KHRDTM, and its k^{th} −order approximate solution is given by

$$S_k(\vartheta,\varphi)=\sum_{r=0}^{k}\Psi_r(\vartheta,\varphi).\tag{3.7}$$

Thus, in the following theorem, we prove that the approximate series solution (3.7) converges to the exact solution if $k\to\infty$, i.e.

$$\psi(\vartheta,\varphi)=\lim_{k\to\infty}S_k(\vartheta,\varphi)=\sum_{r=0}^{\infty}\Psi_r(\vartheta,\varphi).\tag{3.8}$$

4. Convergence analysis of the KHRDTM

This section discusses the necessary condition for convergence of the solution. We follow these theorems applied to the series solutions obtained by KHRDTM.

Theorem 4.1: *Assuming $\mathcal{X} = (C(\mathcal{O}), \|.\|)$ a Banach space of all continuous mappings defined on $\mathcal{O} \subset \mathbb{R}^2 \times \mathbb{R}^+$ having the norm*

$$\|\psi(\vartheta,\varphi,\rho)\|_{\mathcal{X}} = \sup_{(\vartheta,\varphi,\rho)\in\mathcal{O}} |\psi(\vartheta,\varphi,\rho)|.$$

If

$$\exists 0 < \upsilon < 1 \ : \ \|\Psi_r(\vartheta,\varphi)\| \leq \upsilon \|\Psi_{r-1}(\vartheta,\varphi)\|, \forall r \in \mathbb{Z}^+,$$

then the approximate series solution $\sum_{r=0}^{\infty} \Psi_r(\vartheta,\varphi)$ which is obtained by KHRDTM will uniquely converge to $\psi(\vartheta,\varphi,\rho)$.

Proof: Let $S_k(\vartheta,\varphi)$ be the k^{th} partial sums given by the recurrence formula (3.6), i.e.

$$S_k(\vartheta,\varphi) = \sum_{r=0}^{k} \Psi_r(\vartheta,\varphi).$$

Here we prove that $\{S_k(\vartheta,\varphi)\}_{k=0}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$.

As

$$\begin{aligned}\|S_{k+1}(\vartheta,\varphi) - S_k(\vartheta,\varphi)\| &\leq \|\Psi_{r+1}(\vartheta,\varphi)\| \leq \upsilon\|\Psi_r(\vartheta,\varphi)\| \\ &\leq \upsilon^2\|\Psi_{r-1}(\vartheta,\varphi)\| \leq \ldots \leq \upsilon^{n+1}\|\Psi_0(\vartheta,\varphi)\|.\end{aligned}$$

Hence,

$$\begin{aligned}\left\|S_p(\vartheta,\varphi) - S_l(\vartheta,\varphi)\right\| &= \left\|\begin{matrix} S_p(\vartheta,\varphi) - S_{p-1}(\vartheta,\varphi) + S_{p-1}(\vartheta,\varphi) - S_{p-2}(\vartheta,\varphi) \\ +\ldots+S_{l+1}(\vartheta,\varphi) - S_l(\vartheta,\varphi)\end{matrix}\right\| \\ &\leq \left\|S_p(\vartheta,\varphi) - S_{p-1}(\vartheta,\varphi)\right\| + \left\|S_{p-1}(\vartheta,\varphi) - S_{p-2}(\vartheta,\varphi)\right\| \\ &\quad +\ldots+\|S_{l+1}(\vartheta,\varphi) - S_l(\vartheta,\varphi)\| \\ &\leq \upsilon^p\|\Psi_0(\vartheta,\varphi)\| + \upsilon^{p-1}\|\Psi_0(\vartheta,\varphi)\| +\ldots+ \upsilon^{l+1}\|\Psi_0(\vartheta,\varphi)\| \\ &= \upsilon^{l+1}(1+\upsilon+\ldots+\upsilon^{p-l-1})\|\Psi_0(\vartheta,\varphi)\| \\ &\leq \upsilon^{l+1}\left(\frac{1-\upsilon^{p-l}}{1-\upsilon}\right)\|\Psi_0(\vartheta,\varphi)\|, p,l \in \mathbb{N}. \qquad (4.1)\end{aligned}$$

Since $0 < \upsilon < 1$, we have $1 - \upsilon^{p-l} < 1$, so the inequality equation (4.1) can be reduced to

$$\left\|S_p(\vartheta,\varphi) - S_l(\vartheta,\varphi)\right\| \leq \frac{\upsilon^{l+1}}{1-\upsilon}\|\Psi_0(\vartheta,\varphi)\|. \qquad (4.2)$$

As $\Psi_0(\vartheta,\varphi)$ is bounded, then $\left\|S_p(\vartheta,\varphi) - S_l(\vartheta,\varphi)\right\| \to 0$ when $p,l \to \infty$.

Thus $\{S_k(\vartheta,\varphi)\}_{k=0}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$ and it converges to $\psi(\vartheta,\varphi,\rho) \in \mathcal{X}$ such that

$$\lim_{k\to\infty} S_k(\vartheta,\varphi) = \sum_{r=0}^{\infty} \Psi_r(\vartheta,\varphi) = \psi(\vartheta,\varphi,\rho).$$

Now, suppose that the sequence $\{S_k(\vartheta,\varphi)\}_{k\geq 0}$ converges to two functions of $\psi_1(\vartheta,\varphi,\rho),\psi_2(\vartheta,\varphi,\rho)\in\mathcal{X}$, i.e.

$$\lim_{k\to\infty} S_k(\vartheta,\varphi)=\psi_1(\vartheta,\varphi,\rho) \text{ and } \lim_{k\to\infty} S_k(\vartheta,\varphi)=\psi_2(\vartheta,\varphi,\rho). \tag{4.3}$$

Using the triangle inequality with (4.3), we get

$$\begin{aligned}\|\psi_1(\vartheta,\varphi,\rho)-\psi_2(\vartheta,\varphi,\rho)\|&\\ \leq \|\psi_1(\vartheta,\varphi,\rho)-S_k(\vartheta,\varphi)\|+\|S_k(\vartheta,\varphi)-\psi_2(\vartheta,\varphi,\rho)\|&=0,\end{aligned}$$

when $k\to\infty$.

Hence we conclude that $\psi_1(\vartheta,\varphi,\rho)=\psi_2(\vartheta,\varphi,\rho)$.

This ends the proof.

Corollary 4.1: *If the series $\sum_{r=0}^{\infty}\Psi_r(\vartheta,\varphi)$ converges then it is an exact solution of the general NFPDEs (3.1).*

Theorem 4.2: *The maximum absolute truncation error of the series solution given by the recurrence formula (3.6), is calculated as*

$$\left\|\psi(\vartheta,\varphi,\rho)-\sum_{j=0}^{N}\Psi_j(\vartheta,\varphi)\right\|\leq\frac{\upsilon^{N+1}}{1-\upsilon}\|\Psi_0(\vartheta,\varphi)\|.$$

Proof: Using Theorem 4.1 and the inequality equation (4.2), we get

$$\|S_k(\vartheta,\varphi)-S_N(\vartheta,\varphi)\|\leq\frac{\upsilon^{N+1}}{1-\upsilon}\|\Psi_0(\vartheta,\varphi)\|. \tag{4.4}$$

We assume that $S_k(\vartheta,\varphi)=\sum_{j=0}^{k}\Psi_l(\vartheta,\varphi)$ and since $k\to+\infty$, we obtain $S_k(\vartheta,\varphi)\to\psi(\vartheta,\varphi,\rho)$, so the inequality equation (4.4) becomes

$$\|\psi(\vartheta,\varphi,\rho)-S_N(\vartheta,\varphi)\|=\left\|\psi(\vartheta,\varphi,\rho)-\sum_{j=0}^{N}\Psi_j(\vartheta,\varphi)\right\|\leq\frac{\upsilon^{N+1}}{1-\upsilon}\|\Psi_0(\vartheta,\varphi)\|.$$

This ends the proof.

5. Applications and simulations

This section demonstrates the validity and accuracy of the results obtained using the current method to determine the exact solution of some special cases of nonlinear fractional biological population equations (1.1).

Example 5.1: Assume the nonlinear fractional biological population equation described in Caputo's sense as follows

$$D_\rho^\varepsilon\psi=\frac{\partial^2\psi^2}{\partial\vartheta^2}+\frac{\partial^2\psi^2}{\partial\varphi^2}+\psi, 0<\varepsilon\leq 1, \tag{5.1}$$

under the initial condition

$$\psi(\vartheta,\varphi,0)=\sqrt{\sin\vartheta\sinh\varphi}. \tag{5.2}$$

Following the KHRDTM methodology presented in Section 3, we get the series solution of our equation

$$\psi(\vartheta,\varphi,\rho)=\sum_{r=0}^{\infty}\Psi_r(\vartheta,\varphi),$$

where in this case, we set

$$L\psi(\vartheta,\varphi,\rho)=\psi \text{ and } N\psi(\vartheta,\varphi,\rho)=\frac{\partial^2\psi^2}{\partial\vartheta^2}+\frac{\partial^2\psi^2}{\partial\varphi^2}.$$

The values of the RDT terms of $N\psi(\vartheta,\varphi,\rho)$ are given as

$$\begin{aligned}
A_0(\vartheta,\varphi) &= \frac{\partial^2\psi_0^2}{\partial\vartheta^2}+\frac{\partial^2\psi_0^2}{\partial\varphi^2},\\
A_1(\vartheta,\varphi) &= 2\frac{\partial^2\psi_0\psi_1}{\partial\vartheta^2}+2\frac{\partial^2\psi_0\psi_1}{\partial\varphi^2},\\
A_2(\vartheta,\varphi) &= 2\frac{\partial^2(\psi_0\psi_2)}{\partial\vartheta^2}+\frac{\partial^2\psi_1^2}{\partial\vartheta^2}+2\frac{\partial^2(\psi_0\psi_2)}{\partial\varphi^2}+\frac{\partial^2\psi_1^2}{\partial\varphi^2}.
\end{aligned}$$

According to the recurrence formula (3.6), we get

$$\begin{aligned}
\Psi_0(\vartheta,\varphi) &= \sqrt{\sin\vartheta\sinh\varphi},\\
\Psi_1(\vartheta,\varphi) &= \sqrt{\sin\vartheta\sinh\varphi}\frac{\rho^{\varepsilon}}{\Gamma(\varepsilon+1)},\\
\Psi_2(\vartheta,\varphi) &= \sqrt{\sin\vartheta\sinh\varphi}\frac{\rho^{2\varepsilon}}{\Gamma(2\varepsilon+1)},\\
\Psi_3(\vartheta,\varphi) &= \sqrt{\sin\vartheta\sinh\varphi}\frac{\rho^{3\varepsilon}}{\Gamma(3\varepsilon+1)},\\
&\vdots\\
\Psi_k(\vartheta,\varphi) &= \sqrt{\sin\vartheta\sinh\varphi}\frac{\rho^{k\varepsilon}}{\Gamma(k\varepsilon+1)}.
\end{aligned}$$

Thus, the k^{th} −order approximate series solution is derived in the form

$$\begin{aligned}
S_k(\vartheta,\varphi) &= \Psi_0(\vartheta,\varphi)+\Psi_1(\vartheta,\varphi)+\Psi_2(\vartheta,\varphi)+\ldots+\Psi_k(\vartheta,\varphi)\\
&= \sqrt{\sin\vartheta\sinh\varphi}+\sqrt{\sin\vartheta\sinh\varphi}\frac{\rho^{\varepsilon}}{\Gamma(\varepsilon+1)}+\sqrt{\sin\vartheta\sinh\varphi}\frac{\rho^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\\
&\quad+\sqrt{\sin\vartheta\sinh\varphi}\frac{\rho^{3\varepsilon}}{\Gamma(3\varepsilon+1)}+\ldots+\sqrt{\sin\vartheta\sinh\varphi}\frac{\rho^{k\varepsilon}}{\Gamma(k\varepsilon+1)}\\
&= \sqrt{\sin\vartheta\sinh\varphi}\left(1+\frac{\rho^{\varepsilon}}{\Gamma(\varepsilon+1)}+\frac{\rho^{2\varepsilon}}{\Gamma(2\varepsilon+1)}+\frac{\rho^{3\varepsilon}}{\Gamma(3\varepsilon+1)}+\ldots+\frac{\rho^{k\varepsilon}}{\Gamma(k\varepsilon+1)}\right)\\
&= \sqrt{\sin\vartheta\sinh\varphi}\sum_{r=0}^{k}\frac{\rho^{r\varepsilon}}{\Gamma(r\varepsilon+1)}.
\end{aligned}$$

So, the k^{th} −order approximate series solution converges to the exact solution when $k\to\infty$

$$\begin{aligned}
\psi(\vartheta,\varphi,\rho) &= \lim_{k\to\infty}S_k(\vartheta,\varphi)\\
&= \sqrt{\sin\vartheta\sinh\varphi}\lim_{k\to\infty}\sum_{r=0}^{k}\frac{(\rho^{\varepsilon})^r}{\Gamma(r\varepsilon+1)} \qquad (5.3)\\
&= \sqrt{\sin\vartheta\sinh\varphi}E_{\varepsilon}(\rho^{\varepsilon}),
\end{aligned}$$

where $E_{\varepsilon}(\rho^{\varepsilon})$ is the Mittag Leffler function defined by (2.1).

As $\varepsilon=1$ in (5.3), we have

$$\begin{aligned}
\psi(\vartheta,\varphi,\rho) &= \sqrt{\sin\vartheta\sinh\varphi}\sum_{r=0}^{\infty}\frac{\rho^r}{r!}\\
&= \sqrt{\sin\vartheta\sinh\varphi}\exp(\rho),
\end{aligned}$$

which is the same exact solution found in the literature [9].

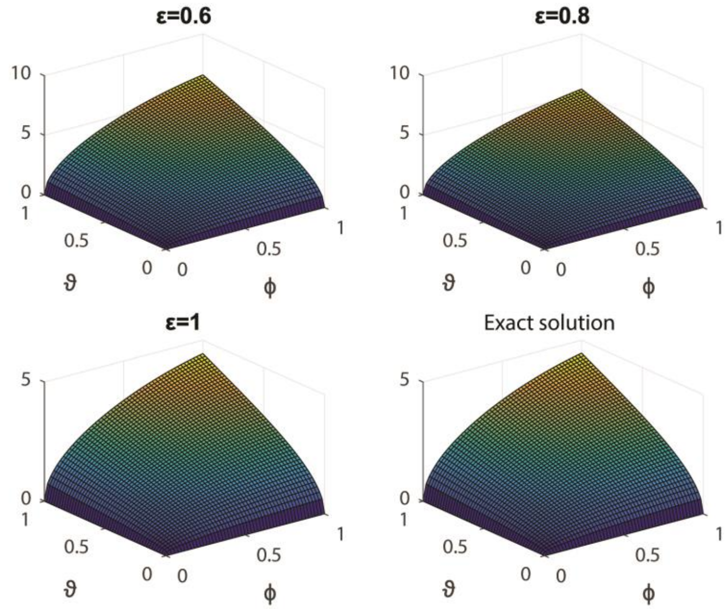

Figure 1
3D-plots of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta, \varphi)$ and exact solution $\psi(\vartheta, \varphi, \rho)$ for (5.1)-(5.2) with $\rho = 1.5$.

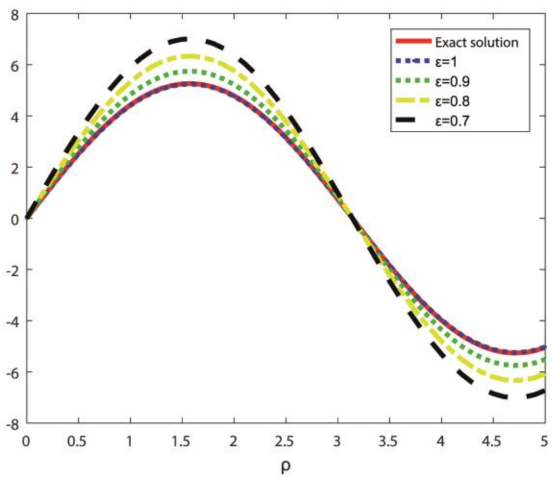

Figure 2
2D-plots of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta, \varphi)$ and exact solution $\psi(\vartheta, \varphi, \rho)$ for (5.1)-(5.2) with $\varphi = 1$ and $\rho = 1.5$.

Table 1
Comparison of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta, \varphi)$ and exact solution $\psi(\vartheta, \varphi, \rho)$ and their associated absolute errors for (5.1)-(5.2) for distinct values of ε.

	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{Exact}	$\lvert\psi_{Exact} - \psi_{KHRDTM}\rvert$
$(\vartheta, \varphi)/\varepsilon$	$\varepsilon = 0.7$	$\varepsilon = 0.8$	$\varepsilon = 0.9$	$\varepsilon = 1$	*Exact solution*	*Absolute errors*
0.1	0.12556	0.11892	0.11408	0.11052	0.11052	1.4090×10^{-10}
0.3	0.37667	0.35673	0.34224	0.33154	0.33154	4.2268×10^{-10}
0.5	0.62759	0.59437	0.57023	0.55239	0.55239	7.0425×10^{-10}
0.7	0.87776	0.83130	0.79753	0.77259	0.77259	9.8497×10^{-10}
0.9	1.12590	1.06630	1.02300	0.99102	0.99102	1.2635×10^{-9}

Example 5.2: Assume the nonlinear fractional biological population equation described in Caputo's sense as follows

$$D_\rho^\varepsilon \psi = \frac{\partial^2 \psi^2}{\partial \vartheta^2} + \frac{\partial^2 \psi^2}{\partial \varphi^2} + h\psi(1 - q\psi), 0 < \varepsilon \leq 1, \tag{5.4}$$

under the initial condition

$$\psi(\vartheta, \varphi, 0) = \exp\left(\sqrt{\frac{hq}{8}}(\vartheta + \varphi)\right). \tag{5.5}$$

Following the KHRDTM methodology presented in Section 3, we get the series solution of our equation

$$\psi(\vartheta, \varphi, \rho) = \sum_{r=0}^{\infty} \Psi_r(\vartheta, \varphi),$$

where in this case, we set

$$L\psi(\vartheta, \varphi, \rho) = h\psi \text{ and } N\psi(\vartheta, \varphi, \rho) = \frac{\partial^2 \psi^2}{\partial \vartheta^2} + \frac{\partial^2 \psi^2}{\partial \varphi^2} + hq\psi^2.$$

The values of the RDT terms of $N\psi(\vartheta, \varphi, \rho)$ are given as

$$\begin{aligned} A_0(\vartheta, \varphi) &= \frac{\partial^2 \psi_0^2}{\partial \vartheta^2} + \frac{\partial^2 \psi_0^2}{\partial \varphi^2} + hq\psi_0^2, \\ A_1(\vartheta, \varphi) &= 2\frac{\partial^2 \psi_0 \psi_1}{\partial \vartheta^2} + 2\frac{\partial^2 \psi_0 \psi_1}{\partial \varphi^2} + 2hq\psi_0 \psi_1, \\ A_2(\vartheta, \varphi) &= 2\frac{\partial^2 (\psi_0 \psi_2)}{\partial \vartheta^2} + \frac{\partial^2 \psi_1^2}{\partial \vartheta^2} + 2\frac{\partial^2 (\psi_0 \psi_2)}{\partial \varphi^2} + \frac{\partial^2 \psi_1^2}{\partial \varphi^2} + 2hq\psi_0 \psi_2 + hq\psi_1^2. \end{aligned}$$

According to the recurrence formula (3.6), we get

$$\begin{aligned} \Psi_0(\vartheta, \varphi) &= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta + \varphi)\right), \\ \Psi_1(\vartheta, \varphi) &= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta + \varphi)\right)\frac{h\rho^\varepsilon}{\Gamma(\varepsilon+1)}, \end{aligned}$$

$$
\begin{aligned}
\Psi_2(\vartheta,\varphi) &= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\frac{h^2\rho^{2\varepsilon}}{\Gamma(2\varepsilon+1)},\\
\Psi_3(\vartheta,\varphi) &= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\frac{h^3\rho^{3\varepsilon}}{\Gamma(3\varepsilon+1)},\\
&\vdots\\
\Psi_k(\vartheta,\varphi) &= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\frac{h^k\rho^{k\varepsilon}}{\Gamma(k\varepsilon+1)}.
\end{aligned}
$$

Thus, the k^{th} −order approximate series solution is derived in the form

$$
\begin{aligned}
S_k(\vartheta,\varphi) &= \Psi_0(\vartheta,\varphi)+\Psi_1(\vartheta,\varphi)+\Psi_2(\vartheta,\varphi)+\ldots+\Psi_k(\vartheta,\varphi)\\
&= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)+\exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\frac{h\rho^{\varepsilon}}{\Gamma(\varepsilon+1)}\\
&\quad+\exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\frac{(h\rho^{\varepsilon})^2}{\Gamma(2\varepsilon+1)}+\exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\frac{(h\rho^{\varepsilon})^3}{\Gamma(3\varepsilon+1)}+\ldots\\
&\quad+\exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\frac{(h\rho^{\varepsilon})^k}{\Gamma(k\varepsilon+1)}\\
&= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\left(1+\frac{h\rho^{\varepsilon}}{\Gamma(\varepsilon+1)}+\frac{(h\rho^{\varepsilon})^2}{\Gamma(2\varepsilon+1)}+\frac{(h\rho^{\varepsilon})^3}{\Gamma(3\varepsilon+1)}\right.\\
&\quad\left.+\ldots+\frac{(h\rho^{\varepsilon})^k}{\Gamma(k\varepsilon+1)}\right)\\
&= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\sum_{r=0}^{k}\frac{(h\rho^{\varepsilon})^r}{\Gamma(r\varepsilon+1)}.
\end{aligned}
$$

So, the k^{th} −order approximate series solution converges to the exact solution when $k\to\infty$

$$
\begin{aligned}
\psi(\vartheta,\varphi,\rho) &= \lim_{k\to\infty}S_k(\vartheta,\varphi)\\
&= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\lim_{k\to\infty}\sum_{r=0}^{k}\frac{(h\rho^{\varepsilon})^r}{\Gamma(r\varepsilon+1)}\\
&= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)E_{\varepsilon}(h\rho^{\varepsilon}),
\end{aligned}
\tag{5.6}
$$

where $E_{\varepsilon}(h\rho^{\varepsilon})$ is the Mittag Leffler function defined by (2.1).

As $\varepsilon=1$ in (5.6), we have

$$
\begin{aligned}
\psi(\vartheta,\varphi,\rho) &= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)\right)\sum_{r=0}^{\infty}\frac{(h\rho)^r}{r!}\\
&= \exp\left(\sqrt{\frac{hq}{8}}(\vartheta+\varphi)+h\rho\right),
\end{aligned}
\tag{5.7}
$$

which is the same exact solution found in the literature [9].

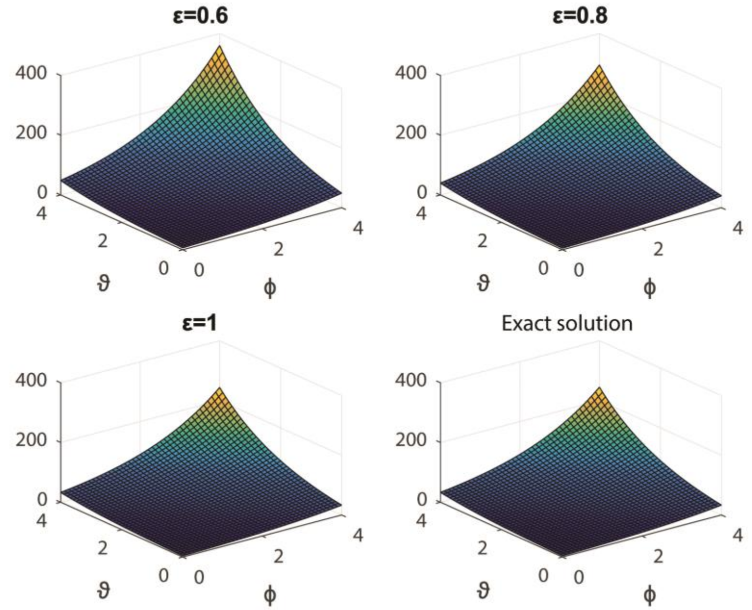

Figure 3
3D-plots of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta,\varphi)$ and exact solution $\psi(\vartheta,\varphi,\rho)$ for (5.4)-(5.5) with $h=1$ and $\rho=1.5$.

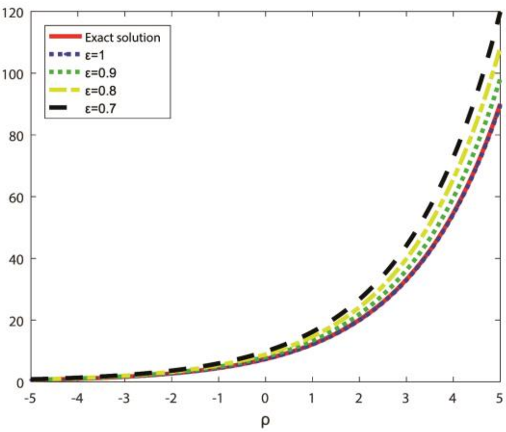

Figure 4
2D-plots of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta,\varphi)$ and exact solution $\psi(\vartheta,\varphi,\rho)$ for (5.4)-(5.5) with $h=\varphi=1$ and $\rho=1.5$.

Table 2
Comparison of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta,\varphi)$ and exact solution $\psi(\vartheta,\varphi,\rho)$ and their associated absolute errors for (5.4)-(5.5) for distinct values of ε.

	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{Exact}	$\lvert\psi_{Exact} - \psi_{KHRDTM}\rvert$
$(\vartheta,\varphi)/\varepsilon$	$\varepsilon = 0.7$	$\varepsilon = 0.8$	$\varepsilon = 0.9$	$\varepsilon = 1$	*Exact solution*	*Absolute errors*
0.1	1.3877	1.3142	1.2608	1.2214	1.2214	1.5572×10^{-9}
0.3	1.6949	1.6052	1.5400	1.4918	1.4918	1.9019×10^{-9}
0.5	2.0702	1.9606	1.8809	1.8221	1.8221	2.3230×10^{-9}
0.7	2.5285	2.3947	2.2974	2.2255	2.2255	2.8373×10^{-9}
0.9	3.0883	2.9249	2.8060	2.7183	2.7183	3.4655×10^{-9}

Example 5.3: Assume the nonlinear fractional biological population equation described in Caputo's sense as follows

$$D_\rho^\varepsilon \psi = \frac{\partial^2\psi^2}{\partial\vartheta^2} + \frac{\partial^2\psi^2}{\partial\varphi^2} + h\psi^{-1}(1 - q\psi), 0 < \varepsilon \leq 1, \tag{5.8}$$

under the initial condition

$$\psi(\vartheta,\varphi,0) = \sqrt{\frac{hq}{4}\vartheta^2 + \frac{hq}{4}\varphi^2 + \varphi + 5}. \tag{5.9}$$

Following the KHRDTM methodology presented in Section 3, we get the series solution of our equation

$$\psi(\vartheta,\varphi,\rho) = \sum_{r=0}^{\infty} \Psi_r(\vartheta,\varphi), \tag{5.10}$$

where in this case, we set

$$L\psi(\vartheta,\varphi,\rho) = hq \text{ and } N\psi(\vartheta,\varphi,\rho) = \frac{\partial^2\psi^2}{\partial\vartheta^2} + \frac{\partial^2\psi^2}{\partial\varphi^2} + h\psi^{-1}. \tag{5.11}$$

The values of the RDT terms of $N\psi(\vartheta,\varphi,\rho)$ are given as

$$\begin{aligned}
A_0(\vartheta,\varphi) &= \frac{\partial^2\psi_0^2}{\partial\vartheta^2} + \frac{\partial^2\psi_0^2}{\partial\varphi^2} + h\psi_0^{-1},\\
A_1(\vartheta,\varphi) &= 2\frac{\partial^2\psi_0\psi_1}{\partial\vartheta^2} + 2\frac{\partial^2\psi_0\psi_1}{\partial\varphi^2} - h\psi_0^{-2}\psi_1,\\
A_2(\vartheta,\varphi) &= 2\frac{\partial^2(\psi_0\psi_2)}{\partial\vartheta^2} + \frac{\partial^2\psi_1^2}{\partial\vartheta^2} + 2\frac{\partial^2(\psi_0\psi_2)}{\partial\varphi^2} + \frac{\partial^2\psi_1^2}{\partial\varphi^2}\\
&\quad - h\psi_0^{-2}\psi_2 + h\psi_0^{-3}\psi_1^2.
\end{aligned}$$

According to the recurrence formula (3.6), we get

$$\begin{aligned}
\Psi_0(\vartheta,\varphi) &= \sqrt{\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5},\\
\Psi_1(\vartheta,\varphi) &= h\left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{-1/2}\tfrac{\rho^{\varepsilon}}{\Gamma(\varepsilon+1)},\\
\Psi_2(\vartheta,\varphi) &= -2h^2\left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{-3/2}\tfrac{\rho^{2\varepsilon}}{\Gamma(2\varepsilon+1)},\\
\Psi_3(\vartheta,\varphi) &= 3h^3\left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{-5/2}\tfrac{\rho^{3\varepsilon}}{\Gamma(3\varepsilon+1)},\\
&\vdots\\
\Psi_k(\vartheta,\varphi) &= (-1)^{k-1}kh^k\left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{1/2-k}\tfrac{\rho^{k\varepsilon}}{\Gamma(k\varepsilon+1)}.
\end{aligned}$$

Thus, the k^{th} $-$order approximate series solution is derived in the form

$$\begin{aligned}
S_k(\vartheta,\varphi) &= \Psi_0(\vartheta,\varphi)+\Psi_1(\vartheta,\varphi)+\Psi_2(\vartheta,\varphi)+\ldots+\Psi_k(\vartheta,\varphi)\\
&= \left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{1/2}+h\left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{-1/2}\tfrac{\rho^{\varepsilon}}{\Gamma(\varepsilon+1)}\\
&\quad -2h^2\left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{-3/2}\tfrac{\rho^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\\
&\quad +3h^3\left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{-5/2}\tfrac{\rho^{3\varepsilon}}{\Gamma(3\varepsilon+1)}\\
&\quad +\ldots+(-1)^k(k+1)h^{k+1}\left(\tfrac{hq}{4}\vartheta^2+\tfrac{hq}{4}\varphi^2+\varphi+5\right)^{1/2-(k+1)}\\
&\qquad \tfrac{\rho^{(k+1)\varepsilon}}{\Gamma((k+1)\varepsilon+1)}\\
&= \Psi_0(\vartheta,\varphi)+\tfrac{h\rho^{\varepsilon}}{\Psi_0(\vartheta,\varphi)}\textstyle\sum_{r=0}^{k}\tfrac{r+1}{\Gamma((r+1)\varepsilon+1)}\left(\tfrac{-h\rho^{\varepsilon}}{\Psi_0^2(\vartheta,\varphi)}\right)^r.
\end{aligned}$$

So, the k^{th} $-$order approximate series solution converges to the exact solution when $k\to\infty$

$$\begin{aligned}
\psi(\vartheta,\varphi,\rho) &= \lim_{k\to\infty}S_k(\vartheta,\varphi)\\
&= \Psi_0(\vartheta,\varphi)+\tfrac{h\rho^{\varepsilon}}{\Psi_0(\vartheta,\varphi)}\lim_{k\to\infty}\textstyle\sum_{r=0}^{k}\tfrac{r+1}{\Gamma((r+1)\varepsilon+1)}\left(\tfrac{-h\rho^{\varepsilon}}{\Psi_0^2(\vartheta,\varphi)}\right)^r\\
&= \Psi_0(\vartheta,\varphi)+\tfrac{h\rho^{\varepsilon}}{\Psi_0(\vartheta,\varphi)}\textstyle\sum_{r=0}^{\infty}\tfrac{r+1}{\Gamma((r+1)\varepsilon+1)}\left(\tfrac{-h\rho^{\varepsilon}}{\Psi_0^2(\vartheta,\varphi)}\right)^r,
\end{aligned}\tag{5.12}$$

As $\varepsilon=1$ in (5.12), we have

$$\begin{aligned}
\psi(\vartheta,\varphi,\rho) &= \Psi_0(\vartheta,\varphi)+\tfrac{h\rho}{\Psi_0(\vartheta,\varphi)}\textstyle\sum_{r=0}^{\infty}\tfrac{1}{r!}\left(\tfrac{-h\rho}{\Psi_0^2(\vartheta,\varphi)}\right)^r\\
&= \Psi_0(\vartheta,\varphi)+\tfrac{h\rho}{\Psi_0(\vartheta,\varphi)}\exp\left(\tfrac{-h\rho}{\Psi_0^2(\vartheta,\varphi)}\right).
\end{aligned}\tag{5.13}$$

which is the same exact solution found in the literature [9].

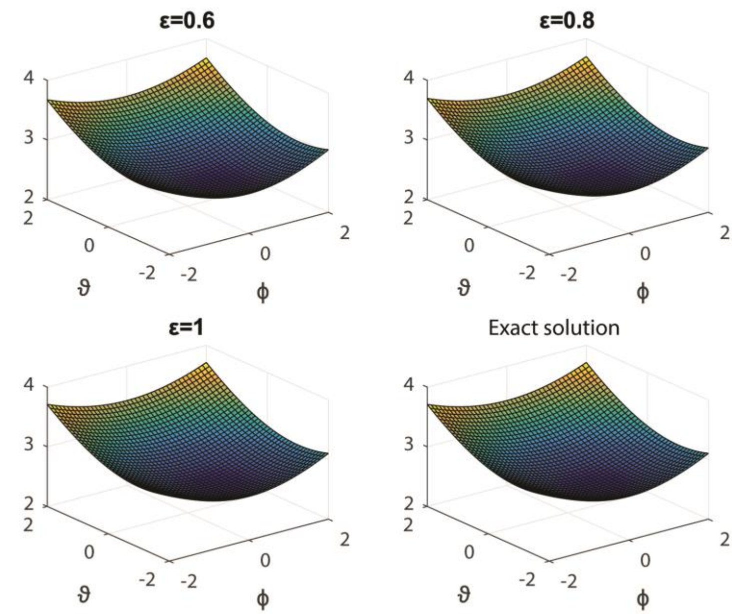

Figure 5
3D-plots of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta, \varphi)$ and exact solution $\psi(\vartheta, \varphi, \rho)$ for (5.8)-(5.9) with $h = 1$ and $\rho = 1.5$.

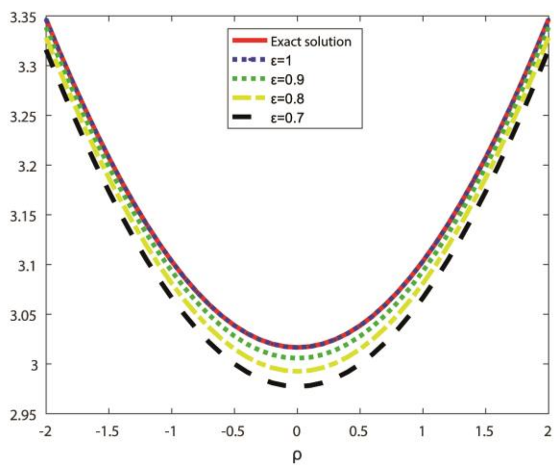

Figure 6
2D-plots of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta, \varphi)$ and exact solution $\psi(\vartheta, \varphi, \rho)$ for (5.8)-(5.9) with $h = \varphi = 1$ and $\rho = 1.5$.

Table 3
Comparison of 6^{th}-order approximate series solution $\sum_{r=0}^{6} \Psi_r(\vartheta, \varphi)$ and exact solution $\psi(\vartheta, \varphi, \rho)$ and their associated absolute errors for (5.8)-(5.9) for distinct values of ε.

	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{KHRDTM}	ψ_{Exact}	$\lvert\psi_{Exact} - \psi_{KHRDTM}\rvert$
$(\vartheta, \varphi)/\varepsilon$	$\varepsilon = 0.7$	$\varepsilon = 0.8$	$\varepsilon = 0.9$	$\varepsilon = 1$	*Exact solution*	*Absolute errors*
0.1	2.3523	2.3328	2.3168	2.3039	2.3039	2.6928×10^{-10}
0.3	2.4113	2.3922	2.3765	2.3639	2.3639	2.1185×10^{-10}
0.5	2.4850	2.4664	2.4511	2.4389	2.4389	1.5841×10^{-10}
0.7	2.5755	2.5567	2.5413	2.5275	2.5275	1.1370×10^{-10}
0.9	2.6715	2.6541	2.6398	2.6284	2.6284	7.9112×10^{-11}

Note: Figures 1-6 illustrate the 2D and 3D plots of the approximate and exact solutions for three specific applications of the nonlinear fractional biological population equations (1.1) using the KHDTM. Furthermore, the numerical values in resolving the proposed applications are presented in Tables 1-3. The numerical simulation indicates that the fractional numerical solution accurately approximates the exact solution.

6. Conclusions

In this manuscript, we used a new hybrid approach called KHRDTM to resolve a special cases of NFPDEs that are nonlinear fractional biological population equations. The numerical explanation of the proposed method has been supported by the presentation of three illustrative applications. The computed approximation solutions are presented in the closed form of the Mittag-Leffler function. From the obtained results, we notice that the approximate solutions are excellent agreement with the exact solutions. This indicates that the KHRDTM is an accurate and suitable analytical technique for solving NFPDEs.

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