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# The $\mathcal{L}_G^p(\Omega)$-continuity of solutions to neutral stochastic differential equations driven by G -Brownian motion

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**Abstract**

This article explores the $\mathcal{L}_G^p(\Omega)$-continuity of the solution to neutral stochastic differential equations, under nonlinear growth and contraction conditions, in the context of *G*-expectation theory. A practical example is presented to demonstrate the findings.

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## 1. Introduction

Recently, the *G*-expectation theory developed by Peng [11] has gained widespread applications in addressing uncertainty problems, strategies for achieving complete hedging in finance, risk assessment, and more related areas. In such *G*-framework, Peng defined a continuous stochastic process with homogeneous increments called *G*-Brownian motion (*G*-Bm), which serves an analogous role to that of Wiener process. However, it features that the process of quadratic variation is a non-deterministic process. After that he developed the corresponding stochastic calculus [12]. At a later point, stochastic differential equations influenced by this *G*-Bm were initially emerged [10], and aroused the interest of many authors, see [14, 16, 7, 6].

Numerous stochastic dynamical systems depend not only on current and previous states, but also on the dynamics of the past, namely, involve

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derivatives with delays. Consequently, a common approach to describe such systems is by using neutral stochastic differential equations (NSDEs). Neutral SDEs were first applied in the "aeroelasticity theory and chemical engineering systems", as noted by Kolmanovskii [4, 5]. In the *G*-framework, NSDEs are also studied by several authors, as referenced in [1, 3, 9].

Most often than not, there is no analytical method to solve non-linear stochastic differential equations. Thus, we have recourse to numerical methods. For instance, the work of Hejazi et al. [13] represents the first derivation of a numerical solution for the time-fractional partial differential equation method regarding the density function for joint transition probability in the context of the geometric Brownian motion equation. Therefore, some of the characteristics and regularity conditions of the solutions, like stability and continuity, must be verified to ensure the reliability of the numerical methods. For example, the stability of the numerical solution determines the consistency and accuracy of the numerical solution and encompasses the behavior of the numerical method under various conditions and inputs over time; in other words, the errors produced by the numerical method do not grow too quickly as the number of iterations increases. The stability of the solutions was studied by numerous authors, including, for instance, Xiao, Zhang, and Fang [15] and Mao [8].

Continuity of solutions under different probabilistic scenarios, represented by *G*-expectation, is also a very important result, because continuity and stability are related, particularly in the context of numerical analysis and differential equations. They basically describe how minor alterations in inputs affect the corresponding outputs. Thus, numerical methods such as finite difference or finite element methods rely on the continuity of the solution; if the solution is not continuous, these numerical methods may fail to converge or produce incorrect results.

In the present research, we are concerned in the continuity of solutions for *G*-NSDEs. The difficulties arise mainly from the presence of the delay term and the loss of the linearity property of the *G*-exepectation. However, under suitable assumptions, the $\mathcal{L}_G^p(\Omega)$-continuity of the solutions can be established.

This article follows this structure. Section 2, focuses on the needed notions about *G*-expectation theory. In Section 3, the principal results are presented where we prove the $\mathcal{L}_G^p(\Omega)$-continuity of $p$-th moment of the solution, and then we give an example that supports the obtained results.

## 2. Basic settings

In this section we give the needed basics about the $G$-expectation theory, (see [10, 11, 12]). In the next, we denote by

- $\Omega$ Indicates a set that is not null.
- $\mathcal{X}$ A class of random functions from $\Omega$ to $\mathbb{R}$.

**Definition 2.1:** A sublinear expectation serves as a mapping function $\mathcal{E} : \mathcal{X} \to \mathbb{R}$, that accomplishes:

(i) Monotone property:

$$\text{if } X_1 \leq X_2 (\text{quasi} - \text{surely}), \text{ then } \tilde{\mathcal{E}}(X_1) \leq \tilde{\mathcal{E}}(X_2);$$

(ii) Preserves constancy:

$$\tilde{\mathcal{E}}(c) = c \quad \forall c \in \mathbb{R};$$

(iii) Subadditive:

$$\tilde{\mathcal{E}}(X_1) - \tilde{\mathcal{E}}(X_2) \leq \tilde{\mathcal{E}}(X_1 - X_2) \text{ for each } X_1, X_2 \in \mathcal{X};$$

(iv) Homogeneity under positive factor:

$$\tilde{\mathcal{E}}(\alpha X_1) = \alpha \tilde{\mathcal{E}}(X_1) \text{ for any } \alpha \geq 0.$$

The space $(\Omega, \mathcal{X}, \tilde{\mathcal{E}})$ is referred as a sub-linear expectation space. We make the assumption that if $X \in \mathcal{X}$, then $\phi(X) \in \mathcal{X}$ for every $\phi \in \mathbb{L}_{lip}(\mathbb{R})$, where $\mathbb{L}_{lip}(\mathbb{R})$ is the set of function $\phi$ locally Lipschitz, taking values in $\mathbb{R}$, and satisfies:

$$|\phi(x_1) - \phi(x_2)| \leq K(1 + |x_1|^n + |x_2|^n)|x_1 - x_2|, \forall x_1, x_2 \in \mathbb{R}$$

where $K$ is a positive constant and $n \in \mathbb{N}$ depends on $\phi$.

**Definition 2.2:** Let $(\Omega_1, \mathcal{X}_1, \tilde{\mathcal{E}}_1)$ and $(\Omega_2, \mathcal{X}_2, \tilde{\mathcal{E}}_2)$ be sub-linear expectation spaces, Random variables $X_1 \in \mathcal{X}_1$ and $X_2 \in \mathcal{X}_2$ are identically distributed (we write $X_1 \stackrel{d}{=} X_2$) if for each $\phi \in \mathbb{L}_{lip}(\mathbb{R})$, the following holds:

$$\tilde{\mathcal{E}}_1(\phi(X_1)) = \tilde{\mathcal{E}}_2(\phi(X_2)).$$

**Definition 2.3:** We say that $X_2 \in \mathcal{X}$ is independent of $X_1 \in \mathcal{X}$ with respect to $\tilde{\mathcal{E}}$, if for all $\phi \in \mathbb{L}_{lip}(\mathbb{R}^2)$,

$$\tilde{\mathcal{E}}(\phi(X_1, X_2)) = \tilde{\mathcal{E}}(\tilde{\mathcal{E}}(\phi(x, X_2))|_{x=X_1}).$$

If $X_1 \overset{d}{=} X_2$ and $X_2$ is independent of $X_1$, then we say $X_2$ is a copy of $X_1$.

**Definition 2.4:** Assume that $X \in \mathcal{X}$. We say that $X$ is $G$-normally distributed, if $V(t,x) = \tilde{\mathcal{E}}(\phi(x+\sqrt{t}X))$ is the only viscosity solution of the next non-linear PDE:

$$\begin{cases} \partial_t V - G(\partial_{xx}^2 V) = 0, \\ V(0,x) = \phi(x), \end{cases} (t,x) \in [0,T] \times \mathbb{R}$$

for each $\phi \in \mathbb{L}_{lip}(\mathbb{R})$.

**Definition 2.5:** The process $(B_t^G)_{t \geqslant 0}$ defined on $(\Omega, \mathcal{X}, \tilde{\mathcal{E}})$ is said to be a $G$-Bm if for $t, s \geqslant 0$, we have:

(i) $B_0^G(\omega) = 0$;

(ii) The increments $(B_{t+s}^G - B_t^G)$ are $N(0;[\sigma_{\min} s, \sigma_{\max} s])$-distributed, independent of $(B_{t_1}^G, B_{t_2}^G, \ldots, B_{t_n}^G)$, $\forall n \in \mathbb{N}$ and $t_1, t_2, \ldots, t_n$, within [0, $T$] with

$$\sigma_{\min} = -\tilde{\mathcal{E}}[-(B_1^G)^2] \text{ and } \sigma_{\max} = \tilde{\mathcal{E}}[(B_1^G)^2].$$

Now, let $\Omega = C_0(\mathbb{R}^+)$ denote the space of all continuous trajectories $(\psi_t)_{t \in \mathbb{R}^+}$, taking values in $\mathbb{R}$, with $\omega_0 = 0$ and endowed by the next distance:

$$\tau(\psi^1, \psi^2) := \sum_{n \geq 1} \frac{1}{2^n} \left[ \left( \max_{t \in [0,n]} |\psi_t^1 - \psi_t^2| \right) \wedge 1 \right].$$

We put $\Omega_t := \{\psi_{\cdot \wedge t} : \omega \in \Omega\}$ for $t \geq 0$. Let the canonical process $B_t^G(\psi) = \psi_t$ for $\psi \in \Omega$ and $t \geq 0$. Let $T \in [0, \infty[$, and put

$$\mathcal{L}_{ip}(\Omega_T) = \{\phi(B_{t_1}^G, \ldots, B_{t_n}^G) : n \geq 1, t_1, \ldots, t_n \in [0,T], \phi \in \mathbb{L}_{lip}(\mathbb{R})\},$$

It is not difficult to see that in case $t \leqslant T$ then, $\mathcal{L}_{ip}(\Omega_t)$ is included in $\mathcal{L}_{ip}(\Omega_T)$. We additionally place

$$Lip(\Omega) = \bigsqcup_{k=1}^{\infty} Lip(\Omega_k).$$

Peng established a sub-linear expectation $\mathbb{E}^G$ called $G$-expectation on $(\Omega, \mathcal{L}_{ip}(\Omega))$ ensuring this canonical process $(B_t^G)_{t\geqslant 0}$ becomes a $G$-Bm. In what follows we consider this $G$-expectation.

Let the set $\mathcal{L}_G^p(\Omega)$ (and $\mathcal{L}_G^p(\Omega_T)$) of random variables defined on $\mathcal{L}_{ip}(\Omega)$ (and $\mathcal{L}_{ip}(\Omega_T)$) for $p \geq 1$, with the norm given as follows:

$$\| U \|_p^G := [\mathbb{E}^G(|U|^p)]^{\frac{1}{p}} < \infty.$$

Note that $\mathcal{L}_G^p(\Omega)$ and $\mathcal{L}_G^p(\Omega_T)$ are Banach spaces. We take $\mu_T = \{t_0, \ldots, t_N\}$ as a partition of the interval $[0, T]$, and considering the elementary processes structured as:

$$\omega_t(\psi) = \sum_{k=1}^{N} \varsigma_k(\psi) \mathcal{I}_{[t_{k-1}, t_k[}(t),$$

in which $\varsigma_k \in \mathcal{L}_G^p(\Omega_{t_k})$ for $k$ within the sequence $\{0, 1, \ldots, N-1\}$. These processes will be symbolized as $\mathcal{M}_G^p(0, T)$ and $\overline{\mathcal{M}_G^p}(0, T)$ is its completion subject to the following norm

$$\| \omega \|_{p,T}^G = \left[ \frac{1}{T} \int_0^T \mathbb{E}^G(|\omega_t|^p) dt \right]^{\frac{1}{p}}.$$

Note that for $1 \leq p \leq q$, we have $\overline{\mathcal{M}_G^q}(0, T)$ is included in $\overline{\mathcal{M}_G^p}(0, T)$. For an elementary process $\omega \in \mathcal{M}_G^2(0, T)$, the $G$-integral of $\omega$ with regard to this $G$-Bm is,

$$I(\omega) = \int_0^T \omega(s) dB_s^G := \sum_{k=1}^{N} \varsigma_k (B_{t_k}^G - B_{t_{k-1}}^G).$$

We can extend continuously the mapping $\omega \to I(\omega)$ to $\overline{\mathcal{M}_G^2}(0, T)$.

**Definition 2.6:** The processes

$$\langle B^G \rangle_t := (B_t^G)^2 - 2\int_0^t B_s^G dB_s^G$$

represents the quadratic variation associated with the $G$-Bm.

For any $\omega \in \mathcal{M}_G^1(0, T)$, consider the mapping

$$\Lambda_{0,T}(\omega) : \mathcal{M}_G^{1,0}(0, T) \mapsto \mathcal{L}_G^1(\mathcal{F}_T),$$

where

$$\Lambda_{0,T}(\omega) = \int_0^T \omega_s d\langle B^G \rangle_s := \sum_{k=1}^{N} \varsigma_k(\psi)(\langle B^G \rangle_{t_k} - \langle B^G \rangle_{t_{k-1}}).$$

Since $\Lambda_{0,T}(\omega)$ is linear and continuous, then we can extended it continuously to $\overline{\mathcal{M}_G^1}(0,T)$.

**Definition 2.7:** We defined on $(\Omega, \mathfrak{T}(\Omega))$ the capacity $\overline{C}(.)$ associated with a set weakly compact of probability measures $\mathcal{U}$ as follows:

$$\overline{C}(A) = \sup_{P \in \mathcal{U}} P(D), \quad A \in \mathfrak{T}(\Omega),$$

and $\mathfrak{T}(\Omega)$ is the $\sigma$ -algebra of $\Omega$.

**Remark 2.8:** We say $D \in \mathfrak{T}(\Omega)$ is polar, if $\overline{C}(D) = 0$, and in connection with this, a property holds quasi surely (q.s.) when it is satisfied everywhere except on a polar.

Now, we give the inequalities of "Burkholder-Davis-Gundy", in the $G$-frame, ($G$-BDG estimates).

**Lemma 2.9:** *For all* $0 \le v \le t \le T$, $p \ge 2$, *and* $\phi \in \overline{\mathcal{M}_G^p}(0,T)$, *it follows that,*

$$\mathbb{E}^G(\sup_{v \le r \le t} |\mathcal{V}_\phi(v,r)|^p) \le C_1 (t-v)^{\frac{p-2}{2}} \int_v^t \mathbb{E}^G(|\phi_u|^p) du,$$

*where* $\mathcal{V}_\phi(v,s) = \int_v^s \phi_u dB_u^G$, *and* $C_1$ *is a positive constant independent of* $\phi$.

**Lemma 2.10:** *For all* $0 \le v \le t \le T$, $p \ge 1$, *and* $\phi \in \overline{\mathcal{M}_G^p}(0,T)$, *we have*

$$\mathbb{E}^G(\sup_{v \le r \le t} |\mathcal{W}_\phi(v,r)|^p) \le C_2 (t-v)^{p-1} \int_v^t \mathbb{E}^G(|\phi_u|^p) du,$$

*where* $\mathcal{W}_\phi(v,s) = \int_v^s \phi_u d\langle B^G \rangle_u$, *and* $C_2$ *is a positive constant independent of* $\phi$.

**Lemma 2.11:** *If* $\chi$ *is a function continuous, defined on* $[a,b]$, *and* $\alpha$, $\beta \ge 0$, *where*

$$\chi(t) \le \alpha + \beta \int_a^t \chi(u) du$$

*wherein* $a \le t \le b$. *Then*

$$\chi(t) \le \alpha e^{\beta(t-a)}.$$

We also need the following estimate.

**Lemma 2.12:** *For* $p>1$, $\eta>0$ *and* $y,z\in\mathbb{R}$, *we have*

$$|y+z|^p \le \rho\left(\frac{\eta|y|^p+|z|^p}{\eta}\right)$$

*where* $\rho=(1+\eta^{\frac{1}{p-1}})^{p-1}$.

2.1 *Notations and assumptions*

In the following we mean by $\mathcal{C}_b([-\tau,0];\mathbb{R})$ the set of continuous functions $\Upsilon$ that are bounded, defined on $[-\tau,0]$ and taking values in $\mathbb{R}$, supplied with the next norm:

$$\|\Upsilon\|=\sup_{-\tau\le u\le 0}|\Upsilon(u)|,\ \tau>0,$$

and $\{B_t^G, t\ge 0\}$ the $G$-Bm defined within $(\Omega,\mathcal{F},P)$, wherein $\mathcal{F}_t := \sigma(B_s^G;\ 0\le s\le t)$ is a filtration that satisfies the usual conditions. The functions $Q$, $D$, $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ are defined as outlined below:

$$\mathcal{N}:\mathcal{C}_b([-\tau,0];\mathbb{R})\mapsto\mathbb{R} \text{ and } \mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3:[0,T]\times\mathcal{C}_b([-\tau,0];\mathbb{R})\mapsto\mathbb{R},$$
$$Q:\mathbb{R}\times\mathcal{C}_b([-\tau,0];\mathbb{R})\mapsto\mathbb{R}.$$

Let us consider the $G$-NSDE

$$dQ(Z(t))=\mathcal{J}_1(t,Z_t)dt+\mathcal{J}_2(t,Z_t)dk\langle B^G\rangle_t+\mathcal{J}_3(t,Z_t)dB_t^G \tag{2.1}$$

Wherein $Q(Z(t)) = (Z(t) - \mathcal{N}(Z_t))$, $Z_t=\{Z(t+u) : -\tau\le u\le 0,\ \tau>0\}\in \mathcal{C}_b([-\tau,0];\mathbb{R})$; Keep in mind that $\mathcal{J}_1,\mathcal{J}_2$ and $\mathcal{J}_3$ are belong to $\overline{\mathcal{M}_G^2}([-\tau, T];\mathbb{R})$. Denote by $\{\langle B^G\rangle_t,\ t\ge 0\}$ the process of quadratic variation of the $G$-Bm. The initial condition of the equation (2.1) is:

$$Z_0=\Upsilon=\{\Upsilon(u):-\tau\le u\le 0\} \tag{2.2}$$

where $\Upsilon$ is an $\mathcal{F}_0$-measurable random variable taking values in $\mathcal{C}_b([-\tau,0];\mathbb{R})$. Additionally, $\Upsilon\in\overline{\mathcal{M}_G^2}([-\tau,0];\mathbb{R})$. Thus, one can write the equation (2.1) here is how

$$Q(Z(t))=Q(Z(0))+\int_0^t\mathcal{J}_1(u,Z_u)du+\int_0^t\mathcal{J}_2(v,Z_u)d\langle B\rangle_u+\int_0^t\mathcal{J}_3(u,Z_u)dB_u \tag{2.3}$$

Assume that

$(\mathbf{A}_1)$**:** For each $\phi\in\mathcal{C}_b([-\tau,0];\mathbb{R})$ and $0\le t\le T$,

$$\sum_{i=1}^{3} |\mathcal{J}_i(t,\phi)|^2 \leq \rho(1+\|\phi\|^2),$$

in which $\rho(.) : \mathbb{R}_+ \to \mathbb{R}_+$ is a function which non descending, concave, and vanishes at the origin 0, and $\int_{0^+} \frac{1}{\rho(x)} dx = \infty.$ With $\rho$ being concave and and $\rho(0)=0,$ there exist $\alpha, \beta \in \mathbb{R}_+$ for which $\rho(x) \leq \alpha + \beta x, \quad \forall x \geq 0.$

**$(\mathbf{A}_2)$:** Let $\mathcal{N}(0)=0,$ and there exists $\mu \in (0,1)$ so that

$$|\mathcal{N}(\phi) - \mathcal{N}(\eta)| \leq \mu \|\phi - \eta\|$$

for every $\phi, \eta \in \mathcal{C}_b([-\tau,0];\mathbb{R}).$ This implies

$$|\mathcal{N}(\phi)| \leq \mu \|\phi\|.$$

**Theorem 2.13:** [2] *Assume that conditions* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *are fulfilled. If* $\mathbb{E}^G(\|\Upsilon\|^p) < \infty,$ *wherein* $p \geq 2.$ *then it follows that*

$$\mathbb{E}^G(\sup_{-\tau \leq s \leq T} |Z(s)|^p) \leq K := \mathbb{E}^G \|\Upsilon\|^p + \lambda e^{\mu T},$$

*where* $\lambda$ *and* $\mu$ *positive constants.*

## 3. $\mathcal{L}_G^p(\Omega)$-continuity of the solution

This section provides the primary result of the article, which is the uniform estimate for $\sup_{0 \leq t \leq T} \mathbb{E}^G(|Z(t+\sigma) - Z(t)|^p)$ in terms of the initial condition. More precisely in terms of $\sup_{|t_1-t_2| \leq \sigma} \mathbb{E}^G(|\Upsilon(t_1) - \Upsilon(t_2)|^p).$ To this end, assume that the sample trajectories of the solution are all continuous. We claim that for any $0 < \sigma < \tau,$ there exists $\mathcal{S}_\delta > 0$ in order that

$$\text{if } t_1, t_2 \in [-\tau, 0] \text{ and } |t_1 - t_2| \leq \sigma$$

Then

$$\mathbb{E}^G(|\Upsilon(t_1) - \Upsilon(t_2)|^p) \leq \mathcal{S}_\sigma \tag{3.1}$$

where $\sigma$ is small enough. This assumption is done because of the $L_G^p(\Omega)$-continuity of all the sample trajectories of $\Upsilon$ on $[-\tau,0],$ and hence, the uniform continuity of $\Upsilon$ on $[-\tau,0].$ In addition, we set forth a new space $L_{G,\mathcal{F}}^p([-\tau,0];\mathbb{R}),$ which represents the family of $\mathcal{F}$-measurable random variables $\phi,$ taking values in $\mathcal{C}_b([-\tau,0];\mathbb{R})$ together with $\mathbb{E}^G(\|\phi\|^p) < \infty$ for $p \geq 2.$

**Lemma 3.1:** *Given that* $p \geq 2$. *Then for each* $\Upsilon \in L^p_{G,\mathcal{F}}([-\tau,0];\mathbb{R})$ *and for each* $0 \leq t \leq \sigma$,

$$\mathbb{E}^G(|Q(Z(t+\sigma)) - Q(Z(t))|^p) \leq L\sigma^{\frac{p}{2}},$$

*where* $L$ *represents some positive constant.*

***Proof:*** From equation (2.3), and using the following inequality

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq n^{p-1}\sum_{i=1}^{n} x_i^p \tag{3.2}$$

along with $G$-BDG estimates and Hölder's inequality, one can show that

$$\mathbb{E}^G(|Q(Z(t+\sigma)) - Q(Z(t))|^p) \leq (3\sigma)^{p-1}\int_t^{t+\sigma} \mathbb{E}^G|\mathcal{J}_1(u,Z_u)|^p\, du$$

$$+C_2(3\sigma)^{p-1}\int_t^{t+\sigma}\mathbb{E}^G|\mathcal{J}_2(u,Z_u)|^p\, du + 3^{p-1}C_1\sigma^{\frac{p}{2}-1}\int_t^{t+\sigma}\mathbb{E}^G|\mathcal{J}_3(u,Z_u)|^p\, du.$$

By the hypothesis $(\mathrm{A}_1)$,

$$\mathbb{E}^G(|Q(Z(t+\sigma)) - Q(Z(t))|^p)$$

$$\leq (3^{p-1}C_1\sigma^{\frac{p}{2}-1} + (C_2+1)(3\sigma)^{p-1})\int_t^{t+\delta}\mathbb{E}^G[(\rho(1+\|Z_u\|^2))^{\frac{p}{2}}]du$$

$$\leq (3^{p-1}C_1\sigma^{\frac{p}{2}-1} + (C_2+1)(3\sigma)^{p-1})\int_t^{t+\sigma}\mathbb{E}^G[(a+b+b\|Z_u\|^2)^{\frac{p}{2}}]du.$$

Using inequality (3.2), further we can write

$$\mathbb{E}^G(|Q(Z(t+\sigma)) - Q(Z(t))|^p)$$

$$\leq 2^{\frac{p}{2}-1}(3^{p-1}C_1\sigma^{\frac{p}{2}-1} + (C_2+1)(3\sigma)^{p-1})\int_t^{t+\sigma}\mathbb{E}^G(\sqrt{(a+b)^p} + \sqrt{b^p}\,\|Z_u\|^p)du$$

$$\leq 2^{\frac{p}{2}-1}(3^{p-1}C_1\sigma^{\frac{p}{2}-1} + (C_2+1)(3\sigma)^{p-1})\sigma\sqrt{(a+b)^p}$$

$$+2^{\frac{p}{2}-1}(3^{p-1}C_1\sigma^{\frac{p}{2}-1} + (C_2+1)(3\sigma)^{p-1})\sqrt{b^p}\int_t^{t+\sigma}\mathbb{E}^G(\sup_{-\tau\leq r\leq v}|Z(r)|^p)du.$$

Based on the theorem 2.13, we derive

$$\mathbb{E}^G|Q(Z(t+\sigma)) - Q(Z(t))|^p \leq \frac{(3\sqrt{2})^p}{6}(C_1\sigma^{\frac{p}{2}} + (C_2+1)\sigma^p)\sqrt{(a+b)^p}$$

$$+\frac{(3\sqrt{2})^p}{6}(C_1\sigma^{\frac{p}{2}}+(C_2+1)\sigma^p)\sqrt{b^p}K.$$

Hence,

$$\mathbb{E}^G(\,|Q(Z(t+\sigma))-Q(Z(t))\,|^p)\leq L\sigma^{\frac{p}{2}} \tag{3.3}$$

where $L=\frac{(3\sqrt{2})^p}{6}\,(C_1+(C_2+1)\sigma^{\frac{p}{2}})\,[\sqrt{(a+b)^p}+\sqrt{b^p}K]$, and the proof is complete.

The main result reads as the following.

**Theorem 3.2:** *Let* $\Upsilon\in L^p_{G,\mathcal{F}}([-\tau,0];\mathbb{R})$. *Assume that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold, and there exists* $\mu\in(0,1)$ *in which*

$$\mathbb{E}^G(\,|\mathcal{N}(\phi)-\mathcal{N}(\psi)\,|^p)\leq\mu^p\sup_{-\tau\leq\theta\leq0}\mathbb{E}^G(\,|\phi(\theta)-\psi(\theta)\,|^p) \tag{3.4}$$

for each $\phi,\psi\in L^p_{G,\mathcal{F}}([-\tau,0];\mathbb{R})$. Then for each $0\leq t\leq T$ and $0<\sigma<\tau$,

$$\sup_{0\leq t\leq T}\mathcal{G}_Z(t,\sigma,p)\leq A\mathcal{S}_\sigma+B\sigma^{\frac{p}{2}},$$

where $\mathcal{G}_Z(t,\sigma,p)=\mathbb{E}^G[\,|Z(t+\sigma)-Z(t)\,|^p]$ and $A,B\geq0$.

***Proof:*** By Lemma 2.12, we have

$$\mathcal{G}_Z(t,\sigma,p)=\mathbb{E}^G(\,|Q(Z(t+\sigma))-Q(Z(t))-\mathcal{N}(Z_t)+\mathcal{N}(Z_{t+\sigma})\,|^p)$$
$$\leq\rho\left(\frac{1}{\eta}\mathbb{E}^G(\,|\mathcal{N}(Z_{t+\sigma})-\mathcal{N}(Z_t)\,|^p)+\mathbb{E}^G(\,|Q(Z(t+\sigma))-Q(Z(t))\,|^p)\right).$$

Using (3.4) and (3.3), we get

$$\mathcal{G}_Z(t,\sigma,p)\leq\rho\left(\frac{\mathcal{G}_Z(t+\theta,\sigma,p)}{\eta}\times\mu^p+L\sigma^{\frac{p}{2}}\right).$$

Let $\eta=\left(\frac{\mu}{1-\mu}\right)^{p-1}$. We can write

$$\mathcal{G}_Z(t,\sigma,p)\leq\mu\sup_{-\tau\leq\theta\leq0}\mathcal{G}_Z(t+\theta,\sigma,p)+\frac{L\sigma^{\frac{p}{2}}}{(1-\mu)^{p-1}},$$

which is true for every $0\leq t\leq T$. Hence,

$$\sup_{0\leq t\leq T}\mathcal{G}_Z(t,\sigma,p)\leq\mu\sup_{-\tau\leq t\leq T}\mathcal{G}_Z(t,\sigma,p)+\frac{L\sigma^{\frac{p}{2}}}{(1-\mu)^{p-1}}.$$

Note that

$$\sup_{0\le t\le T}\mathcal{G}_Z(t,\sigma,p)\le \mu\sup_{-\tau\le t\le 0}\mathcal{G}_Z(t,\sigma,p)+\kappa\sup_{0\le t\le T}\mathcal{G}_Z(t,\sigma,p)+\frac{L\sigma^{\frac{p}{2}}}{(1-\mu)^{p-1}}$$

and which leads to

$$\sup_{0\le t\le T}\mathcal{G}_Z(t,\sigma,p)\le \frac{\mu}{1-\mu}\sup_{-\tau\le t\le 0}\mathcal{G}_Z(t,\sigma,p)+\frac{L\sigma^{\frac{p}{2}}}{(1-\mu)^{p}} \tag{3.5}$$

According to (3.1), we get

$$\begin{aligned}
\sup_{-\tau\le t\le 0}\mathcal{G}_Z(t,\sigma,p)&\le \sup_{-\tau\le t\le -\sigma}\mathcal{G}_Z(t,\sigma,p)+\sup_{-\sigma\le t\le 0}\mathcal{G}_Z(t,\sigma,p)\\
&\le \mathcal{S}_\sigma+2^{p-1}\sup_{-\sigma\le t\le 0}[\mathbb{E}^G(|Z(0)-Z(t)|^p)+\mathbb{E}^G(|Z(t+\sigma)-Z(0)|^p)]\\
&\le \mathcal{S}_\sigma+2^{p-1}\mathcal{S}_\sigma+2^{p-1}\sup_{-\sigma\le t\le 0}\mathbb{E}^G(|Z(t+\sigma)-Z(0)|^p)\\
&\le (1+2^{p-1})\mathcal{S}_\sigma+2^{p-1}\sup_{0\le t\le \sigma}\mathbb{E}^G(|Z(t)-\Upsilon(0)|^p).
\end{aligned}$$

Plugging this into (3.5), we obtain

$$\begin{aligned}
\sup_{0\le t\le T}\mathcal{G}_Z(t,\sigma,p)&\le \frac{\mu}{1-\mu}(1+2^{p-1})\mathcal{S}_\sigma\\
&+\frac{\mu\times 2^{p-1}}{1-\mu}\sup_{0\le t\le\sigma}\mathbb{E}^G\left|Z(t)-\Upsilon(0)\right|^p+\frac{L}{(1-\mu)^p}\sigma^{\frac{p}{2}}
\end{aligned} \tag{3.6}$$

It remains to estimate $\sup_{0\le t\le\sigma}\mathbb{E}^G(|Z(t)-\Upsilon(0)|^p)$. In the same way used in Lemma 3.1, we have for all $0\le t\le\sigma$,

$$\mathbb{E}^G(|Q(Z(t))-Q(Z(0))|^p)\le L\sigma^{\frac{p}{2}}, \tag{3.7}$$

On the other hand,

$$\mathbb{E}^G|Z(t)-\Upsilon(0)|^p=\mathbb{E}^G|Q(Z(t))-Q(Z(0))+\mathcal{N}(Z_t)-\mathcal{N}(\Upsilon)|^p,$$

which implies that

$$\begin{aligned}
\mathbb{E}^G(|Z(t)-\Upsilon(0)|^p)&\le \mu\sup_{-\tau\le\theta\le 0}\mathbb{E}^G(|Z(t+\theta)-\Upsilon(\theta)|^p)\\
&+\frac{\mathbb{E}^G|Q(Z(t))-Q(Z(0))|^p}{(1-\mu)^{p-1}}.
\end{aligned}$$

Thus, by (3.7), we get

$$\mathbb{E}^G(|Z(t)-\Upsilon(0)|^p)\le\mu\sup_{-\tau\le\theta\le0}\mathbb{E}^G(|Z(t+\theta)-\Upsilon(\theta)|^p)+\frac{L\sigma^{\frac{p}{2}}}{(1-\mu)^{p-1}} \tag{3.8}$$

According to (3.1),

$$\begin{aligned}\sup_{-\tau\le\theta\le0}\mathbb{E}^G(|Z(t+\theta)-\Upsilon(\theta)|^p)&\le\sup_{-\tau\le\theta\le-t}\mathbb{E}^G(|Z(t+\theta)-\Upsilon(\theta)|^p)\\&\quad+\sup_{-t\le\theta\le0}\mathbb{E}^G(|Z(t+\theta)-\Upsilon(\theta)|^p)\\&\le\mathcal{S}_\sigma+\sup_{-t\le\theta\le0}\mathbb{E}^G(|Z(t+\theta)-\Upsilon(\theta)|^p)\\&\le\mathcal{S}_\sigma+\sup_{-t\le\theta\le0}\mathbb{E}^G(|Z(t+\theta)+\Upsilon(0)-\Upsilon(0)-\Upsilon(\theta)|^p)\end{aligned} \tag{3.9}$$

Using Lemma 2.12, we can write

$$\begin{aligned}&\sup_{-t\le\theta\le0}\mathbb{E}^G(|Z(t+\theta)-\Upsilon(0)+\Upsilon(0)-\Upsilon(\theta)|^p)\\&\le\sup_{-t\le\theta\le0}\left(\frac{\mathbb{E}^G(|Z(t+\theta)-\Upsilon(0)|^p)}{\sqrt{\mu}}+\lambda\mathbb{E}^G(|\Upsilon(0)-\Upsilon(\theta)|^p)\right)\end{aligned}$$

where $\lambda=(1-\mu^{\frac{1}{2(p-1)}})^{1-p}$, it can be concluded that

$$\begin{aligned}&\sup_{-t\le\theta\le0}\mathbb{E}^G(|Z(t+\theta)-\Upsilon(0)+\Upsilon(0)-\Upsilon(\theta)|^p)\\&\le\sup_{-t\le\theta\le0}\left(\frac{\mathbb{E}^G(|Z(t+\theta)-\Upsilon(0)|^p)}{\sqrt{\mu}}+\lambda\mathcal{S}_\sigma\right)\\&\le\frac{\sup_{0\le t\le\sigma}\mathbb{E}^G(|Z(t)-\Upsilon(0)|^p)}{\sqrt{\mu}}+\lambda\mathcal{S}_\sigma.\end{aligned}$$

Substituting the obtained inequality in (3.9), we get

$$\sup_{-\tau\le\theta\le0}\mathbb{E}^G|Z(t+\theta)-\Upsilon(\theta)|^p\le(1+\lambda)\mathcal{S}_\sigma+\frac{\sup_{0\le t\le\sigma}\mathbb{E}^G(|Z(t)-\Upsilon(0)|^p)}{\sqrt{\mu}}$$

which, being put (3.8), leads to

$$\mathbb{E}^G|Z(t)-\Upsilon(0)|^p\le\sqrt{\mu}\sup_{0\le t\le\sigma}\mathbb{E}^G|Z(t)-\Upsilon(0)|^p+(\lambda+1)\mu\mathcal{S}_\sigma+\frac{L\sigma^{\frac{p}{2}}}{(1-\mu)^{p-1}}.$$

This holds for all $0\le t\le\sigma$. Thus,

$$\sup_{0\le t\le\sigma}\mathbb{E}^G(|Z(t)-\Upsilon(0)|^p)\le\frac{(\lambda+1)\mu(1-\mu)^{p-1}\mathcal{S}_\sigma+L\sigma^{\frac{p}{2}}}{(1-\sqrt{\mu})(1-\mu)^{p-1}}$$

Finally, by substituting the inequality above into (3.6), we can write

$$\sup_{0\le t\le T}\mathcal{G}_Z(t,\sigma,p)\le A\mathcal{S}_\sigma+B\sigma^{\frac{p}{2}},$$

where

$$A=\frac{((1+2^{p-1})(1-\sqrt{\mu})+2^{p-1}\mu(\lambda+1))\mu}{(1-\mu)(1-\sqrt{\mu})},$$

and

$$B=\frac{((1+\sqrt{\mu})2^{p-1}\mu+(1-\mu))L}{(1-\mu)^{p+1}}.$$

The proof is complete.

Finally, an example to demonstrate the achieved results.

**Example 3.3:** We take $B^G(t)$ as a real $G$-Bm. Consider the one-dimensional G-NSDE as follows:

$$d\left(Z(t)-\frac{\sin(Z_t)}{3}\right)=Z_t\sin(t)dt+(\cos(Z_t)-t)d\langle B^G\rangle_t$$
$$+\frac{Z_t}{\sqrt{1+(Z_t)^2}}dB_t^G,\ t\in[0,T], \tag{3.10}$$

with the initial data

$$Z_0:=Z(\theta)=\theta+1,\quad \theta\in[-\tau,0].$$

Note that $Z_t=Z(t+\theta)$ represents the past of the process $Z(t)$ within the time interval $[t-\tau,t]$. We have

$$\mathcal{J}_1(t,Z_t)=Z_t\sin(t),\ \mathcal{J}_2(t,Z_t)=\cos(Z_t)-t,\ \mathcal{J}_3(t,Z_t)=\frac{Z_t}{\sqrt{1+(Z_t)^2}}.$$

Thus,

$$
\begin{aligned}
|Z_t \sin(t)|^2 + |\cos(Z_t) - t|^2 + \left| \frac{Z_t}{\sqrt{1+(Z_t)^2}} \right|^2 &\leq |Z_t|^2 + 2 + 2T^2 + 1 \\
&\leq 3 + 2T^2 + \sup_{-\tau \leq \theta \leq 0} |Z(t+\theta)|^2 \\
&\leq 3 + 2T^2 + \| Z_t \|^2 \\
&\leq \rho(1 + \| Z_t \|^2),
\end{aligned}
$$

where $\rho(x) = (3 + 2T^2)x$. Since $\mathcal{N}(Z_t) := \frac{\sin(Z_t)}{3}$, conditions $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ are obviously satisfied. Thus by Theorem 3.2, we have

$$\lim_{\sigma \to 0} (\sup_{0 \leq t \leq T} \mathcal{G}_Z(t, \sigma, p)) = 0 \ \text{(q.s.) for all } p \geq 2.$$

This illustrates the continuity of the moment of order $p$ to the solution of equation (3.10).

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