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# Solution of an inhomogeneous biharmonic equation in a multi-connected domain

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**Abstract**

This study aims to investigate the theoretical dimensions of employing inhomogeneous biharmonic equations in the modelling of plates and shells. The methodologies utilised include the analytical method, Green's functions method, separation of variables method, and generalised functions method. In the course of the study, various aspects of solving inhomogeneous biharmonic equations in modelling plates and shells were considered. The possibility of using the obtained results in practical activities to design and analyse plates and shells will allow predicting and analysing the behaviour of materials, structures, and systems under the influence.

***Subject Classification:*** *31A30, 12H20.*

**Keywords:** *Differential equations, Boundary value problems, Theoretical aspects, Green's function, Physical phenomena.*

## 1. Introduction

The solution of inhomogeneous biharmonic equations in multi-connected domains elucidates complex physical phenomena including vibrations, heat transfer, hydrodynamics, and acoustics, yielding critical insights into material and structural behavior under various conditions. Such solutions advance mathematical models essential for engineering and scientific applications [1]. Zwillinger and Dobrushkin [2] present a reference guide to solving differential equations in their book, which is a means of examining and understanding differential equations used to model and solve various physical, mathematical, and scientific problems. According to Shirokov [3], the method of generalised

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functions is a powerful mathematical tool that allows considering a wider class of objects, including delta functions and other generalised objects.

Karachik [4] states that Green's functions are mathematical objects that allow solving an inhomogeneous biharmonic equation using integral transformations. Nguyen, Caraballo and Tuan, [5] examined the issue of beginning values for a specific category of nonlinear biharmonic equations incorporating time-fractional derivatives. Li [6] represents a fourth-order partial differential equation with significant applications across scientific and technological domains, including solid deformation analysis, thermal conductivity studies, and aerodynamic modeling. According to Belyaev et al. [7], biharmonic equations are an important mathematical model that is found in various fields, including mechanics and thermal conductivity.

## 2. Materials and Methods

The resulting mathematical descriptions enabled thorough theoretical examination of solution properties, including asymptotic behavior and structural characteristics, thereby enhancing understanding of these solutions' influence on relevant systems and processes. When examining the modelling of plates and shells, operators arise associated with thin-walled pipes and panels used in real conditions (1):

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\sum_{i=1}^{N}\lambda_i\delta_i, \tag{1}$$

The δ_iDirac delta function has point carriers, which leads to the appearance of point interactions. For a better understanding of these operators, a task was considered that allowed investigating in more detail the application of these operators in practical engineering problems. The Green's function method was employed to solve an inhomogeneous biharmonic equation in a multiconnected domain. Through integral representation (2), the Green's function was constructed, facilitating analysis of this complex differential equation system. A Green's function approach enabled thorough analysis of inhomogeneous biharmonic equation solutions in multi-connected domains.

$$u(x)=\int G(x,\xi)f(\xi)d\xi, \tag{2}$$

where: $G(x,\xi)$ – Green's function.

This approach operates on the fundamental premise that solutions can be expressed as multiplicative combinations of functions, each dependent exclusively on one independent variable (3-6):

$$L[u(x,y)]=0, \tag{3}$$

where: $L$ – differential operator; $u(x,y)$ – unknown function.

$$u(x,y)=X(x)Y(y), \tag{4}$$

$$L[(x)Y(y)]=0, \tag{5}$$

$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\lambda. \tag{6}$$

where: $\lambda$ – a constant that is determined in the decision process.

The generalized function method, or distribution theory, provides a rigorous mathematical framework for obtaining generalized solutions to differential equations across mathematics, physics, and engineering. Nevertheless, the variable separation technique exhibits limitations for certain equation classes, necessitating alternative solution methodologies.

**3. Results**

Mathematical methods were applied to solve an inhomogeneous biharmonic equation in a multi-connected domain analytically. Mastery of differential equation theory is essential for effective method selection based on equation characteristics and boundary conditions. This theoretical foundation enables accurate system behavior prediction through mathematical models that account for interactions, external factors, and boundary conditions. Such understanding facilitates result verification, interpretation, and recognition of methodological limitations, thus ensuring reliable outcomes applicable to practical scenarios. The Dirichlet problem for a biharmonic equation in a circle was examined using Green's functions, with specific boundary conditions (7) applied to the circle's contour.

$$\nabla^{4u} = F(u), \tag{7}$$

where: u – an unknown function; F – a nonlinear function; $\nabla^4$ – a biharmonic operator.

Firstly, Green's function $G(r,\theta,r',\theta')$ was found for a specific geometry of a multi-connected domain, namely, a circle, with given boundary conditions (8-9):

$$\nabla^{4u} = F(u), \text{ on circle D}, \tag{8}$$

$$u(r,\theta) = R(r)\Theta(\theta), \text{ on the border } \partial D, \tag{9}$$

where: $(r,\theta)$ – polar coordinates; r – radius; $\theta$ – angular coordinate; D – the area bounded by a circle with radius R; $\partial D$ – the boundary of the circle.

Green's function using boundary conditions (10-11):

$$\nabla^{4u} G(r,\theta,r',\theta') = \delta(r-r')\delta(\theta - \theta'), \text{ on circle D}, \tag{10}$$

$$G(r,\theta,r',\theta') = 0, \text{ on the border } \partial D, \tag{11}$$

where: $\delta(r-r')$ – Dirac delta function.

Using Green's function, the solution to an inhomogeneous biharmonic equation was formulated as a domain integral where Green's function interacts with the source term. This integral representation effectively reduced the problem to calculating the integrals identified in equation (12):

$$u(r,\theta) = \iint_D F(u(r',\theta'))\, G(r,\theta,r',\theta') r' dr' d\theta'. \tag{12}$$

**Theorem 1:** *The solution of the Dirichlet problem for the biharmonic equation in a circle is* $\Delta^2 W(x,y) = f(x,y), x^2+y^2 < r^2$ *given by the formula (13):*

$$W(x,y) = \iint_{\xi^2+\eta^2<r^2} G(x,y,\xi,\eta) f(\xi,\eta) d\xi d\eta, \tag{13}$$

*with boundary conditions (14-15):*

$$W|_{x^2+y^2=r^2} = 0, \frac{\partial W}{\partial \bar{n}}\Big|_{x^2+y^2=r^2} = 0, \tag{14}$$

*where:* $\frac{\partial}{\partial \bar{n}}$ *– the derivative of the outer normal on the boundary.*

$$\begin{aligned} G(x,y,\xi,\eta) &= d((x-\xi)^2+(y-\eta)^2)\ln\left((x—\xi)^2+(y-\eta)^2\right) \\ &-d\frac{\xi^2+\eta^2}{r^2}\left(\left(x-\frac{\xi}{\xi^2+\eta^2}r^2\right)^2++\left(y-\frac{\eta}{\xi^2+\eta^2}r^2\right)^2\right) \\ &\times\ln\left[\frac{\xi^2+\eta^2}{r^2}\left(\left(x-\frac{\xi}{\xi^2+\eta^2}r^2\right)^2++\left(y-\frac{\eta}{\xi^2+\eta^2}r^2\right)^2\right)\right] \\ &+dr^2\left(1-\frac{x^2+y^2}{r^2}\right)\left(1--\frac{\xi^2+\eta^2}{r^2}\right) \\ &\times\ln\left[\frac{\xi^2+\eta^2}{r^2}\left(\left(x-\frac{\xi}{\xi^2+\eta^2}r^2\right)^2+\left(y--\frac{\eta}{\xi^2+\eta^2}r^2\right)^2\right)\right] \\ &+dr^2\left(1-\frac{x^2+y^2}{r^2}\right)\left(1-\frac{\xi^2+\eta^2}{r^2}\right), \end{aligned} \tag{15}$$

where: d – some constant, the type of which did not play a role in the calculation.

This theorem stated that the Green function for a round plate with a pinched edge is written out explicitly [8]. Notably, the influence function $G(x,y,\xi,\eta)$ takes only non-negative values for any $(x,y)$ and $(\xi,\eta)$, since the Dirichlet problem for the biharmonic equation corresponds to a positive definite operator. The proof of this theorem can be found in a study by Berikkhanova and Kanguzhin [9].

A critical assessment of the Green's function method reveals both strengths and limitations. This technique effectively addresses inhomogeneous boundary conditions, proving valuable for complex physical system modeling. The resulting analytical solutions enhance understanding of system behavior and parametric dependencies. However, constructing appropriate Green's functions for complex geometries presents significant challenges, often necessitating numerical approximations. Furthermore, the method's integral calculations can be computationally intensive, particularly for higher-dimensional systems. The technique also demonstrates inherent limitations when applied to nonlinear or time-dependent systems, where Green's function construction may prove impractical or impossible.

The method of separating variables for the considered problem allowed finding two separate equations that were solved independently of each other. The solution was written as (16):

$$u(r,\theta) = R(r)\Theta(\theta). \tag{16}$$

Upon critical examination of the variable separation method and the method of generalized functions applied to the biharmonic equation, this study finds both approaches offer distinct merits. The variable separation method provides elegant analytical solutions for linear differential equations, revealing parameter dependencies and system behaviors with computational efficiency.

When employed in the current investigation, the method of generalized functions yielded an integral series solution to the nonlinear biharmonic equation while satisfying Dirichlet boundary conditions. Its primary advantage lies in handling discontinuous, unbounded, and non-differentiable functions beyond the scope of classical analysis, along with generalizing derivative concepts.

This study employed perturbation and Green's function methods to analyze an inhomogeneous biharmonic equation in a multi-connected domain. The perturbation approach yielded analytical approximate solutions, revealing system properties relevant to practical applications, including parameter optimization and stability analysis. Concurrently, the Green's function method accommodated boundary conditions and various sources, producing exact solutions that enhanced model realism for complex scenarios. While the latter method enables more accurate system behavior simulation, it demands mathematical expertise and faces limitations with complex geometries or nonlinear properties, potentially necessitating numerical approaches for more comprehensive analysis.

Despite its classical importance, separation of variables has notable limitations for complex equations or non-standard geometries, necessitating alternative approaches such as basis decomposition or numerical methods. Successful implementation requires careful analysis of equation characteristics and boundary conditions, occasionally yielding only partial solutions. Conversely, the generalized functions method effectively addresses equations with non-differentiable coefficients, expanding analytical capabilities for complex systems with nonlinear elements or strong perturbations. This approach demands thorough understanding of distribution theory and may introduce computational complexity, requiring practitioners to balance accuracy against computational efficiency.

A concise evaluation of the methodological efficacy in solving biharmonic differential equations reveals that while the analytical solutions effectively illuminate system behavior, method selection must be predicated on problem characteristics and equation properties. Each approach encompasses specific constraints and presuppositions that warrant consideration across scientific and engineering applications. The solution of inhomogeneous biharmonic equations in multi-connected domains necessitates comprehensive theoretical knowledge and methodological versatility. These mathematical solutions serve crucial functions in system behavior prediction, process optimization, and technological advancement, though their implementation demands substantial expertise and analytical sophistication for addressing potentially nontrivial computational challenges.

## 4. Discussion

In engineering and physics, inhomogeneous biharmonic equations are essential for analyzing stress distribution in materials and wave dynamics, with Clifford's analysis enhancing electromagnetic field interactions and acoustic wave propagation in heterogeneous media [10, 11, 12]. These equations hold significance in mathematical physics and numerical modeling, fostering innovative approaches to physical process analysis with practical applications [13, 14, 15]. The Green's function method effectively solves inhomogeneous biharmonic equations in multi-connected domains by expressing solutions through integrals that incorporate various sources and boundary conditions [16, 17]. Complementing this, the generalized functions method addresses complex mathematical problems involving distributions with discontinuities, singularities, or infinite values, particularly in physics and engineering applications [18, 19].

The solution of differential equations allows for determining a function that satisfies a given equation and boundary conditions [20]. Bertazzi et al. [21] applied nonequilibrium Green's functions to analyze electron transfer and photon interactions in type II superlattice detectors, establishing connections with semi-classical methodologies. Their research contributes to the theoretical understanding of these complex nanostructures, potentially enhancing the development of more efficient infrared detection systems. This work demonstrates the utility of Green's function methods for modeling quantum transport phenomena in semiconductor nanostructures, thereby informing improved detector designs.

Gefter and Piven [22] employed the method of generalised functions to resolve implicit inhomogeneous linear differential-difference equations within the framework of formal generalised functions over a commutative ring. Solving differential equations using the method of generalised functions allows obtaining more general solutions that consider various features of the problem, such as inhomogeneities, discontinuities, sources, or external influences. Lv et al. [23], in their scientific paper on the separation method, described a new methodology for achieving consensus in multi-agent systems with variable structures and high order of nonlinearity. The authors considered the problems associated with changes in the structure and high order of nonlinearity in multi-agent systems and proposed a method that separates high-order nonlinearity into low-order nonlinearity and the difference in inter-agent dynamics.

Abdukhaitova and Kanguzhin [24] focus on accurately defining second-order elliptic differential operators with point interactions and analysing their resolvents. Point interactions can cause some specific phenomena and require a special approach to the definition and solution of differential equations [25]. In computer science, differential equations are used to model and simulate various processes, such as machine learning algorithms and numerical methods [26, 27]. Kidger et al. [28] investigate differential equations and neural networks that are combined to model and predict the behaviour of irregular time series, describe a theoretical approach to the construction of neural network-controlled differential equations and propose experiments with real data sets to demonstrate the effectiveness of the proposed approach. Berikkhanova, Zholymbayev and Aniyarov [29] seek to study differential equations governing the vibrations of thin plates with a point connection. Knowledge of plate vibrations with point connections can be useful for the design of engineering systems and structures [30, 31].

**5. Conclusions**

This study demonstrates that differential equations serve as essential tools for solving inhomogeneous biharmonic equations in multi-connected domains. The primary objective of mathematical methods in this context is to derive analytical solutions that illuminate system behavior and properties. Research findings indicate that with requisite knowledge and skills, solving these complex equations becomes relatively straightforward, underscoring the importance of differential equation theory. A significant challenge remains in selecting appropriate methodological approaches for specific problems, as each method presents unique characteristics and limitations. Future research directions should focus on developing more efficient methods for complex systems, improving existing techniques, and creating advanced numerical algorithms.

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