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A new plan to study various solutions of the extended Korteweg-de Vries(K-dV) equation with conformable derivative

A. Sharif *
M. Ghojogh #
Department of Mathematics
Faculty of Sciences
Gonbad Kavous University
Gonbad 49717-99151
Iran

Abstract

In this discussion, the extended fractional Korteweg-de Vries(K-dV) equation has been studied using the modified Sardar sub-equation method. The method in question is an efficient method with various answers, which is very practical and expresses different types of answers. The graphical analysis expressed well shows the behavior of the answers.

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1. Introduction

Many important and significant problems in engineering, experimental sciences, and social sciences are expressed in the form of mathematical models. These models usually lead to the determination of a function and should apply to equations that include derivatives of this unknown function. In the 18th century, the partial differential equation began with the work of mathematicians such as Euler, Lagrange, De Albert and Laplace with an analytical study of physical models. The theory of equations, which includes ordinary differential equations and differential

* *ORCID-ID:* `0000-0002-6598-7139`

* *E-mail:* `sharif@gonbad.ac.ir (Corresponding Author)`
# *E-mail:* `mahdiehghojogh@gmail.com`

equations with partial derivatives, is a branch of mathematical analysis that has a close relationship with physics. Physical concepts such as the movement of a vibrating string, gravity, fluid flow, heat conduction, electricity and magnetism have all led to the emergence of different differential equations. The answers of these equations correctly answer the effects of these physical phenomena. Many physical phenomena cannot be expressed with ordinary derivative models, and the observations and results obtained in these models are closer to modeling with fractional derivatives. In addition to this concept of ordinary derivative which is interpreted with speed in mechanical physics, it also takes on another aspect in fractional derivatives. This aspect can be expressed by increasing or decreasing the moving acceleration. In recent years, the study of nonlinear partial differential equations (PDEs) has seen significant advancements, with researchers developing various innovative methods to find exact solutions. These solutions are crucial for understanding the behavior of complex systems in physics, engineering, biology, and other fields. Below are some of the prominent methods that have been presented: one of the best methods is the modified Sardar sub-equation technique [1][2]. The modified Sardar sub-equation method is an efficient and effective solution method for calculating exact wave solutions and one of the most direct and effective algebraic methods for finding exact solutions of differential equations with nonlinear partial derivatives [3][4][5][6][7]. The difference between this method and other old methods is to avoid complex and tedious calculations and more accurate and explicit conclusions to determine the isolated wave solutions with high accuracy. In recent years, this method is useful and useful for determining the exact solutions (soliton, alternating and exponential solutions) of the Schrödinger equation, Eckhaus equation and various differential equations with other non-linear partial derivatives that are widely used in various fields of science and engineering. , has been successfully used [8][9][10][11][12][13][14][15][16]. The present paper applied the modified Sardar sub-equation method to derive traveling wave solutions for the extended fractional Korteweg-de Vries(K-dV) equation

$$D_\tau^\alpha \varphi_1 + \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - (C_2(1-f) + D_2 f\sigma^2)\right] D_\xi^{2\alpha}\varphi_1 +$$
$$\frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right)(D_\xi^\alpha \varphi_1)^3 + \frac{\lambda^3}{2} D_\xi^{3\alpha}\varphi_1 = 0 \tag{1}$$

The significance of this topic has indeed spurred extensive research and studies, driving researchers to explore and develop various effective methods. Over time, this has resulted in the identification and adoption of widely used techniques that have proven to be impactful in addressing the challenges or goals associated with the subject. These methods have not only advanced understanding but have also provided practical solutions, contributing to progress in the field. The continuous evolution of research in this area highlights its ongoing relevance and the need for further innovation. From [17] we have:

$$\left((1-f)C_1 + fD_1\sigma - \frac{1}{\lambda^2}\right)\varphi_2 = \left[\frac{3}{2\lambda^4} - ((1-f)C_2 + fD_2\sigma^2)\right](\varphi_1)^2,$$

$$C_1 = \frac{\kappa_c - \frac{1}{2}}{\kappa_c - \frac{3}{2}}, D_1 = \frac{\kappa_h - \frac{1}{2}}{\kappa_h - \frac{3}{2}}$$

$$C_2 = \frac{\left(\kappa_c - \frac{1}{2}\right)\left(\kappa_c + \frac{1}{2}\right)}{2\left(\kappa_c - \frac{3}{2}\right)^2},\ D_2 = \frac{\left(\kappa_h - \frac{1}{2}\right)\left(\kappa_h + \frac{1}{2}\right)}{2\left(\kappa_h - \frac{3}{2}\right)^2}$$

And

$$\frac{\partial n_1}{\partial \tau} - \lambda\frac{\partial n_3}{\partial \xi} + \frac{\partial(n_2u_1)}{\partial \xi} + \frac{\partial(n_1u_2)}{\partial \xi} + \frac{\partial u_3}{\partial \xi} = 0,$$

$$\frac{\partial u_1}{\partial \tau} + u_2\frac{\partial u_1}{\partial \xi} + u_1\frac{\partial u_2}{\partial \xi} - \lambda\frac{\partial u_3}{\partial \xi} + \frac{\partial \varphi_3}{\partial \xi} = 0,$$

$$\frac{\partial^2\varphi_1}{\partial \xi^2} + \left[\frac{3}{2\lambda^4} - \left((1-f)C_2 + f\sigma^2D_2\right)\right](\varphi_1)^2 = \frac{1}{\lambda^2}\varphi_3 + 2((1-f)C_2 + f\sigma^2D_2)(\varphi_1\varphi_2) + (C_3(1-f) + f\sigma^3D_3)\varphi_1^3 - n_3$$

by solving (1-3) -(1-4) – (1-5) we obtain

$$\frac{\partial \varphi_1}{\partial \tau} + \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - (C_2(1-f) + D_2f\sigma^2)\right]\frac{\partial \varphi_1^2}{\partial \xi} + \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3D_3\right)\frac{\partial \varphi_1^3}{\partial \xi} + \frac{\lambda^3}{2}\frac{\partial^3\varphi_1}{\partial \xi^3} = 0,$$

and

$$C_3 = \frac{\left(\kappa_c - \frac{1}{2}\right)\left(\kappa_c + \frac{1}{2}\right)\left(\kappa_c + \frac{3}{2}\right)}{6\left(\kappa_c - \frac{3}{2}\right)^3},$$

$$D_3 = \frac{\left(\kappa_h - \frac{1}{2}\right)\left(\kappa_h + \frac{1}{2}\right)\left(\kappa_h + \frac{3}{2}\right)}{6\left(\kappa_h - \frac{3}{2}\right)^3}$$

In the following, we present the method in question in the second part, and in the third part, we describe the application of the method along with the graphical analysis of the behavior of the answers. And in the final part, we present the results of the discussion.

2. Sardar sub-equation technique description

The Sardar sub-equation technique is a mathematical method used to solve nonlinear partial differential equations (PDEs). It is particularly effective for finding exact solutions to such equations, which are often encountered in physics, engineering, and other applied sciences. This technique is a generalization or modification of other sub-equation methods, such as the G′/G-expansion method or the Kudryashov method, and is designed to handle a broader class of nonlinear PDEs [1-2]. To use this approach, we assume the following general NPDE as:

$$G(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{2}$$

here Gdenotes a function of u and its derivatives. We consider the wave transformation as

$$u(x,t) = U(\xi)e^{i(\omega_2 x + \eta_2 t)}, \xi = \omega_1 x + \eta_1 t, \tag{3}$$

Its goal is to convert equation (2) into a regular equation is NODE:

$$N(U, U', U'', \ldots) = 0, \tag{4}$$

The solutions of Eq. (4) can be classify as:

$$\psi(\xi) = \delta_0 + \sum_{r=0}^{m} \delta_r Q^r(\xi), \qquad \delta_m \neq 0, \tag{5}$$

with the constants $\delta_r, r = 1,2,...,M$, ω_j and $\eta_j, j = 1,2$ are to be computed later. Balancing the highest-order derivative and the nonlinear terms of Eq. (4), determines the integer M. The function $Q(\xi)$ in Eq. (5) satisfies

$$(Q'(\xi))^2 = \upsilon_2 Q^4(\xi) + \upsilon_1 Q^2(\xi) + \upsilon_0, \tag{6}$$

where υ_0, υ_1, and υ_2 are constants. In addition, the family of solutions for Eq. (6) with constant are listed as below:

1. If $\upsilon_0 = 0$, $\upsilon_1 > 0$, and $\upsilon_2 = 0$, then

$$Q_1^{\pm}(\xi) = \pm\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\,\mathrm{sech}(\sqrt{\upsilon_1}(\xi + \varsigma)),$$

$$Q_2^{\pm}(\xi) = \pm\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\,\mathrm{csch}(\sqrt{\upsilon_1}(\xi + \varsigma)),$$

2. For constants A_1 and A_2. Let $\upsilon_2 = \pm 4A_1A_2$, $\upsilon_1 > 0$, and $\upsilon_0 = 0$, we have

$$Q_3^{\pm}(\xi) = \frac{4\sqrt{\upsilon_1}A_1}{(4A_1^2 - \upsilon_2)\cosh(\sqrt{\upsilon_1}(\xi + \xi_0)) \pm (4A_1^2 + \upsilon_2)\sinh(\sqrt{\upsilon_1}(\xi + \xi_0))},$$

3. If $\upsilon_0 = \dfrac{\upsilon_1^2}{4\upsilon_2}, \upsilon_1 < 0, \upsilon_2 > 0$, with constants B_1, and B_2, then

$$Q_3^{\pm}(\xi) = \pm\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\tanh\left(\sqrt{-\frac{\upsilon_1}{2}}(\xi + \varsigma)\right),$$

$$Q_4^{\pm}(\xi) = \pm\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\coth\left(\sqrt{-\frac{\upsilon_1}{2}}(\xi + \varsigma)\right),$$

$$Q_5^{\pm}(\xi) = \pm\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}(\tanh(\sqrt{-2\upsilon_1}(\xi + \varsigma)) \pm i\,\mathrm{sech}(\sqrt{-2\upsilon_1}(\xi + \varsigma))),$$

$$Q_6^{\pm}(\xi) = \pm\sqrt{-\frac{\upsilon_1}{8\upsilon_2}}\left(\tanh\left(\sqrt{-\frac{\upsilon_1}{8}}(\xi + \varsigma)\right) \pm \coth\left(\sqrt{-\frac{\upsilon_1}{8}}(\xi + \varsigma)\right)\right),$$

$$Q_7^{\pm}(\xi)=\pm\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\pm\sqrt{B_1^2+B_2^2}-B_1\cosh(\sqrt{-2\upsilon_1}(\xi+\varsigma))}{B_1\sinh(\sqrt{-2\upsilon_1}(\xi+\varsigma))+B_2}\right),$$

$$Q_9^{\pm}(\xi)=\pm\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\cosh(\sqrt{-2\upsilon_1}(\xi+\varsigma))}{\sinh(\sqrt{-2\upsilon_1}(\xi+\varsigma))\pm i}\right),$$

4. If $\upsilon_0=0,\ \upsilon_1<0,\ \upsilon_2\neq 0,$ then

$$Q_{10}^{\pm}(\xi)=\pm\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\sec(\sqrt{-\upsilon_1}(\xi+\varsigma)),$$

$$Q_{11}^{\pm}(\xi)=\pm\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\csc(\sqrt{-\upsilon_1}(\xi+\varsigma)),$$

5. If $\upsilon_0=\dfrac{\upsilon_1^2}{4\upsilon_2},\upsilon_1>0,\upsilon_2>0,$ and $B_1^2-B_2^2>0,$ then

$$Q_{12}^{\pm}(\xi)=\pm\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\tan\left(\sqrt{\frac{\upsilon_1}{2}}(\xi+\varsigma)\right),$$

$$Q_{13}^{\pm}(\xi)=\pm\sqrt{\frac{\upsilon_1}{2\upsilon_2}}\cot\left(\sqrt{\frac{\upsilon_1}{2}}(\xi+\varsigma)\right),$$

$$Q_{14}^{\pm}(\xi)=\pm\sqrt{\frac{\upsilon_1}{2\upsilon_2}}(\tan(\sqrt{2\upsilon_1}(\xi+\varsigma))\pm\sec(\sqrt{2\upsilon_1}(\xi+\varsigma))),$$

$$Q_{15}^{\pm}(\xi)=\pm\sqrt{\frac{\upsilon_1}{8\upsilon_2}}\left(\tan\left(\sqrt{\frac{\upsilon_1}{8}}(\xi+\varsigma)\right)-\cot\left(\sqrt{\frac{\upsilon_1}{8}}(\xi+\varsigma)\right)\right),$$

$$Q_{16}^{\pm}(\xi)=\pm\sqrt{\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\pm\sqrt{B_1^2-B_2^2}-B_1\cos(\sqrt{2\upsilon_1}(\xi+\varsigma))}{B_1\sin(\sqrt{-2\upsilon_1}(\xi+\varsigma))+B_2}\right),$$

$$Q_{17}^{\pm}(\xi)=\pm\sqrt{\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\cos(\sqrt{2\upsilon_1}(\xi+\varsigma))}{\sin(\sqrt{2\upsilon_1}(\xi+\varsigma))\pm 1}\right),$$

6. If $\upsilon_0 = 0$ and $\upsilon_1 > 0$, then

$$Q_{18}^{\pm}(\xi) = \pm \frac{4\upsilon_1 e^{\pm\sqrt{\upsilon_1}(\xi+\varsigma)}}{\upsilon_1 e^{\pm 2\sqrt{\upsilon_1}(\xi+\varsigma)} - 4\upsilon_1\upsilon_2},$$

$$Q_{19}^{\pm}(\xi) = \pm \frac{\pm 4\upsilon_1 e^{\pm\sqrt{\upsilon_1}(\xi+\varsigma)}}{1 - 4\upsilon_1 e^{\pm 2\sqrt{\upsilon_1}(\xi+\varsigma)}},$$

7. If $\upsilon_0 = \upsilon_1 = 0$ and $\upsilon_2 > 0$, then

$$Q_{20}^{\pm}(\xi) = \pm \frac{1}{\sqrt{\upsilon_2}(\xi+\varsigma)},$$

8. If $\upsilon_0 = \upsilon_1 = 0$ and $\upsilon_2 < 0$, then

$$Q_{21}^{\pm}(\xi) = \pm \frac{i}{\sqrt{-\upsilon_2}(\xi+\varsigma)},$$

3. Mathematical Analysis

The main idea of introducing equation (6) using fractional derivatives as follows [17]

$$D_\tau^\alpha \varphi_1 + \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - (C_2(1-f) + D_2 f\sigma^2)\right] D_\xi^{2\alpha}\varphi_1 + \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right)(D_\xi^\alpha \varphi_1)^3 + \frac{\lambda^3}{2} D_\xi^{3\alpha}\varphi_1 = 0 \tag{7}$$

We consider

$$\varphi_1(\xi,\tau) = U(\chi), \qquad \chi = \left(\frac{1}{\alpha}\right)\xi^\alpha - V\left(\frac{1}{\alpha}\right)\tau^\alpha \tag{8}$$

so

$$-VU + \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - (C_2(1-f) + D_2 f\sigma^2)\right] U^2 + \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right) U^3 + \frac{\lambda^3}{2} U'' = 0 \tag{9}$$

We consider

$$-VU + s_1 U^2 + s_2 U^3 + \frac{\lambda^3}{2} U'' = 0$$

Where

$$s_1 = \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - (C_2(1-f) + D_2 f\sigma^2)\right]$$
$$s_2 = \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right)$$

In (3-2) by HBM between φ_1^3 and $d^2\varphi_1 / d\chi^2$ we have $M = 1$. So solution takes the following form

$$U(\xi) = \delta_0 + \delta_1 Q(\xi), \qquad \delta_1 \neq 0, \tag{10}$$

Putting Eqs. (6) and (10) into Eqs. (8), gathering all the coefficients of $Q^r(\xi)$ and setting them to zero, we get the algebraic equations and by solving them via Maple package we have following solutions

Set 1:

$$\delta_1 = \frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}, \delta_0 = -\frac{s_1}{3s_2}, V = \frac{-2s_1}{9s_2} \tag{11}$$

Where

$$s_1 = \frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4} - (C_2(1-f) + D_2 f\sigma^2)\right], s_2 = \frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6} - C_3(1-f) - f\sigma^3 D_3\right)$$

By substituting the parameters of the set 1 in equation (10) in addition to the general solutions of equation (6), we will have the following cases:

The bright and singular solitons can be consider as (Image 1- a , b):

$$\varphi_{1-1}(\xi,\tau) = -\frac{s_1}{3s_2} \pm \frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\,\mathrm{sech}\left(\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha + \frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha + \varsigma\right)\right),$$

$$\varphi_{1-2}(\xi,\tau) = -\frac{s_1}{3s_2} \pm \frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\,\mathrm{csch}\left(\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha + \frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha + \varsigma\right)\right),$$

Where $\upsilon_0 = 0, \upsilon_1 > 0,$ and $\upsilon_2 = 0.$

The dark and singular solitons can be considered as (Image 1- c , d):

$$\varphi_{1-3}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\tanh\left(\sqrt{-\frac{\upsilon_1}{2}}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right),$$

$$\varphi_{1-4}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\coth\left(\sqrt{-\frac{\upsilon_1}{2}}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right),$$

Where $\upsilon_0=\frac{\upsilon_1^2}{4\upsilon_2},\upsilon_1<0,\upsilon_2>0,$ with constants B_1, and B_2.

The combo bright-singular solitons are expressed as (Image 1- e):

$$\varphi_{1-5}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\frac{4\sqrt{\upsilon_1}A_1}{(4A_1^2-\upsilon_2)\cosh\left(\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\xi_0\right)\right)\pm (4A_1^2+\upsilon_2)\sinh\left(\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\xi_0\right)\right)},$$

Where $\upsilon_2=\pm 4A_1A_2$, $\upsilon_1>0$, and $\upsilon_0=0.$
The combo solitons are as follows (Image 1- f , g , h , j) :

$$\varphi_{1-6}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\tanh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\right.$$
$$\left.\pm i\,\mathrm{sech}\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\right),$$

$$\varphi_{1-7}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\tanh\left(\sqrt{-\frac{\upsilon_1}{8}}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\pm\right.$$
$$\left.i\coth\left(\sqrt{-\frac{\upsilon_1}{8}}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\right),$$

$$\varphi_{1-8}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\pm\sqrt{B_1^2+B_2^2}-B_1\cosh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)}{B_1\sinh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)+B_2}\right),$$

$$\varphi_{1-9}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\cosh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)}{\sinh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)\pm i}\right),$$

Where $\upsilon_0=\frac{\upsilon_1^2}{4\upsilon_2},\upsilon_1<0,\upsilon_2>0,$

The trigonometric function solutions are acquired as follows (Image 1- k):

$$\varphi_{1-10}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\sec\left(\sqrt{-\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

$$\varphi_{1-11}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\csc\left(\sqrt{-\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

Where $\upsilon_0=0,\upsilon_1<0,\upsilon_2\neq 0.$

$$\varphi_{1-12}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\tan\left(\sqrt{\frac{\upsilon_1}{2}}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

And

$$\varphi_{1-13}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\sqrt{\frac{\upsilon_1}{2\upsilon_2}}\cot\left(\sqrt{\frac{\upsilon_1}{2}}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

The mixed trigonometric function solutions are as follows:

$$\varphi_{1-14}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\sqrt{\frac{\upsilon_1}{2\upsilon_2}}\left(\tan\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)\pm\right.$$
$$\left.\sec\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)\right),$$

$$\varphi_{1-15}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\sqrt{\frac{\upsilon_1}{8\upsilon_2}}\left(\tan\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)-\right.$$
$$\left.\cot\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)\right),$$

$$\varphi_{1-16}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\pm\sqrt{B_1^2+B_2^2}-B_1\cos\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)}{B_1\sin\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)+B_2}\right),$$

$$\varphi_{1-17}(\xi,\tau)=-\frac{s_1}{3s_2}\pm\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2}\times$$
$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\cos\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)}{\sin\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha+\frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)\pm 1}\right),$$

Where $\upsilon_0=\frac{\upsilon_1^2}{4\upsilon_2},\upsilon_1<0,\upsilon_2>0,$

The exponential solutions are:

$$\varphi_{1-18}(\xi,\tau) = -\frac{s_1}{3s_2} \pm \frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2} \times \frac{4\upsilon_1 e^{\pm\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha + \frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha + \varsigma\right)}}{\upsilon_1 e^{\pm 2\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha + \frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha + \varsigma\right)} - 4\upsilon_1\upsilon_2},$$

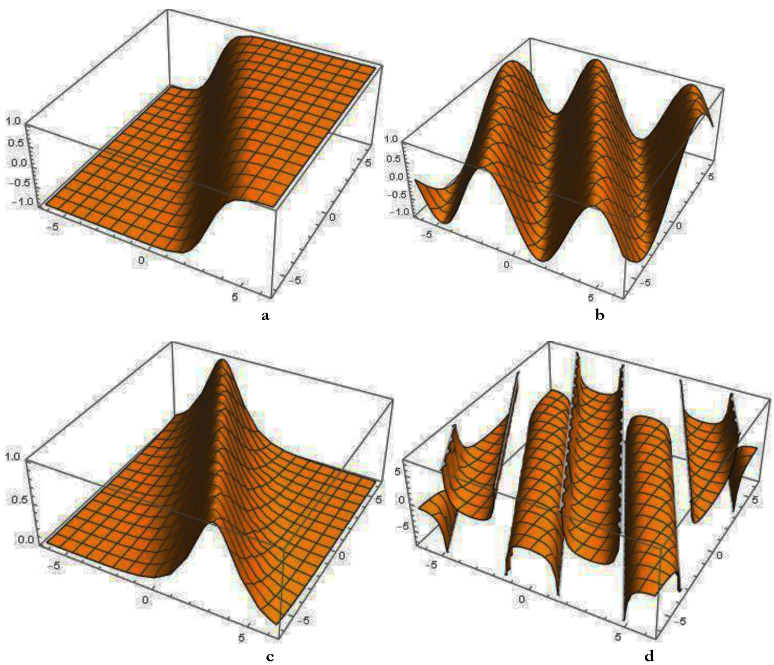
$$\varphi_{1-19}(\xi,\tau) = -\frac{s_1}{3s_2} \pm \frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2} \times \frac{\pm 4\upsilon_1 e^{\pm\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha + \frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha + \varsigma\right)}}{1 - 4\upsilon_1 e^{\pm 2\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha + \frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha + \varsigma\right)}},$$

Where $\upsilon_0 = 0$ and $\upsilon_1 > 0$

The rational solutions are:

$$\varphi_{1-20}(\xi,\tau) = -\frac{s_1}{3s_2} \pm \frac{(-\upsilon_2 s_2{}^2)^{(1/3)}\lambda}{s_2} \times \frac{1}{\sqrt{\upsilon_2}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha + \frac{2s_1}{9s_2}\left(\frac{1}{\alpha}\right)\tau^\alpha + \varsigma\right)},$$

For $\upsilon_0 = \upsilon_1 = 0$ and $\upsilon_2 > 0$



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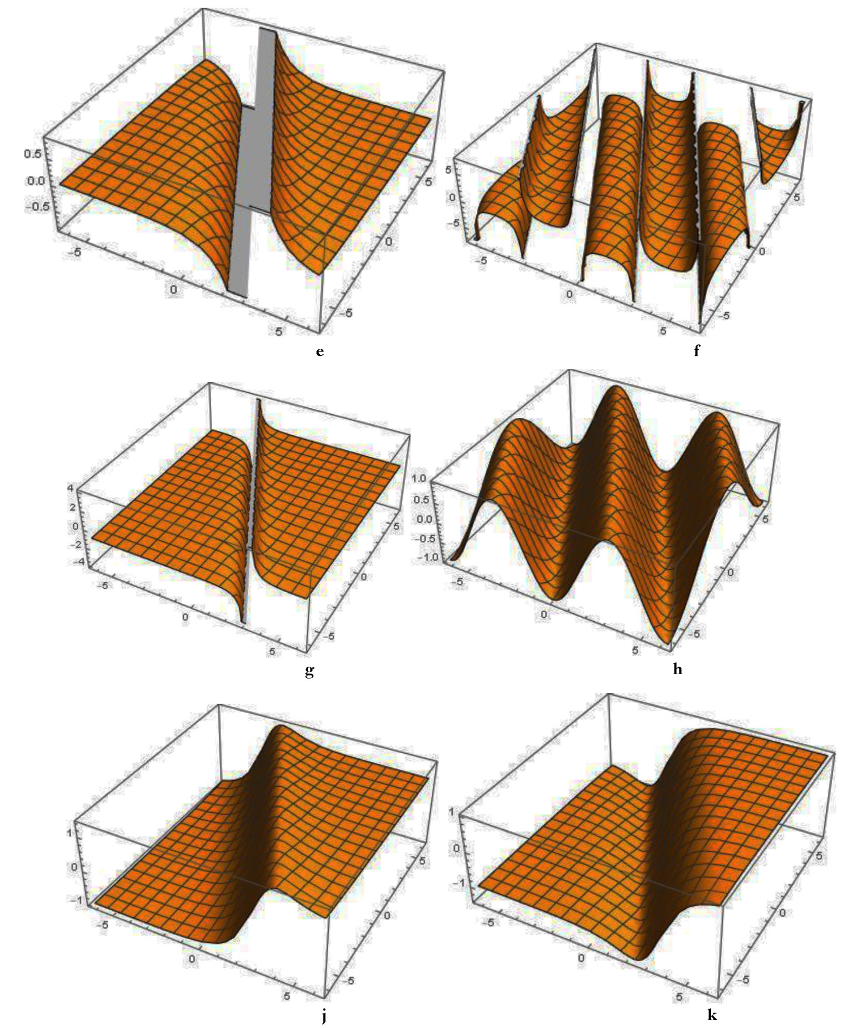

Image 1

Analytical behavior of set 1 (a: φ_{1-1} b: φ_{1-2} c: φ_{1-3} d: φ_{1-4} e: φ_{1-5} f: φ_{1-6} g: φ_{1-7} h: φ_{1-8} g: φ_{1-9} k: φ_{1-10}) for $\alpha = 1$.

Set 2:

$$\delta_1 = \frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2} - \frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}, \delta_0 = 0, V = \frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}$$

Where

$$s_1=\frac{\lambda^3}{2}\left[\frac{3}{2\lambda^4}-\left(C_2(1-f)+D_2 f\sigma^2\right)\right], s_2=\frac{\lambda^3}{2}\left(\frac{5}{2\lambda^6}-C_3(1-f)-f\sigma^3 D_3\right)$$

By substituting the parameters of the second part in equation (10) in addition to the general solutions of equation (6), we will have the following cases:

We obtain bright and singular solitons as:

$$\varphi_{1-1}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times \sqrt{-\frac{\upsilon_1}{\upsilon_2}}\operatorname{sech}\left(\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

$$\varphi_{1-2}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times \sqrt{-\frac{\upsilon_1}{\upsilon_2}}\operatorname{csch}\left(\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

Where $\upsilon_0=0,\ \upsilon_1>0,$ and $\upsilon_2=0.$

The dark and singular solitons can be considered as:

$$\varphi_{1-3}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times \sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\tanh\left(\sqrt{-\frac{\upsilon_1}{2}}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

$$\varphi_{1-4}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times \sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\coth\left(\sqrt{-\frac{\upsilon_1}{2}}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

Where $\upsilon_0=\frac{\upsilon_1^2}{4\upsilon_2}, \upsilon_1<0, \upsilon_2>0,$ with constants $B_1,$ and $B_2.$

Bright-singular solitons as:

$$\varphi_{1-5}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\frac{4\sqrt{\upsilon_1}A_1}{(4A_1^2-\upsilon_2)\cosh\left(\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-V\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\xi_0\right)\right)\pm(4A_1^2+\upsilon_2)\sinh\left(\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-V\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\xi_0\right)\right)},$$

and $V=\dfrac{\lambda^3 \upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}$

Where $\upsilon_2=\pm 4A_1A_2$, $\upsilon_1>0$, and $\upsilon_0=0$.

The combo solitons are as follows:

$$\varphi_{1-6}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\tanh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-V\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\pm i\,\mathrm{sech}\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-V\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\right),$$

and $V=\dfrac{\lambda^3 \upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}$

$$\varphi_{1-7}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\tanh\left(\sqrt{-\frac{\upsilon_1}{8}}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-V\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\pm i\coth\left(\sqrt{-\frac{\upsilon_1}{8}}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-V\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\right),$$

and $V=\dfrac{\lambda^3 \upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}$

$$\varphi_{1-8}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$

$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\pm\sqrt{B_1^2+B_2^2}-B_1\cosh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)}{B_1\sinh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)+B_2}\right),$$

$$\varphi_{1-9}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$

$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\cosh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)}{\sinh\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right)\pm i}\right),$$

Where $\upsilon_0=\frac{\upsilon_1^2}{4\upsilon_2},\upsilon_1<0,\upsilon_2>0,$

The trigonometric function solutions are acquired as follows:

$$\varphi_{1-10}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\sec\left(\sqrt{-\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

$$\varphi_{1-11}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\sqrt{-\frac{\upsilon_1}{\upsilon_2}}\csc\left(\sqrt{-\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^\alpha-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^\alpha+\varsigma\right)\right),$$

Where $\upsilon_0=0,\upsilon_1<0,\upsilon_2\neq 0.$

$$\varphi_{1-12}(\xi,\tau)=\left(\frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\sqrt{-\frac{v_1}{2v_2}}\tan\left(\sqrt{\frac{v_1}{2}}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right),$$

And

$$\varphi_{1-13}(\xi,\tau)=\left(\frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\sqrt{\frac{v_1}{2v_2}}\cot\left(\sqrt{\frac{v_1}{2}}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right),$$

We obtain mixed trigonometric function solutions as:

$$\varphi_{1-14}(\xi,\tau)=\left(\frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\sqrt{\frac{v_1}{2v_2}}\tan\left(\sqrt{2v_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\pm$$
$$\sqrt{\frac{v_1}{2v_2}}\sec\left(\sqrt{2v_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right),$$

$$\varphi_{1-14}(\xi,\tau)=\left(\frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$
$$\sqrt{\frac{v_1}{8v_2}}\tan\left(\sqrt{2v_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)-$$
$$\sqrt{\frac{v_1}{8v_2}}\cot\left(\sqrt{2v_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right),$$

$$\varphi_{1-16}(\xi,\tau)=\left(\frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$

$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\pm\sqrt{B_1^2+B_2^2}-B_1\cos\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)}{B_1\sin\left(\sqrt{-2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)+B_2}\right),$$

$$\varphi_{1-17}(\xi,\tau)=\left(\frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times$$

$$\sqrt{-\frac{\upsilon_1}{2\upsilon_2}}\left(\frac{\cos\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)}{\sin\left(\sqrt{2\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)\right)\pm 1}\right),$$

Where $\upsilon_0=\frac{\upsilon_1^2}{4\upsilon_2},\upsilon_1<0,\upsilon_2>0,$

The exponential solutions are:

$$\varphi_{1-18}(\xi,\tau)=\left(\frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}\right)$$
$$\times\frac{4\upsilon_1 e^{\pm\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)}}{\upsilon_1 e^{\pm 2\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)}-4\upsilon_1\upsilon_2},$$

$$\varphi_{1-19}(\xi,\tau)=\left(\frac{(-v_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-v_2 s_2{}^2)^{(1/3)}}{2s_2}\right)$$
$$\times\frac{\pm 4\upsilon_1 e^{\pm\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)}}{1-4\upsilon_1 e^{\pm 2\sqrt{\upsilon_1}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3 v_1 s_2}{(-v_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)}},$$

Where $\upsilon_0=0$ and $\upsilon_1>0$

The rational solutions are:

$$\varphi_{1-20}(\xi,\tau)=\left(\frac{(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}-\frac{i\sqrt{3}(-\upsilon_2 s_2{}^2)^{(1/3)}}{2s_2}\right)\times \frac{1}{\sqrt{\upsilon_2}\left(\left(\frac{1}{\alpha}\right)\xi^{\alpha}-\frac{\lambda^3\upsilon_1 s_2}{(-\upsilon_2 s_2{}^2)^{(1/3)}(i\sqrt{3}+1)}\left(\frac{1}{\alpha}\right)\tau^{\alpha}+\varsigma\right)},$$

For $\upsilon_0=\upsilon_1=0$ and $\upsilon_2>0$

Conclusion:

This study focuses on obtaining new solutions for Korteweg-de Vries equation using fractional derivatives and the extended Sardar-Sub equation method. The Korteweg-de Vries equation equation is a fundamental nonlinear partial differential equation that describes wave propagation in various physical contexts. By incorporating fractional derivatives, we extend the applicability of the KDV equation to systems with memory effects or anomalous diffusion, which are common in complex systems. This method is a powerful analytical tool for solving NLPDEs. It is known for its ability to generate a wide variety of solutions, including solitary waves, periodic solutions, and other types of wave solutions. The method involves transforming the original equation into a simpler form, which can then be solved using standard techniques.

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