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# Generalized Hyer's-Ulam stability for the cubic functional equation in non-Archimedean normed spaces

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**Abstract**

In this current article, we explore the Hyer's-Ulam Stability for the cubic functional equation over non-Archimedean Normed space and the equation is written as follows:

$$(3\mathfrak{f}(x+3y)-\mathfrak{f}(3x+y)=12[\mathfrak{f}(x+y)+f(x-y)]-48\mathfrak{f}(x)+80\mathfrak{f}(y))$$

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## 1. Introduction

One of the most fascinating problems in the stability issue of functional equation, was first raised by Ulam [4, 24] in 1940.

It states as follows, under what circumstances is it true that a mapping that approximates the solution of a given functional equation has to be near the precise solution of the given functional equation?

In the next year, Hyers [12],[20] gave the headmost positive answer to Ulam's query in the setting of Banach spaces such that

When $\mathfrak{g}: M \to M'$ satisfies the below inequality

$$\|\mathfrak{g}(x+y)-\mathfrak{g}(x)-\mathfrak{g}(y)\| \le \delta \qquad \forall x, y \in M \tag{1.1}$$

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where M and $M'$ are Banach spaces for some $\delta > 0$. Then there is some function T: $M \to M'$ such that

$$\|g(x) - T(x)\| \leq \delta \qquad \forall x \in M \tag{1.2}$$

This is known as Hyer's-Ulam Stability.

The second person to solve this issue under additive mappings was Aoki [1] in 1950. The expanded Hyer's theorem was introduced in 1978 by Rassias [22], who allowed the Cauchy difference to be unbounded. In 1982, Rassias [21] substituted $\|x\|^p\|y\|^p$ for the factor $\|x\|^p + \|y\|^p$ for every $p \in \mathcal{R}$. Gajda [10],[3] provided an answer to this for p $\geq 1$ in 1991. In 1994, Gavruta [11] generalized the Rassias theorem by substituting a general control function $\varphi$ $(x,y)$ for the unbounded Cauchy difference.

Numerous mathematical contexts, such as Abelian groups, Banach spaces, non-Archimedean analysis, and fuzzy normed spaces, have been used to investigate the stability of cubic functional equations. Ebadian et al. extended classical stability results to algebraic structures by examining the approximate stability of $AQ_4Q_2C$ functional equation in two-variable Abelian groups [5]. In 2024, By investigating the n-dimensional cubic functional equation in non-Archimedean Banach spaces. The findings contribute to the broader understanding of stability in functional equations within non-Archimedean contexts [7].

By investigating the stability of both $CQ_4$ functional equation in Non-Archimedean spaces and utilizing fixed-point techniques to prove their results, Eshaghi Gordji and Savadkouhi made further contributions to this field [8].

A generalized additive-cubic-quartic functional equation was studied by Eshaghi-Gordji et al. in a related work. They provided conditions under which approximate solutions result in accurate ones [9]. Najati and Park made early contributions by analyzing the Hyers Ulam Stability of a certain cubic functional equation in Banach spaces, laying the foundation for further study in this area.[17].

In 2020, Najati and Sahoo[6] explored the Hyers-Ulam stability of several functional equations, concentrating on additive and quadratic varieties, and presented significant generalizations that advanced the theoretical comprehension of functional stability. In 2022, Building on this research, El-Fassi[18] examined the stability of two functional equations with fuzzy number values, utilizing fuzzy set theory to address and manage uncertainty in functional behavior. Continuing this line of study, Valdes et al.[19] investigated the fuzzy Hyers-Ulam-Rassias stability of functional equations associated with quadratic forms, offering refined findings that integrate fuzzy logic with sophisticated stability concepts in 2023. Lastly, these stability results were extended to fuzzy normed spaces by Ravi, Rassias, and Narasimman. They demonstrated the behavior of functional equations in uncertain and imprecise circumstances, which has impact on control theory and decision-making[23].

Inspired by modular arithmetic and the structure of integers, Kurt Hensel first proposed the p-adic number system in 1897 (see [13],[14]). Their unique properties, such as non-Archimedean norms and ultrametric topology, make them a prominent area of research in both pure and applied mathematics.

The following cubic functional equation is investigated in this article and can be written as

$$3\mathfrak{f}(x+3y) - \mathfrak{f}(3x+y) = 12[\mathfrak{f}(x+y) + \mathfrak{f}(x-y)] - 48\mathfrak{f}(x) + 80\mathfrak{f}(y) \quad (1.3)$$

It is important to note that $\mathfrak{f}(x) = \kappa x^3$ is the solution of (1.3). Every cubic mapping is a solution to the cubic functional equation.

## 2. Preliminaries

This section focuses on providing most basic results and remarks to understand our main results.

**Definition 2.1:** [25] A non-Archimedean normed space is a vector space embedded with norm called **"non-Archimedean norm",$\|.\|$** defined as follows: that is $\|.\|$ is a function from $\mathcal{X} \to \mathcal{R}$ satisfying the below properties

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$ , $\forall\, x \in \mathcal{X}$

(ii) $\|\lambda x\| = |\lambda|\, \|x\|, \forall\, x \in \mathcal{X}$ and $\lambda \in \mathcal{K}$

(iii) $\|x+y\| \leq max\{\|x\|, \|y\|\}$ , $\forall\, x, y \in \mathcal{X}$

**Definition 2.2:** [15], [16] In the setting of non-Archimedean valuation, a sequence $\{x_n\}$ in $\mathcal{K}$ is considered Cauchy iff

$$|x_{n+1} - x_n| \to 0, \text{as } n \to \infty$$

**Definition 2.3:** [2] Let $\mathfrak{c} \in \mathcal{R}$, where $\mathfrak{c}$ is fixed with $0 < \mathfrak{c} < 1$ and let $p$ be a fixed prime number.

Now, Let $x$ be rational number with $x \neq 0$, $x$ can be written as the form

$$x = p^{\alpha}\frac{\mu}{\nu}$$

where $\mu, \nu, \alpha \in \mathcal{Z}$, the ring of integers and $p$ does not divide both $\mu$ and $\nu$. Here $\alpha$ be neither positive nor negative, or zero according as $x$.

Then, we define

$$|x|_p = \mathfrak{c}^{\alpha} \text{ and } |0|_p = 0$$

called the **p-adic valuation** of $x$. This $p$ −adic valuation $|.|_p$ can be proved to be a non-Archimedean valuation.

## 3. Main Results

In the following section, Let $\mathcal{X}$ represent normed space and $\mathcal{Y}$ represent a complete non-Archimedean normed space. Let us consider the difference operator

$$\mathcal{D}_{\mathfrak{f}}(x,y) := 3\mathfrak{f}(x+3y) - \mathfrak{f}(3x+y) - 12[\mathfrak{f}(x+y) + \mathfrak{f}(x-y)] + 48\mathfrak{f}(x) - 80\mathfrak{f}(y) \quad (3.1)$$

Here, we establish the conditions that are essential to prove the stability for cubic functional equation in terms of non-Archimedean normed spaces.

**Theorem 3.1:** *Assume that* $\varphi: \mathcal{X}^2 \to [0, \infty)$ *is a function that*

$$\lim_{n\to\infty} \frac{1}{27^n}\varphi(3^n x, 3^n y) = 0, \forall ù, y \in \mathcal{X}. \tag{3.2}$$

*and assume that* $\mathfrak{f}: \mathcal{X} \to \mathcal{Y}$ *satisfies the below inequality*

$$\mathcal{D}_{\mathfrak{f}}(x, y) \leq \varphi(x, y), \forall x, y \in \mathcal{X}. \tag{3.3}$$

Next, there is a one and only one cubic mapping $T: \mathcal{X} \to \mathcal{Y}$ in such a way that

$$\|\mathfrak{f}(x) - T(x)\| \leq \tilde{\varphi}(x) \tag{3.4}$$

$$\text{Where}, T(x) := \lim_{n\to\infty} \frac{1}{27^n}\mathfrak{f}(3^n x), \forall x \in \mathcal{X}. \tag{3.5}$$

$$\text{and}, \tilde{\varphi}(x) = \lim_{n\to\infty} max\left\{\frac{1}{|27^{\iota+1}|}\varphi(3^\iota x, 0): 0 \leq \iota < n\right\}$$

Moreover, if

$$\lim_{\kappa\to\infty}\lim_{n\to\infty} \frac{1}{|27|} max\left\{\frac{1}{|27^\iota|}\varphi(3^\iota x, 0): \kappa \leq \iota < n + \kappa\right\} = 0, \forall x \in \mathcal{X} \tag{3.6}$$

where $n$, $\kappa$ are non-negative integers ; then $T$ is a only cubic mapping which satisfies (3.4)

***Proof:*** Substituting $y = 0$ in equation (3.1), we get

$$\|\mathfrak{f}(3x) - 27\mathfrak{f}(x)\| \leq \varphi(x, 0)$$

$$\left\|\frac{\mathfrak{f}(3x)}{27} - \mathfrak{f}(x)\right\| \leq \frac{1}{|27|}\varphi(x, 0) \tag{3.7}$$

Replace $x$ by $3x$ and multiply $\frac{1}{27}$ on both sides in equation (3.7), and Continuing to '$n$' terms one can get

$$\left\|\frac{1}{27^{n+1}}\mathfrak{f}(3^{n+1}x) - \frac{\mathfrak{f}(3^n x)}{27^n}\right\| \leq \frac{1}{|27^{n+1}|}\varphi(3^n x, 0), \forall x \in \mathcal{X}, \tag{3.8}$$

for all non - negative integer '$n$'. Taking $n \to \infty$ in (3.8) we get

$$\left\|\frac{1}{27^{n+1}}\mathfrak{f}(3^{n+1}x) - \frac{\mathfrak{f}(3^n x)}{27^n}\right\| = 0 \; by \; (3.2). \tag{3.9}$$

Consequently $\{\frac{1}{27^n}\mathfrak{f}(3^n x)\}$ is Cauchy.

Hence we can conclude that the sequence $\{\frac{1}{27^n}\mathfrak{f}(3^n x)\}$ converges, $\forall$ $x \in \mathcal{X}$, since $\mathcal{Y}$ is complete.

Hence we consider a function $\mathrm{T}: \mathcal{X} \to Y$ as the limit by

$$\mathrm{T}(x) = \lim_{n\to\infty} \frac{1}{27^n}\mathfrak{f}(3^n x) \text{ exists.}$$

Then, one can infer that by applying induction on 'n' in equation (3.8) to demonstrate (3.4)

$$\left\|\frac{1}{27}\mathfrak{f}(3x) - \mathfrak{f}(x) + \frac{1}{27^2}\mathfrak{f}(3^2x) - \frac{1}{27}\mathfrak{f}(3x) + \frac{1}{27^3}\mathfrak{f}(3^3x) - \frac{1}{27^2}\mathfrak{f}(3^2x) + \ldots\right.$$

$+\frac{1}{27^n}\mathfrak{f}(3^n x)-\frac{1}{27^{n-1}}\mathfrak{f}(3^{n-1}x)\Big\|$

$\leq max\left\{\frac{1}{|27|}\varphi(x,0),\frac{1}{|27^2|}\varphi(3x,0)\ldots\ldots\frac{1}{|27^n|}\varphi(3^{n-1}x,0)\right\}$

$\left\|\frac{1}{27^n}\mathfrak{f}(3^n x)-\mathfrak{f}(x)\right\|\leq max\left\{\frac{1}{|27|}\varphi(x,0),\frac{1}{|27^2|}\varphi(3x,0)\ldots\ldots\frac{1}{|27^n|}\varphi(3^{n-1}x,0)\right\}$

$$\left\|\frac{1}{27^n}\mathfrak{f}(3^n x)-\mathfrak{f}(x)\right\|\leq max\left\{\frac{1}{|27^{\iota+1}|}\varphi(3^\iota x,0):0\leq\iota<n\right\},\forall n\in\mathcal{N}\ and\ x\in\mathcal{X}. \tag{3.10}$$

$\left\|\frac{1}{27^n}\mathfrak{f}(3^n x)-\mathfrak{f}(x)\right\|\leq \frac{1}{|27|}max\left\{\frac{1}{|27^\iota|}\varphi(3^\iota x,0):0\leq\iota<n\right\},\forall n\in\mathcal{N}\ and\ x\in\mathcal{X}.$

By applying the limit $n\to\infty$ in (3.10), we get

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\lim_{n\to\infty}max\left\{\frac{1}{|27^{\iota+1}|}\varphi(3^\iota x,0):0\leq\iota<n\right\}\leq\tilde{\varphi}(x)$$

Now, we show that $T$ is a cubic function.

From (3.3), we have

$\left\|\mathcal{D}_{\mathfrak{f}}(x,y)\right\|=\Big\|3\lim_{n\to\infty}\frac{1}{27^n}\mathfrak{f}(3^n x+3^{n+1}y)-\lim_{n\to\infty}\frac{1}{27^n}\mathfrak{f}(3^{n+1}x+3^n y)$

$-12\lim_{n\to\infty}[\frac{1}{27^n}\mathfrak{f}(3^n x+3^n y)+\frac{1}{27^n}\mathfrak{f}(3^n x-3^n y)]-80\lim_{n\to\infty}\frac{1}{27^n}\mathfrak{f}(3^n y)$

$+48\lim_{n\to\infty}\frac{1}{27^n}\mathfrak{f}(3^n x)\Big\|$

$\left\|\mathcal{D}_{\mathfrak{f}}(x,y)\right\|=\lim_{n\to\infty}\frac{1}{27^n}\|3\mathfrak{f}(3^n x+3^{n+1}y)-\mathfrak{f}(3^{n+1}x+3^n y)$

$$-12[\mathfrak{f}(3^n x+3^n y)+\mathfrak{f}(3^n x-3^n y)]-80\mathfrak{f}(3^n x)+48\mathfrak{f}(3^n x)\| \tag{3.11}$$

$$\left\|\mathcal{D}_{\mathfrak{f}}(x,y)\right\|\leq\lim_{n\to\infty}\frac{1}{27^n}\varphi(3^n x,3^n y)=0 \text{ by using (3.2)}, \forall x,y\in\mathcal{X}. \tag{3.12}$$

Therefore T is cubic.

**Uniqueness**: Let us consider another function $\mathrm{T}':\mathcal{X}\to\mathcal{Y}$ such that

$$\mathrm{T}'(x)=\lim_{n\to\infty}\frac{1}{27^n}\mathfrak{f}'(3^n x)$$

Then,

$$\|\mathrm{T}(x)-\mathrm{T}'(x)\|=\lim_{n\to\infty}\left\|\frac{1}{27^n}\mathfrak{f}(3^n x)-\frac{1}{27^n}\mathfrak{f}'(3^n x)\right\|$$

$$\|\mathrm{T}(x)-\mathrm{T}'(x)\|=\lim_{n\to\infty}|27|^{-n}\|\mathfrak{f}(3^n x)-\mathfrak{f}'(3^n x)\|$$

$$\|\mathrm{T}(x)-\mathrm{T}'(x)\|=\lim_{n\to\infty}|27|^{-n}\|\mathfrak{f}(3^n x)-27^n\mathfrak{f}(x)+27^n\mathfrak{f}(x)-\mathfrak{f}'(3^n x)\|$$

$$\|\mathrm{T}(x)-\mathrm{T}'(x)\|=\lim_{n\to\infty}|27|^{-n}\|(\mathfrak{f}(3^n x)-27^n\mathfrak{f}(x))+(27^n\mathfrak{f}(x)-\mathfrak{f}'(3^n x))\|$$

$$\|\mathrm{T}(x)-\mathrm{T}'(x)\|\leq\lim_{n\to\infty}|27|^{-n}max\{\|\mathfrak{f}(3^n x)-27^n\mathfrak{f}(x)\mathrm{P},\mathrm{P}27^n\mathfrak{f}(x)-\mathfrak{f}'(3^n x)\|\}$$

$$\|\mathrm{T}(x)-\mathrm{T}'(x)\|\leq\lim_{n\to\infty}max\{|27|^{-n}\|\mathfrak{f}(3^n x)-\mathfrak{f}(x)\|,|27|^{-n}\|\mathfrak{f}(x)-\mathfrak{f}'(3^n x)\|\}$$

$$\|\mathrm{T}(x)-\mathrm{T}'(x)\|\leq\lim_{n\to\infty}\frac{1}{|27|}\lim_{\iota\to\infty}max\left\{\frac{1}{|27|^\iota}\varphi(3^\iota x,0),\frac{1}{|27|^\iota}\varphi(3^\iota x,0)\right\}$$

$$\|\mathrm{T}(x)-\mathrm{T}'(x)\| \leq \lim_{\kappa\to\infty}\lim_{n\to\infty}\frac{1}{|27|}max\left\{\frac{1}{27^{\iota}}\varphi(3^{\iota}x,0):\kappa\leq\iota<n+\kappa\right\}=0,\forall x\in\mathcal{X}.$$

$$\Rightarrow \mathrm{T}=\mathrm{T}'$$

Hence proved.

**Corollary 3.2:** *Let p be the prime and* $p\neq 3$*, and for* $\xi\geq 0$*, define the function* $\mathfrak{f}:\mathcal{X}\to\mathcal{Y}$ *fulfills*

$$\mathcal{D}_{\mathfrak{f}}(x,y)\leq\xi,\forall\, x,y\in\mathcal{X}. \tag{3.13}$$

Next, there exists one and only cubic function $T:\mathcal{X}\to\mathcal{Y}$ in such a way that

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\xi \tag{3.14}$$

*Proof.* By Theorem (3.1),

$$\mathcal{D}_{\mathfrak{f}}(x,y)\leq\varphi(x,y) \tag{3.15}$$

Hence we have $\varphi(\mathcal{X},y)=\xi$,

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\tilde{\varphi}(x)$$

where $\tilde{\varphi}(x)=\lim_{n\to\infty}max\left\{\frac{1}{|27^{\iota+1}|}\varphi(3^{\iota}x,0):0\leq\iota<n\right\}$

Therefore, $\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\lim_{n\to\infty}max\left\{\frac{1}{|27^{\iota+1}|}\xi:0\leq\iota<n\right\}$

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq max\left\{\frac{\xi}{|27|},\frac{\xi}{|27^2|},\frac{\xi}{|27^3|},\ldots,\frac{\xi}{|27^{n-1}|}\right\}$$

(where $p$ is a fixed prime other than 3)

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\xi \tag{3.16}$$

**Corollary 3.3:** *Let p be the prime and* $p\neq 3$*. Let* $\alpha,\gamma,\xi>0$ *and* $\alpha+\gamma>0$ *and define a mapping* $\mathfrak{f}:\mathcal{X}^2\to[0,\infty)$ *fulfills*

$$\mathcal{D}_{\mathfrak{f}}(x,y)\leq\xi(\|x\|^{\alpha+\gamma}+\|y\|^{\alpha+\gamma}),\forall x,y\in\mathcal{X}.$$

Next, there is a cubic function $\mathrm{T}:\mathcal{X}\to\mathcal{Y}$ such that

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\xi\|x\|^{\alpha+\gamma},\forall x\in\mathcal{X}$$

and T is unique.

***Proof:*** By Theorem (3.1), we have $\varphi(x,y)=\xi(\|x\|^{\alpha+\gamma}+\|y\|^{\alpha+\gamma})$, then

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\tilde{\varphi}(x)$$

where $\tilde{\varphi}(x)=\lim_{n\to\infty}max\left\{\frac{1}{|27^{\iota+1}|}\varphi(3^{\iota}x,0):0\leq\iota<n\right\}$

Now,

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\lim_{n\to\infty}max\left\{\frac{1}{|27^{\iota+1}|}\xi\|(3^{\iota}x)\|^{\alpha+\gamma}:0\leq\iota<n\right\}$$

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\|\leq\lim_{n\to\infty}max\left\{\frac{1}{|27^{\iota+1}|}\xi|3^{\iota(\alpha+\gamma)}|\|x\|^{\alpha+\gamma}:0\leq\iota<n\right\}$$

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\| \leq \max\left\{\frac{1}{|27|}\xi\|x\|^{\alpha+\gamma}, \frac{1}{|27^2|}\xi|3^{(\alpha+\gamma)}|\|x\|^{\alpha+\gamma}, \ldots\ldots\ldots \frac{1}{|27^n|}\xi|3^{(n-1)(\alpha+\gamma)}|\|x\|^{\alpha+\gamma}\right\}$$

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\| \leq \xi\|x\|^{\alpha+\gamma} max\left\{\frac{1}{|27|}, \frac{|3^{(\alpha+\gamma)}|}{|27^2|}, \frac{|3^{2(\alpha+\gamma)}|}{|27^3|}, \ldots\ldots\ldots \frac{|3^{(n-1)(\alpha+\gamma)}|}{|27^n|}\right\}$$

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\| \leq \xi\|x\|^{\alpha+\gamma} max\left\{\frac{1}{|27|}, \frac{1}{|27^2|}, \frac{1}{|27^3|}, \ldots\ldots\ldots \frac{1}{|27^n|}\right\}$$

(where $p$ is a fixed prime other than)

$$\|\mathfrak{f}(x)-\mathrm{T}(x)\| \leq \xi\|x\|^{\alpha+\gamma} \tag{3.17}$$

**Example 3.1:** Let $\mathfrak{f}: \mathcal{X} \to \mathcal{Y}$ be defined by $\mathfrak{f}(x) = 2x^3 + 1$ and let $p > 3$, be a prime number. since for every $n \in \mathcal{N}$, $|27^n| = 1$. Then there is no cubic function with

$$g(x) := \lim_{n\to\infty} \frac{1}{27^n} g(3^n x), \forall x \in \mathcal{X}. \tag{3.18}$$

and $g(x) = \mathfrak{f}(3x) - 27\mathfrak{f}(x)$ such that

$$\|\mathfrak{f}(x) - g(x)\| \leq \varepsilon. \tag{3.19}$$

***Proof:*** Let $\mathfrak{f}(x) = 2x^3 + 1$ From (3.1), we have

$$\mathcal{D}_{\mathfrak{f}}(x,y) = 3.2.(x+3y)^3 + 3 - (2.(3x+y)^3+1) - 12[2.(x+y)^3+1$$
$$+2.(x-y)^3+1] - 80.2.y^3 - 80 + 48.2.x^3 + 48$$
$$\mathcal{D}_{\mathfrak{f}}(x,y) = 6 \neq 0$$

Then, we have

$$g(3^n x) = \mathfrak{f}(3^{n+1}x) - 27\mathfrak{f}(3^n x) \text{ and } g(3^{n-1}x) = \mathfrak{f}(3^n x) - 27\mathfrak{f}(3^{n-1}x).$$

$$\left\|\frac{g(3^n x)}{27^n} - \frac{g(3^{n-1}x)}{27^{n-1}}\right\| = \left\|\frac{\mathfrak{f}(3^{n+1}x) - 27\mathfrak{f}(3^n x)}{27^n} - \left\{\frac{\mathfrak{f}(3^n x) - 27\mathfrak{f}(3^{n-1}x)}{27^{n-1}}\right\}\right\|$$

$$= \left\|\frac{2.(3^{n+1}x)^3 + 1 - (54.(3^n x)^3 + 27)}{27^n} - \left\{\frac{2.(3^n x)^3 + 1 - (54.(3^{n-1}x)^3 + 27)}{27^{n-1}}\right\}\right\|$$

$$\left\|\frac{g(3^n x)}{27^n} - \frac{g(3^{n-1}x)}{27^{n-1}}\right\| = \frac{2.3^3.3^{3n}.x^3 + 1 - 54.3^{3n}.x^3 - 27 - 54.3^{3n}.x^3 - 27 + 54.3^{3n}.x^3 - 27.27}{27^n}$$

$$\left\|\frac{g(3^n x)}{27^n} - \frac{g(3^{n-1}x)}{27^{n-1}}\right\| = |782| \neq 0$$

hence, $\{27^{-n} g(3^n x)\}$ is not a Cauchy sequence.

**Theorem 3.4:** *Assume that* $\varphi: \mathcal{X}^2 \to [0, \infty)$ *is a function that*

$$\lim_{n\to\infty} 27^n \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = 0, \forall x, y \in \mathcal{X}. \tag{3.20}$$

and assume that $\mathfrak{f}: \mathcal{X} \to \mathcal{Y}$ satisfies the below inequality

$$\mathcal{D}_{\mathfrak{f}}(x,y) \leq \varphi(x,y), \forall x, y \in \mathcal{X}. \tag{3.21}$$

Next, there is a only one cubic function $\mathrm{T}: \mathcal{X} \to \mathcal{Y}$ such that

$$\|\mathfrak{f}(x) - \mathrm{T}(x)\| \leq \tilde{\varphi}(x) \tag{3.23}$$

$$\text{Where } T(x) = \lim_{n\to\infty} 27^n \mathfrak{f}\left(\frac{x}{3^n}\right), \forall x \in \mathcal{X} \tag{3.23}$$

$$\text{and } \tilde{\varphi}(x) = \lim_{n\to\infty} max\left\{|27^\iota|\varphi\left(\frac{x}{3^{\iota+1}}, 0\right): 0 \le \iota < n\right\}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} \frac{1}{|27|} max\left\{|27^\kappa|\varphi\left(\frac{x}{3^\kappa}, 0\right): j \le \kappa < n+j\right\} = 0, \forall x \in \mathcal{X} \tag{3.24}$$

where $n$, $j$ are non-negative integers; then T is an only cubic mapping satisfies (3.22)

***Proof:*** Substituting $y = 0$ in (3.1), we get

$$\|(3x) - 27\mathfrak{f}(x)\| \le \varphi(x, 0)$$

Replace $x$ by $\frac{x}{3}$ and multiply 27 on both sides in above equation, and Continuing to '$n$' terms we get

$$\left\|27^{n+1}\mathfrak{f}\left(\frac{x}{3^{n+1}}\right) - 27^n f\left(\frac{x}{3^n}\right)\right\| \le |27^n|\varphi\left(\frac{x}{3^{n+1}}, 0\right), \forall x \in \mathcal{X} \tag{3.25}$$

and all non - negative integers '$n$'. Taking $n \to \infty$ in (3.25)

$$\left\|27^{n+1}\mathfrak{f}\left(\frac{x}{3^{n+1}}\right) - 27^n\mathfrak{f}\left(\frac{x}{3^n}\right)\right\| = 0 \text{ by (3.20)} \tag{3.26}$$

Consequently, $\{27^n\mathfrak{f}(\frac{x}{3^n})\}$ is Cauchy. Hence we can conclude that the sequence $\{27^n\mathfrak{f}(\frac{x}{3^n})\}$ converges, $\forall\ x \in \mathcal{X}$, since $\mathcal{Y}$ is complete.

Hence we consider a function $T: \mathcal{X} \to \mathcal{Y}$ as the limit by

$$T(x) = \lim_{n\to\infty} 27^n\mathfrak{f}\left(\frac{x}{3^n}\right)$$

Then, one can infer that by applying induction on 'n' in (3.25) to demonstrate (3.22)

$$\left\|27\mathfrak{f}\left(\frac{x}{3}\right) - \mathfrak{f}(x) + 27^2\mathfrak{f}\left(\frac{x}{3^2}\right) - 27\mathfrak{f}\left(\frac{x}{3}\right) + 27^3\mathfrak{f}\left(\frac{x}{3^3}\right) - 27^2\mathfrak{f}\left(\frac{x}{3^2}\right) + \ldots\ldots + 27^n\mathfrak{f}\left(\frac{x}{3^n}\right) - 27^{n-1}\mathfrak{f}\left(\frac{x}{3^{n-1}}\right)\right\|$$

$$\le max\left\{\varphi\left(\frac{x}{3}, 0\right), 27\varphi\left(\frac{x}{3^2}, 0\right)\ldots\ldots 27^{n-1}\varphi\left(\frac{x}{3^n}, 0\right)\right\}$$

$$\left\|27^n\mathfrak{f}\left(\frac{x}{3^n}\right) - \mathfrak{f}(x)\right\| \le max\left\{\varphi\left(\frac{x}{3}, 0\right), 27\varphi\left(\frac{x}{3^2}, 0\right)\ldots\ldots 27^{n-1}\varphi\left(\frac{x}{3^n}, 0\right)\right\}$$

$$\left\|27^n\mathfrak{f}(\frac{x}{3^n}) - \mathfrak{f}(x)\right\| \le max\left\{27^\iota\varphi\left(\frac{x}{3^{\iota+1}}, 0\right): 0 \le \iota < n\right\}, \forall n \in \mathcal{N} \text{ and } x \in \mathcal{X}. \tag{3.27}$$

$$\left\|27^n\mathfrak{f}\left(\frac{x}{3^n}\right) - \mathfrak{f}(x)\right\| \le \frac{1}{|27|} max\left\{27^\kappa\varphi\left(\frac{x}{3^\kappa}, 0\right): 0 \le \kappa < n\right\}, \forall n \in \mathcal{N} \text{ and } x \in \mathcal{X}.$$

By applying the limit $n \to \infty$ in (3.27), we get

$$\|\mathfrak{f}(x) - T(x)\| \le \lim_{n\to\infty} max\left\{27^\iota\varphi(\frac{x}{3^{\iota+1}}, 0): 0 \le \iota < n\right\} \le \tilde{\varphi}(x).$$

Now, we show that $T$ is a cubic function. From (3.21) we have

$$\mathcal{D}_{\mathfrak{f}}(x,y) = \Big\| 3\lim_{n\to\infty} 27^n \mathfrak{f}\left(\tfrac{x+3y}{3^n}\right) - \lim_{n\to\infty} 27^n \mathfrak{f}\left(\tfrac{3x+y}{3^n}\right)$$
$$-12\lim_{n\to\infty} 27^n \left[\mathfrak{f}\left(\tfrac{x+y}{3^n}\right) + \mathfrak{f}\left(\tfrac{x-y}{3^n}\right)\right] - 80\lim_{n\to\infty} 27^n \mathfrak{f}\left(\tfrac{y}{3^n}\right) + 48\lim_{n\to\infty} 27^n \mathfrak{f}\left(\tfrac{x}{3^n}\right)\Big\|$$

$$\begin{aligned}\mathcal{D}_{\mathfrak{f}}(x,y) = \lim_{n\to\infty} 27^n \Big\| 3\mathfrak{f}\left(\tfrac{x+3y}{3^n}\right) - \mathfrak{f}\left(\tfrac{3x+y}{3^n}\right)\\ -12\left[\mathfrak{f}\left(\tfrac{x+y}{3^n}\right) + \mathfrak{f}\left(\tfrac{x-y}{3^n}\right)\right] - 80\mathfrak{f}\left(\tfrac{y}{3^n}\right) + 48\,\mathfrak{f}\left(\tfrac{x}{3^n}\right)\Big\|\end{aligned} \tag{3.28}$$

$$\mathcal{D}_{\mathfrak{f}}(x,y) \le \lim_{n\to\infty} 27^n \varphi\left(\tfrac{x}{3^n}, \tfrac{y}{3^n}\right) = 0 \; by\; using\; (3.20), \forall x,y \in \mathcal{X}. \tag{3.29}$$

Therefore T is cubic.

**Uniqueness**: Let us consider another function $\mathrm{T}': \mathcal{X} \to \mathcal{Y}$ such that

$$T'(x) = \lim_{n\to\infty} 27^n \mathfrak{f}'\left(\tfrac{x}{3^n}\right)$$

Then,

$$\|T(x) - T'(x)\| = \lim_{n\to\infty} \mathrm{P}|27^n|\mathfrak{f}\left(\tfrac{x}{3^n}\right) - |27^n|\mathfrak{f}'\left(\tfrac{x}{3^n}\right)$$
$$\|T(x) - T'(x)\| = \lim_{n\to\infty} |27^n| \left\|\mathfrak{f}\left(\tfrac{x}{3^n}\right) - \mathfrak{f}'\left(\tfrac{x}{3^n}\right)\right\|$$
$$\|T(x) - T'(x)\| = \lim_{n\to\infty} |27^n| \left\|\mathfrak{f}\left(\tfrac{x}{3^n}\right) - \tfrac{1}{27^n}\mathfrak{f}(x) + \tfrac{1}{27^n}\mathfrak{f}(x) - \mathfrak{f}'\left(\tfrac{x}{3^n}\right)\right\|$$
$$\|T(x) - T'(x)\| = \lim_{n\to\infty} |27^n| \left\|\left(\mathfrak{f}\left(\tfrac{x}{3^n}\right) - \tfrac{1}{27^n}\mathfrak{f}(x)\right) + \left(\tfrac{1}{27^n}\mathfrak{f}(x) - \mathfrak{f}'\left(\tfrac{x}{3^n}\right)\right)\right\|$$
$$\|T(x) - T'(x)\| \le \lim_{n\to\infty} |27^n| max\left\{\left\|\mathfrak{f}\left(\tfrac{x}{3^n}\right) - \tfrac{1}{27^n}\mathfrak{f}(x)\right\|, \left\|\tfrac{1}{27^n}\mathfrak{f}(x) - \mathfrak{f}'\left(\tfrac{x}{3^n}\right)\right\|\right\}$$
$$\|T(x) - T'(x)\| \le \lim_{n\to\infty} max\left\{|27^n|\left\|\mathfrak{f}\left(\tfrac{x}{3^n}\right) - \tfrac{1}{27^n}\mathfrak{f}(x)\right\|, |27^n|\left\|\tfrac{1}{27^n}\mathfrak{f}(x) - \mathfrak{f}'\left(\tfrac{x}{3^n}\right)\right\|\right\}$$
$$\|T(x) - T'(x)\| \le \lim_{n\to\infty} \tfrac{1}{|27|} \lim_{\kappa\to\infty} max\left\{|27^\kappa|\varphi\left(\tfrac{x}{3^\kappa}, 0\right), |27^\kappa|\varphi\left(\tfrac{x}{3^\kappa}, 0\right)\right\}$$
$$\|T(x) - T'(x)\| \le \lim_{n\to\infty} \tfrac{1}{|27|} \lim_{j\to\infty} max\left\{|27^\kappa|\varphi\left(\tfrac{x}{3^\kappa}, 0\right) : j \le \kappa < n + j\right\} = 0$$
$$\Rightarrow T = T'$$

Hence proved.

**Corollary 3.5:** *Let p be the prime and* $p \ne 3$, *and for* $\xi \ge 0$, *define the function* $\mathfrak{f}: \mathcal{X}^2 \to [0,\infty)$ *fulfills*

$$\mathcal{D}_{\mathfrak{f}}(x,y) \le \xi, \forall\, x,y \in \mathcal{X}. \tag{3.30}$$

Then, there exists one and only cubic function $T: \mathcal{X} \to \mathcal{Y}$ which ensures that

$$\|\mathfrak{f}(x) - \mathrm{T}(x)\| \le \xi \tag{3.31}$$

***Proof:*** By Theorem 3.4, we have

$$\mathcal{D}_{\mathfrak{f}}(x,y) \le \varphi(x,y) \tag{3.32}$$

Hence we have $\varphi(x,y) = \xi$,

$$\|\mathfrak{f}(x) - T(x)\| \leq \tilde{\varphi}(x)$$
$$\text{where } \tilde{\varphi}(x) = \lim_{n\to\infty} max\left\{|27^{\iota}|\varphi(\tfrac{x}{3^{\iota+1}}, 0): 0 \leq \iota < n\right\}$$
$$\text{Therefore, } \|\mathfrak{f}(x) - T(x)\| \leq \lim_{n\to\infty} max\{|27^{\iota}|\xi: 0 \leq \iota < n\}$$
$$\|\mathfrak{f}(x) - T(x)\| \leq max\{|27|\xi, |27^2|\xi, |27^3|\xi, \ldots\ldots |27^{n-1}|\xi\}$$

(where $p$ is a fixed prime other than 3)

$$\|\mathfrak{f}(x) - \mathrm{T}(x)\| \leq \xi$$

**Corollary 3.6:** Let p be the prime and $p \neq 3$. Let $\alpha, \gamma, \xi > 0$ and $\alpha + \gamma > 0$ and define a function $\mathfrak{f}: \mathcal{X}^2 \to [0, \infty)$ fulfills

$$\mathcal{D}_{\mathfrak{f}}(x, y) \leq \xi(\|x\|^{\alpha+\gamma} + \|y\|^{\alpha+\gamma}), \forall x, y \in \mathcal{X}.$$

Next, there is one and only cubic function $T: \mathcal{X} \to \mathcal{Y}$ which ensures that

$$\|\mathfrak{f}(x) - \mathrm{T}(x)\| \leq \xi\|x\|^{\alpha+\gamma}, \forall x \in \mathcal{X}.$$

***Proof:*** By Theorem (3.4), we have $\varphi(x, y) = \xi(\|x\|^{\alpha+\gamma} + \|y\|^{\alpha+\gamma})$, then

$$\|\mathfrak{f}(x) - \mathrm{T}(x)\| \leq \tilde{\varphi}(x)$$
$$where\ \tilde{\varphi}(x) = \lim_{n\to\infty} max\left\{|27^{\iota}|\varphi\left(\tfrac{x}{3^{\iota+1}}, 0\right): 0 \leq \iota < n\right\}$$
$$\text{Therefore, } \|\mathfrak{f}(x) - \mathrm{T}(x)\| \leq \lim_{n\to\infty} max\left\{|27^{\iota}|\xi \left\|\tfrac{x}{3^{\iota+1}}\right\|^{\alpha+\gamma}: 0 \leq \iota < n\right\}$$
$$\|\mathfrak{f}(x) - T(x)\| \leq max\left\{|27^{\iota}|\xi \frac{1}{|3^{(\iota+1)(\alpha+\gamma)}|}\|x\|^{\alpha+\gamma}: 0 \leq \iota < n\right\}$$
$$\|\mathfrak{f}(x) - T(x)\| \leq \xi\|x\|^{\alpha+\gamma} max\left\{\frac{1}{3^{\alpha+\gamma}}, \frac{|27|}{3^{2(\alpha+\gamma)}}, \frac{|27^2|}{3^{3(\alpha+\gamma)}}, \ldots\ldots \frac{|27^{n-1}|}{3^{n(\alpha+\gamma)}}\right\}$$
$$\|\mathfrak{f}(x) - T(x)\| \leq \xi\|x\|^{\alpha+\gamma} max\{1, |27|, |27^2|, \ldots\ldots |27^{n-1}|\},$$

Where $p$ is a fixed prime other than 3

$$\|\mathfrak{f}(x) - T(x)\| \leq \xi\|x\|^{\alpha+\gamma} \tag{3.33}$$

## 4. Experimental Results

Table 1 highlights the similarities and differences between approximate $\mathfrak{f}(x)$ and exact solution $T(x)$ for equation (1.3) through a graphical examination. Equation (1.3) is precisely satisfied by the cubic function $T(\mathcal{X}) = \mathcal{X}^3$, which provides the exact answer. An alternate cubic function, $\mathfrak{f}(\mathcal{X}) = \mathcal{X}^3 + \varepsilon$, is provided for experimental validation, where $\varepsilon$ stands for a little perturbation or correction factor. Both functions are plotted using Origin software to provide a detailed examination of their graphical trends and to visually compare their behavior. The findings show that the $\mathfrak{f}(\mathcal{X})$ and $T(\mathcal{X})$ curves strongly overlap at several places (see figure 1), suggesting that the approximate function offers a reliable approximation of the precise answer. The closeness of the two graphs indicates that $\mathfrak{f}(\mathcal{X})$ is a good approximation for equation (1.3), despite slight variations caused by the perturbation factor $\varepsilon$. The approximate function's utility in theoretical and practical applications is validated by this graphical representation, which supports the idea that it retains a significant association with the precise solution despite minor deviations.

**Table 1**
**Comparison between Exact and Approximate Solution**

| $\varkappa$ | Exact solution $\top(\varkappa)$ | Approximate Solution $\mathfrak{f}(\varkappa)$ | $\lvert\mathfrak{f}(\varkappa) - \top(\varkappa)\rvert$ |
|---|---|---|---|
| -1 | -1 | -0.99 | 0.01 |
| -0.9 | -0.729 | -0.719 | 0.01 |
| -0.8 | -0.512 | -0.502 | 0.01 |
| -0.7 | -0.343 | -0.333 | 0.01 |
| -0.6 | -0.216 | -0.206 | 0.01 |
| -0.5 | -0.125 | -0.115 | 0.01 |
| -0.4 | -0.064 | -0.054 | 0.01 |
| -0.3 | -0.027 | -0.017 | 0.01 |
| -0.2 | -0.008 | 0.002 | 0.01 |
| -0.1 | -0.001 | 0.009 | 0.01 |
| 0 | 0 | 0.01 | 0.01 |
| 0.1 | 0.001 | 0.011 | 0.01 |
| 0.2 | 0.008 | 0.018 | 0.01 |
| 0.3 | 0.027 | 0.037 | 0.01 |
| 0.4 | 0.064 | 0.074 | 0.01 |
| 0.5 | 0.125 | 0.135 | 0.01 |
| 0.6 | 0.216 | 0.226 | 0.01 |
| 0.7 | 0.343 | 0.353 | 0.01 |
| 0.8 | 0.512 | 0.522 | 0.01 |
| 0.9 | 0.729 | 0.739 | 0.01 |
| 1 | 1 | 1.01 | 0.01 |

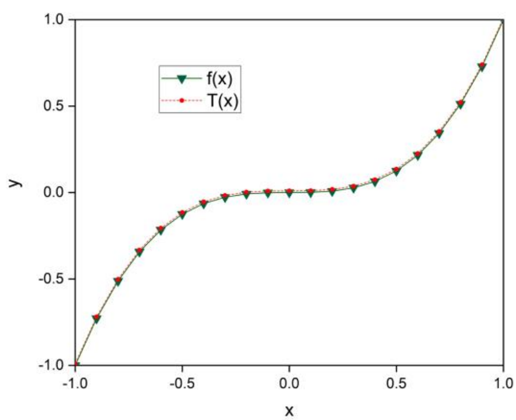


**Figure 1**
**Comparison between Exact and Approximate Solution**

## 5. Conclusion

This paper establishes the Hyers-Ulam stability of the cubic functional equation in Non-Archimedean normed spaces. Our results verify that there is an exact solution that is near the approximate solution for a given function satisfying an approximation version of the cubic functional equation (1.3). This illustrates the basic stability property, which makes sure that minor changes in the functional equation don't cause the exact solution to change significantly.

We examined and provided a graphical depiction of the link between the approximate and exact solutions in Non-Archimedean normed space in order to prove these findings. This graphic highlights the stability of the cubic functional equation under minor perturbations and offers an easy explanation of how functional equation perturbations affect solution behavior.The findings emphasize the structural integrity of functional equations under approximation and enhance our knowledge of stability theory in non-Archimedean contexts.

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