

A novel computational expansion of the fractional kinetic equation with incomplete H-functions

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Abstract

This study introduces a novel and more generalized form of the fractionalized kinetic equation (KE), expressed using the incomplete H -function combined with an exponential function. The significance of the fractional KE, which plays a crucial role in various engineering and scientific applications, is analyzed through the of incomplete special functions, including the incomplete H -function, H -function, Fox-Wright function, and Meijer's G -function. Solutions are efficiently derived using the modified degenerate Laplace transform, which incorporates these incomplete special functions. Additionally, graphical simulations are presented to illustrate the results. The obtained conclusions are broad in scope, providing a framework for generating solutions to a wide range of fractional kinetic equations.

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1. Introduction

To elucidate the memory impacts of intricate systems, it is noteworthy that fractional derivatives introduce a convolution integral featuring an exponential-law and power-law memory kernel, thereby enhancing the significance of fractional differential equations. Furthermore, it is demonstrable that over the recent decades, fractional calculus operators have given rise to remarkably innovative and transformative applications across diverse domains such as chemistry, physics, engineering, biology, finance, and other fields of inquiry [2, 15, 17]. The KE models a system of differential equations that describe the reaction rates governing production and destruction processes, allowing for analyzing changes in a star's chemical composition across different orders.

The generalization and extension of kinetic fractional equations [20], incorporating various fractional operators, have gained significant attention in applied mathematics, physics, engineering, chemistry, heat transfer, biology, control theory, and dynamical systems. This advancement has facilitated the mathematical modeling of diverse physical phenomena, including diffusion in porous media and kinetics in viscoelastic materials [11]. Over the past few decades, kinetic fractional equations have existed in multiple forms and have been widely employed to describe and solve complex problems in astrophysics and physics (see, e.g., [13]). Applying special functions plays a crucial role in solving fractional differential equations. Given their efficiency and practical significance in astrophysical computations, kinetic fractional equations are deeply interconnected with numerous problems across various domains of mathematical physics.

Including incomplete special functions in this newly developed fractional generalization of the KE adds a new dimension to the research. The Riemann-Liouville fractional KE [7] has been formulated using polynomials along with incomplete special functions, such as the H -function, Meijer's G -function, and the Fox-Wright function. The Laplace transformation method is employed to obtain solutions for these fractional KEs. Additionally, several exceptional cases are briefly discussed.

Numerous authors are actively working on the development and wide variety of implications for these incomplete functions, such as in [1, 16]. The authors of [12] investigated applications of the incomplete H -function on the influence of environmental pollution on the occurrence of biological populations.

The earlier study of kinetic equation provides the rate $d\mathcal{K}/dt$ as

$$\frac{d\mathcal{K}}{dt} = -\Lambda + Y,$$

here, $\mathcal{K} = \mathcal{K}(t)$ denotes the reaction rate, $\Lambda = \Lambda(\mathcal{K}_t)$ denotes the destruction rate, and $Y = Y(\mathcal{K}_t)$ signifies the production rate.

Or
$$\frac{d\mathcal{K}}{dt} = -\Lambda(\mathcal{K}_t) + Y(\mathcal{K}_t),$$

moreover, $\mathcal{K}_t$ is stated as $\mathcal{K}_t(t^*) = \mathcal{K}(t - t^*)$ for $t^* > 0$.

Haubold and Mathai [6] proposed a fractional KE, which is expressed as follows (refer to [5]):

$$\frac{d\mathcal{K}}{dt} = -\Lambda(\mathcal{K}_t) + \Upsilon(\mathcal{K}_t), \quad (1)$$

we also derived specific cases of (1) under the condition where the homogeneity or spatial variation of $\mathcal{K}(t)$ is neglected. The resulting equation is given as follows:

$$\frac{d\mathcal{K}_k}{dt} = -\varepsilon_k \mathcal{K}_k(t), \quad (2)$$

here, $\mathcal{K}_k(t=0) = \mathcal{K}_0$ represents the initial density of species k at $t=0$, with $\varepsilon_k > 0$ denoting time. By disregarding the index k and integrating Eq. (2), we obtain:

$$\mathcal{K}(t) - \mathcal{K}_0 = -m {}_0\mathcal{D}_t^{-1} \mathcal{K}(t),$$

here, ${}_0\mathcal{D}_t^{-1}$ represents a special case of the Riemann-Liouville (R-L) fractional integral. The R-L operator ${}_0\mathcal{D}_t^{-\mu}$ is defined as:

$${}_0\mathcal{D}_t^{-1} \ell(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{\ell(\varphi)}{(t-\varphi)^{1-\mu}} d\varphi, \quad \Re(\mu), t > 0.$$

Furthermore, Saxena and Kalla [14] proposed the following fractional KE:

$$\mathcal{K}(t) - \mathcal{K}_0 \ell(t) = -m {}_0\mathcal{D}_t^{-\mu} \mathcal{K}(t),$$

here, $\ell(t) \in \mathfrak{L}(0, \infty)$, $\mathcal{K}(t)$ represents the population density of the species at t time, and $\mathcal{K}_0 = \mathcal{K}(0)$ denotes the initial population density at $t=0$.

In this article, we employ the modified degenerate Laplace transform (MDLT) to solve the fractionalized KE in the R-L sense. This approach integrates the incomplete H -function, H -function, Fox-Wright function, and Meijer's G -function, following the framework outlined in [4, 19].

2. Preliminaries

The following definitions represent the incomplete gamma function $\gamma(a, \mathfrak{S})$ and $\Gamma(a, \mathfrak{S})$ as described in [3]:

$$\gamma(a, \mathfrak{S}) = \int_0^{\mathfrak{S}} \vartheta^{a-1} e^{-\vartheta} d\vartheta, \quad \mathfrak{S} \geq 0, \quad \Re(a) > 0,$$

and

$$\Gamma(a, \mathfrak{S}) = \int_{\mathfrak{S}}^{\infty} e^{-\vartheta} \vartheta^{a-1} d\vartheta, \quad \mathfrak{S} \geq 0, \quad \Re(a) > 0.$$

These functions satisfy the following decomposition rule:

$$\gamma(a, \mathfrak{S}) + \Gamma(a, \mathfrak{S}) = \Gamma(a), \quad \Re(a) > 0,$$

where $\Re(a)$ denotes the real part of the parameter a . Furthermore, by setting $\mathfrak{S} = 0$, we obtain $\Gamma(a, 0) = \Gamma(a)$.

The incomplete H -function

The incomplete H -functions $\gamma_{r,s}^{u,v}$ and $\Gamma_{r,s}^{u,v}$ have studied and defined by Srivastava, Saxena, and Parmar, [19] in the form of Mellin-Barnes contour integral as follow:

$$\Gamma_{r,s}^{u,v}(z) = \Gamma_{r,s}^{u,v} \left[z \left| \begin{matrix} (g_1, \nu_1, y), (g_j, \nu_j)_{2,r} \\ (b_j, \omega_j)_{1,s} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{E}} G(\vartheta, y) z^{-\vartheta} d\vartheta, \quad (3)$$

and

$$\gamma_{r,s}^{u,v}(z) = \gamma_{r,s}^{u,v} \left[z \left| \begin{matrix} (g_1, v_1, y), (g_i, v_i)_{2,r} \\ (b_j, \omega_j)_{1,s} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_E g(\vartheta, y) z^{-\vartheta} d\vartheta, \quad (4)$$

where,

$$G(\vartheta, y) = \frac{\Gamma(1-g_1-v_1\vartheta:y) \prod_{j=2}^V \Gamma(1-g_j-v_j\vartheta) \prod_{j=1}^U \Gamma(b_j+\omega_j\vartheta)}{\prod_{j=u+1}^S \Gamma(1-b_j-\omega_j\vartheta) \prod_{j=v+1}^F \Gamma(g_j+v_j\vartheta)},$$

and

$$g(\vartheta, y) = \frac{\gamma(1-g_1-v_1\vartheta:y) \prod_{j=1}^U \Gamma(b_j+\omega_j\vartheta) \prod_{j=2}^V \Gamma(1-g_j-v_j\vartheta)}{\prod_{j=v+1}^F \Gamma(g_j+v_j\vartheta) \prod_{j=u+1}^S \Gamma(1-b_j-\omega_j\vartheta)}.$$

This family of incomplete H -functions characterized as (3) and (4) exist for $z \geq 0$, according to the conditions specified by Srivastava, Saxena, and Parmar, [19]. The incomplete H -functions satisfy the following relation, commonly referred to as the decomposition formula:

$$\gamma_{r,s}^{u,v}(z) + \Gamma_{r,s}^{u,v}(z) = H_{r,s}^{u,v}(z).$$

Laplace transform

The fundamental definition of the Laplace transform (LT) for a piecewise continuous function $\mathfrak{f}(t)$, defined for $t \geq 0$, is given as follows:

$$\mathcal{L}[\mathfrak{f}(t):p] = \int_0^\infty e^{-pt} \mathfrak{f}(t) dt.$$

Let $\mathfrak{f}(t)$ be a function defined for $t \geq 0$. The integral

$$\mathcal{L}_\lambda[\mathfrak{f}(t):p] = \int_0^\infty (1+\lambda t)^{-\frac{p}{\lambda}} \mathfrak{f}(t) dt, \quad \lambda \in (0, \infty),$$

is referred to as the degenerate LT [8] of $\mathfrak{f}$, provided the integral converges.

Let $\mathfrak{f}(t)$ be a function defined for $t \geq 0$. The integral

$$\mathcal{ML}_\lambda[\mathfrak{f}(t):p] = \int_0^\infty (1+\lambda p)^{-\frac{t}{\lambda}} \mathfrak{f}(t) dt, \quad \lambda \in (0, \infty),$$

is referred to as the MDLT [9] of $\mathfrak{f}$, provided the integral converges. It is also defined by $\mathcal{ML}_\lambda[\mathfrak{f}(t):p] = \overline{F}(p)$.

Modified degenerate Laplace transform convolution

Numerous physical applications involve the convolution of the functions $\mathfrak{f}(t)$ and $g(t)$, defined for $t > 0$. The integral below represents the modified degenerate Laplace convolution of these functions:

$$\mathcal{ML}_\lambda\{\mathfrak{f}(t) * g(t):p\} = \mathcal{ML}_\lambda\{\mathfrak{f}(t);p\} \mathcal{ML}_\lambda\{g(t);p\}, \quad \Re(p) > 0, \quad \lambda \in (0,1].$$

The modified degenerate Laplace transform for R-L fractional integral:

$$\mathcal{ML}_\lambda[\mathcal{D}_0^{-\mu} g(t):p] = \left(\frac{\lambda}{\log(1+\lambda p)}\right)^\mu \overline{G}(p),$$

by using the identity

$$\mathcal{ML}_\lambda\{t^u:p\} = \left(\frac{\lambda}{\log(1+\lambda p)}\right)^{u+1} \Gamma(1+u), \quad (u \in \mathbb{R}, p > 0). \quad (5)$$

$$\Leftarrow \mathcal{ML}_\lambda^{-1} \left(\frac{1}{\log(1+\lambda p)} \right)^{u+1} = \frac{t^u}{\Gamma(1+u)\lambda^{u+1}}, \quad (6)$$

in which $\mathcal{ML}_\lambda^{-1}$ considered as the inverse MDLT.

3. Solution of fractionalized kinetic equations

Theorem 3.1: For all $\varphi, \rho, \beta, \varpi, \alpha, c > 0$, then the solution of

$$\mathcal{K}(t) - \mathcal{K}_0 t^{\varphi-1} \exp(\alpha t^\rho) \Gamma_{r,s}^{u,v}[\beta t^\varpi] = -c^\mu \mathcal{D}_0^{-\mu} \mathcal{K}(t) \quad (7)$$

is provided as

$$\begin{aligned} \mathcal{K}(t) = & \mathcal{K}_0 t^{\varphi-1} \sum_{k=0}^{\infty} (-ct^\mu)^k \sum_{j=0}^{\infty} \frac{(\alpha t^\rho)^j}{\Gamma(1+j)} \\ & \times \Gamma_{r+1,s+1}^{u,v+1} \left[\beta t^\varphi \left| \begin{matrix} (\mathfrak{g}_1, \nu_1; y), (1-\varphi-\rho j, \varpi), (\mathfrak{g}_j, \nu_j)_{2,r} \\ (\mathfrak{h}_j, \omega_j)_{1,s}, (1-\varphi-\rho j-\mu k, \varpi) \end{matrix} \right. \right]. \end{aligned} \quad (8)$$

Proof: Applying MDLT both side of Equation (7), we obtain:

$$\mathcal{ML}_\lambda[\mathcal{K}(t):p] + \mathcal{ML}_\lambda[c^\mu \mathcal{D}_0^{-\mu} \mathcal{K}(t):p] = \mathcal{ML}_\lambda[\mathcal{K}_0 t^{\varphi-1} \exp(\alpha t^\rho) \Gamma_{r,s}^{u,v}[\beta t^\varpi]:p]. \quad (9)$$

Use Equation (5) in Equation (9) and replace incomplete H -function by Equation (3), we get:

$$\begin{aligned} \overline{\mathcal{K}}(p) \left(1 + \left(\frac{c\lambda}{\log(1+\lambda p)} \right)^\mu \right) = & \mathcal{K}_0 \sum_{j=0}^{\infty} \frac{(\alpha t)^\rho}{\Gamma(1+j)} \\ & \frac{1}{2\pi i} \int_E G(\vartheta, y) (\beta)^{-\vartheta} \Gamma(\rho j - \varpi \vartheta + \varphi) \left(\frac{\lambda}{\log(1+\lambda p)} \right)^{\rho j - \varpi \vartheta + \varphi} d\vartheta. \end{aligned} \quad (10)$$

After adjusting the terms and applying $(1+\ell)^{-1} = \sum_{j=0}^{\infty} (-1)^j \ell^j$ in Equation (10), we can express it as follows:

$$\begin{aligned} \overline{\mathcal{K}}(p) = & \mathcal{K}_0 \sum_{j=0}^{\infty} \frac{(\alpha t)^\rho}{\Gamma(1+j)} \frac{1}{2\pi i} \int_E G(\vartheta, y) (\beta)^{-\vartheta} \Gamma(\rho j - \varpi \vartheta + \varphi) \\ & \sum_{k=0}^{\infty} (-ct^\mu)^k \left(\frac{\lambda}{\log(1+\lambda p)} \right)^{\rho j - \varpi \vartheta + \varphi + \mu k} d\vartheta. \end{aligned} \quad (11)$$

Applying the inverse MDLT to both sides of Equation (11) and using Equation (6), we obtain:

$$\begin{aligned} \mathcal{K}(t) = & \mathcal{K}_0 t^{\varphi-1} \sum_{k=0}^{\infty} (-ct^\mu)^k \sum_{j=0}^{\infty} \frac{(\alpha t)^\rho}{\Gamma(1+j)} \\ & \frac{1}{2\pi i} \int_E G(\vartheta, y) (\beta)^{-\vartheta} t^{-\varpi \vartheta} \frac{\Gamma(\rho j - \varpi \vartheta + \varphi)}{\Gamma(\rho j - \varpi \vartheta + \varphi + \mu k)} d\vartheta. \end{aligned}$$

After adjusting the terms, we obtain the desired results.

Theorem 3.2: For all $\varphi, \alpha, \rho, \beta, \varpi, c > 0$, then the solution of

$$\mathcal{K}(t) - \mathcal{K}_0 t^{\varphi-1} \exp(\alpha t^\rho) \gamma_{r,s}^{u,v}[\beta t^\varpi] = -c^\mu \mathcal{D}_0^{-\mu} \mathcal{K}(t) \quad (12)$$

is provided as

$$\begin{aligned} \mathcal{K}(t) &= \mathcal{K}_0 t^{\varphi-1} \sum_{k=0}^{\infty} (-ct^{\mu})^k \sum_{j=0}^{\infty} \frac{(\alpha t^{\rho})^r}{\Gamma(1+j)} \\ &\quad \times \gamma_{r+1,s+1}^{u,v+1} \left[\beta t^{\varphi} \left| \begin{matrix} (g_1, v_1: y), (1 - \varphi - \rho j, \varpi), (g_j, v_j)_{2,r} \\ (h_j, \omega_j)_{1,s}, (1 - \rho j - \varphi - \mu k, \varpi) \end{matrix} \right. \right]. \end{aligned}$$

Proof: The proof follows directly from the definitions in (4) and is similar to Theorem 3.1. Therefore, we omit the proof.

- (i) **H-function:** We taking $y = 0$ in (3), then it becomes to the H -function [18]:

$$\Gamma_{r,s}^{u,v} \left[z \left| \begin{matrix} (g_1, v_1: 0), (g_j, v_j)_{2,r} \\ (h_j, \omega_j)_{1,s} \end{matrix} \right. \right] = H_{r,s}^{u,v} \left[z \left| \begin{matrix} (g_j, v_j)_{1,r} \\ (h_j, \omega_j)_{1,s} \end{matrix} \right. \right].$$

Corollary 3.3: For all $\rho, \beta, \varphi, \alpha, \varpi, c > 0$, then the solution of

$$\mathcal{K}(t) - \mathcal{K}_0 t^{\varphi-1} \exp(\alpha t^{\rho}) H_{r,s}^{u,v} [\beta t^{\varpi}] = -c^{\mu} \mathcal{D}_0^{-\mu} \mathcal{K}(t) \quad (13)$$

is provided as

$$\begin{aligned} \mathcal{K}(t) &= \mathcal{K}_0 t^{\varphi-1} \sum_{k=0}^{\infty} (-ct^{\mu})^k \sum_{j=0}^{\infty} \frac{(\alpha t^{\rho})^r}{\Gamma(1+j)} \\ &\quad \times H_{r+1,s+1}^{u,v+1} \left[\beta t^{\varphi} \left| \begin{matrix} (1 - \varphi - \rho j, \varpi), (g_j, v_j)_{1,r} \\ (h_j, \omega_j)_{1,s}, (1 - \varphi - \rho j - \mu k, \varpi) \end{matrix} \right. \right]. \end{aligned}$$

- (ii) Incomplete Meijer G -function: Next, we taking $v_j = 1$ and $\omega_j = 1$ in (3), then it becomes to the G -function suggested by Meijer [10]:

$$\Gamma_{r,s}^{u,v} \left[z \left| \begin{matrix} (g_1, 1: y), (g_j, 1)_{2,r} \\ (h_j, 1)_{1,s} \end{matrix} \right. \right] = {}^{\Gamma} G_{r,s}^{u,v} \left[z \left| \begin{matrix} (g_1: y) (g_j)_{2,r} \\ (h_j)_{1,s} \end{matrix} \right. \right].$$

Corollary 3.4: For all $\varphi, \alpha, \rho, \beta, \varpi, c > 0$, then the solution of

$$\mathcal{K}(t) - \mathcal{K}_0 t^{\varphi-1} \exp(\alpha t^{\rho}) {}^{\Gamma} G_{r,s}^{u,v} [\beta t^{\varpi}] = -c^{\mu} \mathcal{D}_0^{-\mu} \mathcal{K}(t) \quad (14)$$

is provided as

$$\begin{aligned} \mathcal{K}(t) &= \mathcal{K}_0 t^{\varphi-1} \sum_{k=0}^{\infty} (-ct^{\mu})^k \sum_{j=0}^{\infty} \frac{(\alpha t^{\rho})^r}{\Gamma(1+j)} \\ &\quad \times {}^{\Gamma} G_{r+1,s+1}^{u,v+1} \left[\beta t^{\varphi} \left| \begin{matrix} (g_1: y), (1 - \varphi - \rho j, \varpi), (g_j)_{2,r} \\ (h_j)_{1,s}, (1 - \varphi - \rho j - \mu k, \varpi) \end{matrix} \right. \right]. \end{aligned}$$

- (iii) Incomplete Fox-Wright function: The incomplete type functions, such as H -functions and Fox-Wright functions, are related through the following relations [19].

If we make the substitution $z = -z, u = 1, v = r, s = s + 1, g_j \rightarrow (1 - g_j), h_j \rightarrow (1 - h_j), j = 1, 2, \dots, s$, and multiply by $\Gamma(\vartheta)$.

$$\Gamma_{r,s+1}^{1,r} \left[-z \left| \begin{matrix} (1 - g_1, v_1: y), (1 - g_j, v_j)_{2,r} \\ (0, 1), (1 - h_j, \omega_j)_{1,s} \end{matrix} \right. \right] = {}_r \Psi_s^{(\Gamma)} \left[z \left| \begin{matrix} (g_1, v_1: y), (g_j, v_j)_{2,r} \\ (h_j, \omega_j)_{1,s} \end{matrix} \right. \right],$$

and

$$\gamma_{r,s+1}^{1,r} \left[-z \left| \begin{matrix} (1 - g_1, v_1: y), (1 - g_j, v_j)_{2,r} \\ (0,1), (1 - h_j, \omega_j)_{1,s} \end{matrix} \right| \right] = {}_r \Psi_s^{(\gamma)} \left[z \left| \begin{matrix} (g_1, v_1: y), (g_j, v_j)_{2,r} \\ (h_j, \omega_j)_{1,s} \end{matrix} \right| \right].$$

Corollary 3.5: For all $\varphi, \alpha, \rho, \beta, \varpi, c > 0$, then the solution of

$$\mathcal{K}(t) - \mathcal{K}_0 t^{\varphi-1} \exp(\alpha t^\rho) {}_r \Psi_s^{(\Gamma)}[\beta t^\varpi] = -c^\mu \mathcal{D}_0^{-\mu} \mathcal{K}(t) \quad (15)$$

is provided as

$$\begin{aligned} \mathcal{K}(t) = & \mathcal{K}_0 t^{\varphi-1} \sum_{k=0}^{\infty} (-ct^\mu)^k \sum_{j=0}^{\infty} \frac{(\alpha t^\rho)^j}{\Gamma(1+j)} \\ & \times {}_{r+1} \Psi_{s+1}^{(\Gamma)} \left[\beta t^\varphi \left| \begin{matrix} (g_1, v_1: y), (\varphi + \rho j, \varpi), (g_j, v_j)_{2,r} \\ (h_j, \omega_j)_{1,s}, (\varphi + \rho j + \mu k, \varpi) \end{matrix} \right| \right]. \end{aligned}$$

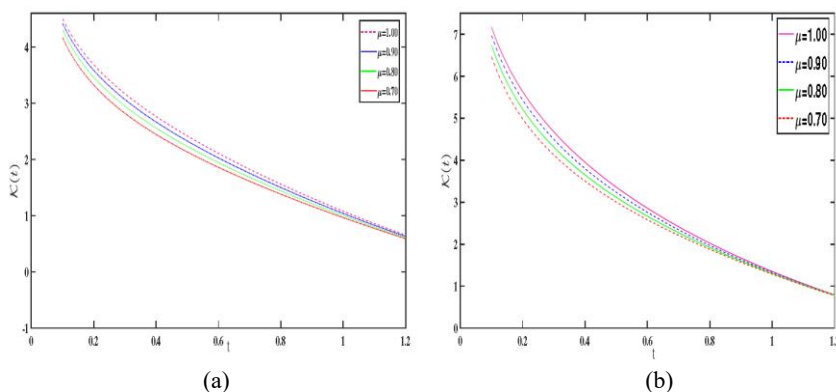


Figure 1

The plot between t and $\mathcal{K}(t)$ for various μ values, when $\varphi = 0.8, \alpha = 1$, (a) $\mathcal{K}_0 = 3, c = 0.5$, and (b) $\mathcal{K}_0 = 5, c = 1$.

4. Applications and numerical simulation

In this section, we explore the implications and applications of the aforementioned discoveries, delving into a detailed analysis. We employ MATLAB for a visual examination, simulating the numerical outcomes. Focusing on a specific exponential function can yield distinctive instances of new findings. To illustrate this point, consider the following example:

Example 4.1: The solution of the problem

$$\mathcal{K}(t) - \mathcal{K}_0 t^{\varphi-1} \Gamma_{r,s}^{u,v}[\beta t^\varpi] = -c^\mu \mathcal{D}_0^{-\mu} \mathcal{K}(t). \quad (16)$$

Solution: If we set $\alpha = 0$ in Equation (7), then solution of the Eq. (16) is

$$\begin{aligned} \mathcal{K}(t) = & \mathcal{K}_0 t^{\varphi-1} \sum_{k=0}^{\infty} (-ct^\mu)^k \\ & \times \Gamma_{1+r,1+s}^{u,1+v} \left[\beta t^\varphi \left| \begin{matrix} (g_1, v_1: y), (1 - \varphi, \varpi), (g_j, v_j)_{2,r} \\ (h_j, \omega_j)_{1,s}, (1 - \varphi - \mu k, \varpi) \end{matrix} \right| \right]. \end{aligned}$$

Example 4.2: The solution of the problem

$$\mathcal{K}(t) - \mathcal{K}_0 t^{\varphi-1} \cos(t) \Gamma_{r,s}^{u,v}[\beta t^{\varpi}] = -c^{\mu} \mathcal{D}_0^{-\mu} \mathcal{K}(t). \quad (17)$$

Solution: If we set $\rho = 1$ in Equation (7), and take the real part of $\exp(\alpha it)$, then solution of the Eq. (17) is

$$\begin{aligned} \mathcal{K}(t) = & \mathcal{K}_0 t^{\varphi-1} \sum_{k=0}^{\infty} (-ct^{\mu})^k \sum_{j=0}^{\infty} \frac{(-1)^r (\alpha t)^{2r}}{\Gamma(1+2r)} \\ & \times \Gamma_{r+1,s+1}^{u,v+1} \left[\beta t^{\varphi} \left| (g_1, v_1; \gamma), (1-\varphi-2r, \varpi), (g_j, v_j)_{2,r} \right| \right. \\ & \left. (h_j, \omega_j)_{1,s}, (1-\varphi-2r-\mu k, \varpi) \right]. \end{aligned}$$

Similar to the above example, we can obtain assertions following Theorem 3.2, Corollary 3.3-3.5, respectively.

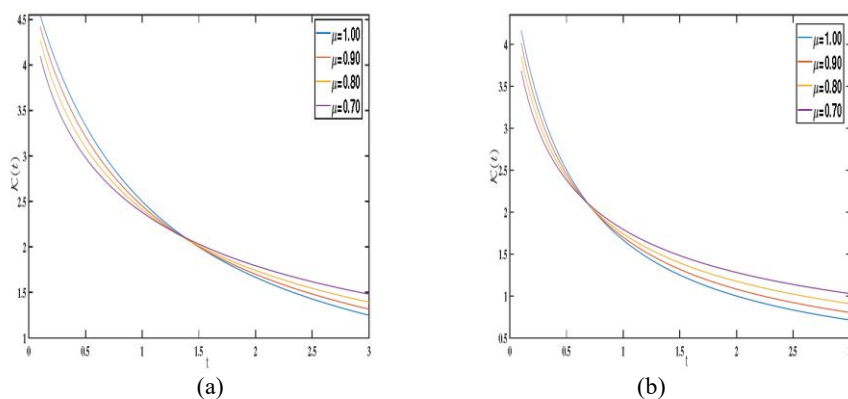


Figure 2

The plot between t and $\mathcal{K}(t)$ for various μ values, when $\mathcal{K}_0 = 5, \varphi = 1, \alpha = 0$, (a) $c = 1$, and (b) $c = 2$.

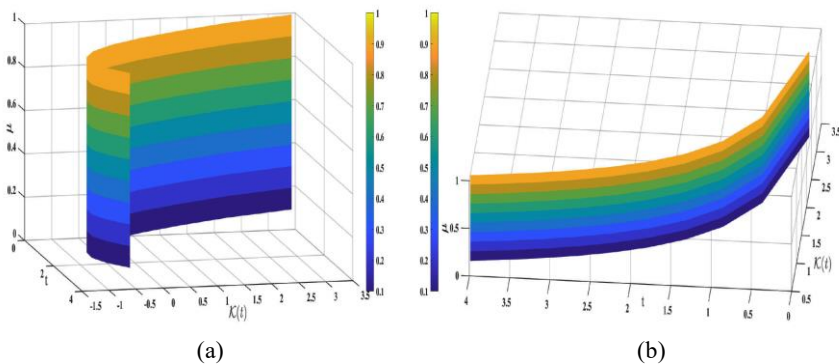


Figure 3

The solution surface between $t, \mathcal{K}(t), \mu$, when $\mathcal{K}_0 = 4, \varphi = 1$, (a) $c = 1, \alpha = 1$, and (b) $c = 2, \alpha = 0$.

Using MATLAB, we simulate numerical results for fractional KE (7) at various parameter values, as shown in Figures 1 to 3. Particular emphasis is given to the exponential function in MATLAB simulations to obtain impressive graphical results. The impact of time and the order of the R-L operator of arbitrary order on the reaction rate is shown in Figures 1 to 2. Figures 1 to 3 demonstrate that the reaction rate decreases as time increases and is influenced by the value of the fractional parameter constant. The graphical results indicate that the convergence region of the solution depends on the parameter μ . As a result, similar observations can be made for the behavior of solutions (12), (14)-(15), and (16)-(17) for different parameters and time intervals.

5. Conclusion

This study aims to introduce an innovative fractional extension of the traditional kinetic equation and explore its solution through the application of the integral transformation method. The investigation involves examining a range of functions, such as the incomplete H -function, H -function, incomplete Fox-Wright function, Meijer's G -function, and the exponential function. Additionally, various new fractional KEs are explored, along with their corresponding solutions. The main conclusions are general, as presented in Theorems 3.1, 3.2, and their corollaries.

These key findings offer a foundation for generating diverse KE and their potential solutions, allowing for the customization of arbitrary sequences and constraints with appropriate fundamental values. Subsequent research will extend this exploration to more generalized Kinetic Equations and explore additional suggested solutions.

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