

A new stochastic integral created for air pollution

Tayebe Waezizadeh *

Mohammad Reza Molaei †

Department of Pure Mathematics

Faculty of Mathematics and Computer

Shahid Bahonar University of Kerman & Afzalipour

Research Institute- Mahani Math Center

Kerman 7616914111

Iran

Abstract

In this paper, using an order creator for integral, a new random variable is introduced. Also, some basic properties are investigated. As follow, a new integral that is a random process is introduced. This new random process shows the amount of air pollution caused by airborne particles in an area. By studying the properties of the random process, some results are obtained in relation to the amount of air pollution.

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1. Introduction

Air pollution is one of the biggest problems in many countries [1], [7] and [9]. This problem effects on economics, health and etc. At the same time, several factors, such as cars, factories and etc. effect on the concentration of pollutants. Measuring of the concentration of substances is difficult and so expensive, because it needs special equipments for ecological monitoring systems [4], [13]and [8]. One of the most important methods which describes the rate of air pollution in an area is mathematical modeling [10]. On the other hand, the movement of airborne particles is random. Therefore, an stochastic mathematical model describes the behaviour of particles suspended in the air [2].

* ORCID-ID: 0000-0003-4382-2162

† ORCID-ID: 0000-0003-4808-8382

* E-mail: waezizadeh@uk.ac.ir (Corresponding Author)

† E-mail: mrmolaei@uk.ac.ir

In this paper, the desired area is considered as a probability space. Also, the rate of pollution produced per unit time is considered a random variable. To obtain, the a mount of total air pollution, stochastic integral is used.

The stochastic integral has been considered first by Itô in 1944 [5], and he applied it in mathematical finance, then Stratonovich in 1966 [12] introduced another type of stochastic integral with applications in physics. In this paper, a new type of stochastic integral via the maps change the order is introduced that is based on "order creators for integral" which is defined by Karami, Molaei and Asgari [6]. To define, we assume $(\Omega, \mathbb{F}, P)$ is a probability triple, and M is the collection of random variables from Ω to $\mathbb{R}^n$.

The "order creator for integral" [6] is a mapping $S: \Omega \times M \rightarrow \mathbb{R}$ with these conditions

- (i) If $\chi, \eta \in M$ and $\chi(p) = \eta(p)$, $S(p, \chi) = S(p, \eta)$ for all $p \in \Omega$.
- (ii) $S(p, -\chi) = -S(p, \chi)$ for each $(p, \chi) \in \Omega \times M$.
- (iii) If any component of $\chi(p)$ and $\eta(p)$ satisfies in $\chi(p)(i) \leq \eta(p)(i)$, $S(p, \chi) \leq S(p, \eta)$ for all $p \in \Omega$.

To illustrate S we assume χ is the velocity vector of the air flow generated by an air pollution source. For a point p in the region $\Omega \subseteq \mathbb{R}^3$, we take $S(p, \chi)$ as the density of pollution of the air at point p .

By a mapping S we can define the following two partially orders on Ω and M . For $\chi, \eta \in M$, the inequality $\chi \geq \eta$ is defined by $S(p, \chi) \geq S(p, \eta)$ for all $p \in \Omega$. For $p, q \in \Omega$, the inequality $p \geq q$ is defined by $S(p, \chi) \geq S(q, \chi)$ for all $\chi \in M$.

In the next section, using this notion, a new concept of stochastic integral is introduced. Also, the stochastic integral properties are investigated. As follow, this resulted integral denotes the air pollution in a region created by a flow on it in section 3. In the last section, some properties of mathematical expectation in this sense are investigated.

2. Integral

In this section, an order creator for integrals, S , is considered. Using of S , an order on M is defined.

Let $\chi \in M$, define χ^+ and χ^- by:

$$\chi^+(p) = \begin{cases} \chi(p) & \text{if } S(p, \chi) \geq S(p, 0) \\ 0 & \text{if } S(p, 0) > S(p, \chi) \end{cases}$$

$$\chi^-(p) = \begin{cases} -\chi(p) & \text{if } S(p, \chi) \leq S(p, 0) \\ 0 & \text{if } S(p, 0) < S(p, \chi) \end{cases}$$

for all $p \in \Omega$.

Theorem 1: χ^+ and χ^- are stochastic variables on Ω and $\chi = \chi^+ - \chi^-$.

Proof: First we prove, χ^+ is a measurable map. For a given open subset U of $\mathbb{R}^n$, $\chi^{+^{-1}}(U)$ is the set $\{p \in \Omega: \chi^+(p) \in U\}$. If $0 \in U$, then

$$\chi^{+^{-1}}(U) = \chi^{+^{-1}}\{0\} \cup \chi^{+^{-1}}(U - \{0\}).$$

Where, $\chi^{+^{-1}}\{0\} = \{p \in \Omega: \chi(p)(i) < 0, \quad i = 1, 2, \dots, n\}$. Since, χ is a measurable function so, $\chi^{+^{-1}}\{0\}$ and $\chi^{+^{-1}}(U - \{0\})$ are measurable. If $0 \in U^c$, then $\chi^{+^{-1}}(U) = \chi^{-1}(U)$. Therefore, χ^+ is a stochastic variable.

The same deduction implies that χ^- is measurable. The equality $\chi = \chi^+ - \chi^-$ follows from the definitions of χ^+ and χ^- .

Definition 1: For $\chi, \eta \in M$ with $\chi(q) = \{\chi(q)(i)\}_{i=1}^n$ and $\eta(q) = \{\eta(q)(i)\}_{i=1}^n$, the variable $\chi \vee \eta$ from Ω to $\mathbb{R}^n$ is defined by $\chi \vee \eta(q) = \{\chi \vee \eta(q)(i)\}_{i=1}^n$. Where the i component of $(\chi \vee \eta)(q)$ is the maximum of the set $\{\chi(q)(i), \eta(q)(i)\}$.

Theorem 2: If $\chi, \eta \in M$, $\chi \vee \eta$ is a stochastic variable.

Proof: Let $\chi, \eta: \Omega \rightarrow \mathbb{R}$ be two one-dimensional stochastic variables. Let $U \subseteq \mathbb{R}$ be a Borel set. Take Borel sets U_1 and U_2 such that $U = U_1 \cup U_2$ with $\chi(p) > \eta(p)$ for all $p \in \chi^{-1}(U_1)$ and $\chi(q) < \eta(q)$ for all $q \in \eta^{-1}(U_2)$. Thus $(\chi \vee \eta)^{-1}(U) = \chi^{-1}(U_1) \cup \eta^{-1}(U_2)$ and $\chi \vee \eta$ is measurable.

Now consider $\chi, \eta: \Omega \rightarrow \mathbb{R}^n$ are n -dimensional stochastic variables. Considering that all components of the map $\chi \vee \eta$ are one-dimensional stochastic variable, using the above theorems, it is measurable, so $(\chi \vee \eta)(p)$ is a stochastic variable.

The probability space $(\Omega, \mathbb{F}, P)$ is constructed by taking $\Omega = \bigcup_{j=1}^k A_j$, with $A_j \in \mathbb{F}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$. If $C_j: \mathbb{F} \rightarrow \mathbb{R}^n$ be a constant map with $j \in \{1, 2, \dots, k\}$. Then the stochastic variable φ of the form $\varphi = \sum_{j=1}^k C_j \chi_{A_j}$ is a simple stochastic variable on Ω . In this situation, we denote the stochastic integral of φ with respect to the probability measure P by $\int_{\Omega}^S \varphi dP$ and we define it by $\int_{\Omega}^S \varphi dP = \sum_{j=1}^k S(\cdot, C_j)P(A_j)$. We see that $\int_{\Omega}^S \varphi dP: \Omega \rightarrow \mathbb{R}$ is a mapping which is defined by $(\int_{\Omega}^S \varphi dP)(q) = \sum_{j=1}^k S(q, C_j)P(A_j)$.

If $S(\cdot, C_j)$ is a random variable, then $\sum_{j=1}^k S(\cdot, C_j)P(A_j)$ is a finite summation of measurable functions. Thus $\int_{\Omega}^S \varphi dP$ is measurable, and this means that $\int_{\Omega}^S \varphi dP: \Omega \rightarrow \mathbb{R}$ is a stochastic variable.

Definition 1 If $\chi \in M$, then we define the integral of χ with measure P by

$$\int_{\Omega}^S \chi dP = \int_{\Omega}^S \chi^+ dP - \int_{\Omega}^S \chi^- dP,$$

where

$$\int_{\Omega}^S \chi^{\pm} dP = \bigvee_{0 \leq \varphi \leq \chi^{\pm}, \varphi \text{ simple}} \int_{\Omega}^S \varphi dP.$$

Now we show that this new concept of integral is a kind of stochastic integral.

Theorem 3: $\int_{\Omega}^s \chi dP$ is a stochastic variable.

Proof: The proof is obtained easily by lemma 2.6 in [3].

3. Air pollution

Let $X(t)(q)$ be the rate of pollution flow at point q produced by a source of air pollution at time t . $\{X(t)\}_{t \in T}$ is a random process. The map $S(q, X_t)$ is Density of airborne particles in q at time t . Also, $\int_{\Omega}^s X(t)dP$ is the total air pollution at time t , which is a random process.

In this case, Ω is considered the study area and the measure of this area is one, $P(\Omega) = 1$. Also, we assume that $\mathbb{F}$ is power set of Ω . Thus, $(\Omega, \mathbb{F}, P)$ is a probability space.

Theorem 4: Let $X: \Omega \rightarrow \mathbb{R}^n$ be a Wiener process. Assume $S(\cdot, X) = u(X, t)$ where u is a continuous map. Also, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$ and that $\frac{\partial^2 u}{\partial x_i \partial x_j}$ for all $i, j = 1, 2, \dots, n$ exist and are continuous. Then

$$S(\cdot, X_r) = S(\cdot, X_s) + \sum_{i=1}^n \int_s^r \frac{\partial u}{\partial x_i} dX_i + \frac{1}{2} \sum_{i=1}^n \int_s^r \frac{\partial^2 u}{\partial x_i^2} dt. \quad 0 \leq s \leq r$$

Proof: $X(\cdot) = (X_1(\cdot), X_2(\cdot), \dots, X_n(\cdot))$ is a Wiener process so, assume $Y(t) = u(X, t)$. By Itô formula

$$dY = \frac{\partial u}{\partial t} dt + \sum_{i=1}^n \frac{\partial u}{\partial x_i} dX_i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} dt.$$

Hence,

$$Y(r) - Y(s) = \sum_{i=1}^n \int_s^r \frac{\partial u}{\partial x_i} dX_i + \frac{1}{2} \sum_{i=1}^n \int_s^r \frac{\partial^2 u}{\partial x_i^2} dt.$$

The proof is complete.

According to the findings of the preceding theorem, the density of air pollution at $p \in \Omega$ at time r , $S(p, X_r)$, is equal to the sum of the density of air pollution at $p \in \Omega$ at time s with $\sum_{i=1}^n \int_s^r \frac{\partial u}{\partial x_i} dX_i + \frac{1}{2} \sum_{i=1}^n \int_s^r \frac{\partial^2 u}{\partial x_i^2} dt$.

Example 1: Suppose $\Omega \subseteq \mathbb{R}^3$ and the source of air pollution is at point q_0 . Define

$$S(q, X_t) = \frac{1}{d(q, q_0)+1} g(t) X(t).$$

Where, $d(q, q_0)$ is Euclidean distance between q and q_0 . The function g , is the effect of weather conditions such as wind, rainfall, etc. on the concentration of pollutants.

If $X(t)$ is a simple stochastic variable, $X(t) = \sum_{j=1}^n C_j \chi_{A_j}$.

$$\int_{\Omega}^s X(t)dP = \sum_{j=1}^n S(\cdot, C_j)P(A_j) = \frac{1}{d(q, q_0)+1} g(t) \sum_{j=1}^n C_j P(A_j)$$

$$= \frac{1}{d(q, q_0)+1} g(t) \int X(t) dP.$$

Thus, if $X(t)$ is a positive stochastic variable, then

$$\begin{aligned} \int_{\Omega}^S X(t) dP &= \int_{0 \leq \varphi \leq X^{\pm}, \varphi \text{ simple}} \int_{\Omega}^S \varphi dP \\ &= \frac{1}{d(q, q_0)+1} g(t) \left(\int_{0 \leq \varphi \leq X^{\pm}, \varphi \text{ simple}} \int_{\Omega}^S \varphi dP \right) = \frac{1}{d(q, q_0)+1} g(t) \int X(t) dP. \end{aligned}$$

The following example shows that, if there are two air pollution sources. The total pollution which is produced by these two sources is not equal the sum of air pollution that is generated by each one. The equality is established if the condition of the second part of the next theorem is established.

Example 2: Let $\Omega = [0, 1]$ and X, Y be two one-dimensional random variables defined by $X = \frac{1}{2}\chi_{[0, \frac{1}{2})} + \frac{1}{3}\chi_{[\frac{1}{2}, 1]}$ and $Y = \chi_{[0, \frac{1}{3})} + \frac{1}{3}\chi_{[\frac{1}{3}, \frac{1}{2})} + \frac{1}{4}\chi_{[\frac{1}{2}, 1]}$. If we take $S(p, X) = (X(p))^3$, then

$$\begin{aligned} \int_{\Omega}^S X dP &= S\left(\cdot, \frac{1}{2}\right) P\left(\left[0, \frac{1}{2}\right)\right) + S\left(\cdot, \frac{1}{3}\right) P\left(\left[\frac{1}{2}, 1\right]\right) = 0.08. \\ \int_{\Omega}^S Y dP &= S(\cdot, 1) P\left(\left[0, \frac{1}{3}\right)\right) + S\left(\cdot, \frac{1}{3}\right) P\left(\left[\frac{1}{3}, \frac{1}{2}\right)\right) + S\left(\cdot, \frac{1}{4}\right) P\left(\left[\frac{1}{2}, 1\right]\right) = 0.346. \end{aligned}$$

On the other hand, $X + Y = \frac{3}{2}\chi_{[0, \frac{1}{3})} + \frac{5}{6}\chi_{[\frac{1}{3}, \frac{1}{2})} + \frac{7}{12}\chi_{[\frac{1}{2}, 1]}$. Hence,

$$\begin{aligned} \int_{\Omega}^S (X + Y) dP &= S\left(\cdot, \frac{3}{2}\right) P\left(\left[0, \frac{1}{3}\right)\right) + S\left(\cdot, \frac{5}{6}\right) P\left(\left[\frac{1}{3}, \frac{1}{2}\right)\right) + S\left(\cdot, \frac{7}{12}\right) P\left(\left[\frac{1}{2}, 1\right]\right) \\ &= 2.411. \end{aligned}$$

$$\text{Thus } \int_{\Omega}^S (X + Y) dP \neq \int_{\Omega}^S X dP + \int_{\Omega}^S Y dP.$$

Theorem 5: [6] Suppose $\chi, \eta: \Omega \rightarrow \mathbb{R}^n$ are stochastic variables. By using of S for "order creator for integral" we have the following properties:

- (i) If $0 \leq \chi \leq \eta$, then $\int_{\Omega}^S \chi dP \leq \int_{\Omega}^S \eta dP$.
- (ii) If $S(\cdot, \chi + \eta) = S(\cdot, \chi) + S(\cdot, \eta)$, then $\int_{\Omega}^S (\chi + \eta) dP = \int_{\Omega}^S \chi dP + \int_{\Omega}^S \eta dP$.
- (iii) $\int_{\Omega}^S \alpha \chi dP = \alpha \int_{\Omega}^S \chi dP$ for all $\alpha \in \mathbb{R}$.

Remark 1: The set $\{\int_{\Omega}^S \chi dP: \chi \text{ is a stochastic variable on } \Omega\}$ is a vector space, when for all random variables χ, η on Ω , $S(\cdot, \chi + \eta) = S(\cdot, \chi) + S(\cdot, \eta)$.

Theorem 6: If $\{X_n\}_{n \geq 1}$ is a sequence of increasing random variables, then

$$\lim_{n \rightarrow \infty} \int_{\Omega}^S X_n dP = \int_{\Omega}^S X dP \text{ where } \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \text{ for all } \omega \in \Omega.$$

Proof: Since the sequence $\{X_n\}_{n \geq 1}$ is increasing, then the first part of theorem 5 implies $\int_{\Omega}^S X_n dP \leq \int_{\Omega}^S X dP$. Therefore, $\lim_{n \rightarrow \infty} \int_{\Omega}^S X_n dP \leq \int_{\Omega}^S X dP$. To show

the opposite inequality we assume $\int_{\Omega}^S X dP < \infty$. Given $\varepsilon > 0$ there exists a simple random variable ψ with $\psi(\omega) \leq X(\omega)$ for all $\omega \in \Omega$ and $\int_{\Omega}^S \psi dP > \int_{\Omega}^S X dP - \varepsilon$. For $0 < c < 1$ we define the sets $\{A_n\}_{n \geq 1}$ by

$$A_n = \{\omega: X_n(\omega) \geq c\psi(\omega)\}, n \in \mathbb{N}.$$

The increasing property of the sequence $\{X_n\}_{n \geq 1}$ implies to the increasing property of the sequence $\{A_n\}_{n \geq 1}$ and $\cup_{n=1}^{\infty} A_n = \Omega$. We have $c\psi(\omega) \leq \psi(\omega) \leq X_n(\omega)$ for all $\omega \in \Omega$ and $c\psi(\omega) \leq X(\omega)$ when $X(\omega) > 0$, and $\cup_{n=1}^{\infty} A_n = \Omega$. Thus,

$$\int_{\Omega}^S X_n dP \geq \int_{\Omega}^S X_n \chi_{A_n} dP \geq c \int_{\Omega}^S \psi \chi_{A_n} dP$$

so,

$$\sup_n \int_{\Omega}^S X_n dP \geq c(\sup_n \int_{\Omega}^S \psi \chi_{A_n} dP).$$

Let $\psi(\omega) = \sum_{i=1}^k a_i \chi_{B_i}(\omega)$, where $B_i \subseteq \Omega$ for all $i = 1, 2, \dots, k$. Therefore,

$$\int_{\Omega}^S \psi \chi_{A_n} dP = \int_{\Omega}^S \sum_{i=1}^k a_i \chi_{B_i \cap A_n} dP = \sum_{i=1}^k S(\cdot, a_i) P(B_i \cap A_n).$$

Since the sequence $\{A_n\}$ converges to Ω monotonically, we have $\lim_{n \rightarrow \infty} P(A_n \cap B_i) = P(B_i)$. Therefore, $\lim_{n \rightarrow \infty} \int_{\Omega}^S \psi \chi_{A_n} dP = \int_{\Omega}^S \psi \chi_{B_i} dP$.

Consequently,

$$\lim_{n \rightarrow \infty} \int_{\Omega}^S X_n dP = \sup_n \int_{\Omega}^S X_n dP \geq c \int_{\Omega}^S \psi dP > \int_{\Omega}^S X dP - \varepsilon.$$

The proof is complete when $c \rightarrow 1$ and $\varepsilon \rightarrow 0$.

4. Mathematical expectation

In general, if two random variables, X_1, X_2 , are independent, $\int_{\Omega}^S X_1 dP$ and $\int_{\Omega}^S X_2 dP$ are not necessarily independent. In the next example we present two independent random variables X_1 and X_2 that $E(\int_{\Omega}^S X_1 dP \cdot \int_{\Omega}^S X_2 dP)$ is not equal to $E(\int_{\Omega}^S X_1 dP)E(\int_{\Omega}^S X_2 dP)$.

Example 3: Let Ω be a probability space and $X_i: \Omega \rightarrow \mathbb{R}$ for $i = 1, 2$ be stochastic variables where

$$X_i(\omega) = \begin{cases} 1 & \omega \in B_i \\ 0 & \omega \in \Omega - B_i \end{cases}$$

B_i 's are measurable sets and $P(B_i) = p$. X_1 and X_2 are independent random variables. But $\int_{\Omega}^S X_1 dP = \int_{\Omega}^S X_2 dP = S(\cdot, 1)p$. The straightforward calculations implies

$$E(\int_{\Omega}^S X_1 dP \cdot \int_{\Omega}^S X_2 dP) \neq E(\int_{\Omega}^S X_1 dP)E(\int_{\Omega}^S X_2 dP).$$

Theorem 7: Let φ and ψ be simple random variables which, $\varphi = \sum_{j=1}^k C_j \chi_{A_j}$ and $\psi = \sum_{i=1}^m D_i \chi_{B_i}$. If $S(\cdot, C_j)$ and $S(\cdot, D_i)$ are independent for $j = 1, 2, \dots, k$ and $i = 1, 2, \dots, m$ then, $E(\int_{\Omega}^S \varphi dP, \int_{\Omega}^S \psi dP) = E(\int_{\Omega}^S \varphi dP)E(\int_{\Omega}^S \psi dP)$.

Proof: By the definition of integral

$$\begin{aligned} E(\int_{\Omega}^S \varphi dP, \int_{\Omega}^S \psi dP) &= E(\sum_{j=1}^k S(\cdot, C_j)P(A_j), \sum_{i=1}^m S(\cdot, D_i)P(B_i)) \\ &= E(\sum_{j=1}^k \sum_{i=1}^m S(\cdot, C_j)S(\cdot, D_i)P(A_j)P(B_i)) \\ &= \sum_{j=1}^k \sum_{i=1}^m E(S(\cdot, C_j)S(\cdot, D_i))P(A_j)P(B_i) \\ &= E(\int_{\Omega}^S \varphi dP)E(\int_{\Omega}^S \psi dP). \end{aligned}$$

Remark 2: Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of simple random variables which $\lim_{n \rightarrow \infty} \varphi_n(p) = X(p)$ and $\lim_{n \rightarrow \infty} \psi_n(p) = Y(p)$ for all $p \in \Omega$. Also consider $S(\cdot, \varphi_n)$ and $S(\cdot, \psi_n)$ are independent then $\int_{\Omega}^S X dP$ and $\int_{\Omega}^S Y dP$ are independent.

Theorem 8: Let S be a continuous function and for any finite sequence $\{X_i\}_{i=1}^n$ random variables that $S(\cdot, \sum_{i=1}^n \lambda_i X_i) = \sum_{i=1}^n \lambda_i S(\cdot, X_i)$ where $\sum_{i=1}^n \lambda_i = 1$. if X be a positive stochastic variable, then $\int_{\Omega}^S X dP = S(\cdot, E(X))$.

Proof: Let $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ be a simple random variable. By theorem assumption, $S(\cdot, E(\varphi)) = \sum_{i=1}^n S(\cdot, a_i)P(A_i) = \int_{\Omega}^S \varphi dP$. Using monotone convergence theorem [11], there is a sequence of simple random variables $\{\varphi_n\}$ such that $\lim_{n \rightarrow \infty} \varphi_n = X$. Hence, by theorem 6, $\lim_{n \rightarrow \infty} \int_{\Omega}^S \varphi_n dP = \int_{\Omega}^S X dP$. On the other hand, since S is continuous function so, $\lim_{n \rightarrow \infty} S(\cdot, E(\varphi_n)) = S(\cdot, E(X))$. Therefore, $\int_{\Omega}^S X dP = S(\cdot, E(X))$.

Example 4: The motion of particles in the air is Brownian motion. A possible improvement of the mathematical model is frictional forces. The one-dimensional stochastic model is

$$\dot{X} = -bX + \sigma W,$$

where $W(\cdot)$ is "white noise". $b > 0$ and σ are factor of friction and diffusion factor respectively. $X(\cdot)$ is the velocity of Brownian particles. The stochastic differential equation is

$$dX = -bXdt + \sigma dW, \quad X(0) = X_0.$$

The solution is

$$X(t) = e^{-bt}X_0 + \sigma \int_0^t e^{-b(t-s)} dW \quad t \geq 0$$

Consider $S(\cdot, X(t)) = X(t)$. Set $\{\varphi_n = \sum_{i=1}^{m_n} a_i \chi_{A_i}\}_{n=1}^{\infty}$ be the sequence of simple stochastic variables which convergence to X . Therefore,

$$\begin{aligned} E\left(\int_{\Omega}^S X dP\right) &= \lim_{n \rightarrow \infty} E\left(\int_{\Omega}^S \varphi_n dP\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^{m_n} S(\cdot, a_i) P(A_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{m_n} a_i P(A_i) = \lim_{n \rightarrow \infty} E(\varphi_n) = E(X). \end{aligned}$$

Thus, the expectation of the total air pollution is equal the expectation of the random variable X .

Conclusion

Since the movement of airborne particles is random. An stochastic mathematical model is a suitable choice to describe amount of air pollution in an area. Using order creators for integral, total amount of air pollution expressed with a random variable. Also, some basic properties are investigated.

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