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Pseudospectra and condition pseudospectra of multivalued linear operators on non-Archimedean Banach spaces

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Abstract

In the present paper, we extend the concepts of pseudo-spectra and condition pseudo-spectra of any closed relations in non-Archimedean Banach spaces. Moreover, many results are proved about them.

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1. Introduction

In the Archimedean context, Neumann [11] presented linear relations in relation to the diffirential operators with non-dense domain. Recently, Arens [3] had developed the linear relations in complex spaces. There are numerous works on linear relations, see, e.g. [1], [2], [5] and [13]. Sookoo [9] generalized several results concerning norms and linear operators on complex normed spaces and he considered terms for continuity of an F-norm on a complex locally convex space. In the end, he gave a few properties related to the absorbing subsets in complex F-normed spaces. Recently, Atiya, Ibrahem, and Adeeb [4] investigated the relation between faithful, finitely generated, separated acts and the one-to-one operators and examined the associated S-act of $\cosh A$ and its attributes. On the other hand, they established that if A is a continuous linear operator, hence

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$V_{\cosh A}$ is faithful and separated S-act, and if a complex Banach space V is finite-dimensional, then $V_{\cosh A}$ is infinitely generated.

In the non-Archimedean framework, we will extend several results of spectral theory of linear (bounded) operators to linear (closed) relations in non-Archimedean Banach spaces. Also the concept of pseudo-spectra of any linear (closed) relations is extended and studied.

In the rest of this study, $\mathbb{Q}_p$ will be the collection of p-adic numbers, **K** is a complete field with a nontrivial non-Archimedean valuation $|\cdot|$, $\mathcal{F}$ and $\mathcal{E}$ denote non-Archimedean (**n.A**) Banach spaces over **K** and $\mathcal{CR}(\mathcal{F},\mathcal{E})$ is the space of any closed linear relations from $\mathcal{F}$ to $\mathcal{E}$. If $\mathcal{F}=\mathcal{E}$, we put $\mathcal{LR}(\mathcal{F},\mathcal{F}) = \mathcal{LR}(\mathcal{F})$, $\mathcal{B}(\mathcal{F})$ is the domain of any linear (bounded) operators on $\mathcal{F}$ and $\mathcal{F}'$ will be the dual space of the space $\mathcal{F}$.

2. Preliminaries

We recall various elementary results.

Definition 2.1: [6] A domain **K** is called **n.A** if it is furnished with a function $|\cdot|:\mathbf{K}\to\mathbf{R}_+$ satisfying

(i) $|f|=0$ if, and only if, $f=0$;

(ii) $|f\mu|=|f|\,|\mu|$ for each $f,\mu\in\mathbf{K}$;

(iii) $|f+\mu|\le\max\{|f|,|\mu|\}$ for all $f,\mu\in\mathbf{K}$.

Definition 2.2: [6] The domain **K** is **sc** if all decreasing succession of balls $(B_n)_n$ in **K** has a non-empty intersection.

Definition 2.3: [6] If $\mathcal{F}$ is a **ves** over **K**, hence the function $\|\cdot\|:\mathcal{F}\to\mathbf{R}_+$ is **n.A** norm when:

(i) $\|z\|=0$ if and only if $z=0$;

(ii) $\|\beta z\|=|\beta|\,\|z\|$ for each $z\in\mathcal{F}$ and $\beta\in\mathbf{K}$;

(iii) $\|z+f\|\le\max(\|z\|,\|f\|)$ for any $z,f\in\mathcal{F}$.

Definition 2.4: [6] A complete **n.A** normed linear space will be called a **n.A** Banach space.

We obtain:

Lemma 2.5: **[8]** *Let* $(x'_l)_{1\le l\le n} \in \mathcal{E}'$ *be linearly independent, hence there are elements* $x_1,\cdots,x_n$ *in* $\mathcal{E}$ *with*

$$x'_k(x_l) = \delta_{k,l} = \begin{cases} 1, & \text{if } k = l; \\ 0, & \text{if } k \neq l. \end{cases} \quad 1 \le l, k \le n. \tag{2.1}$$

On the other hand, if the family $(x_l)_{1\le l\le n} \in \mathcal{E}$ is linearly independent, hence there are elements $x'_1,\cdots,x'_n \in \mathcal{E}'$ with (2.1) holds.

Theorem 2.6: [10] *Presume that* $\mathbf{K}$ *is* **sc**. *For any* $g \in \mathcal{F} \setminus \{0\}$, *there is* $g' \in \mathcal{F}'$ *with* $g'(g) = 1$ *and* $\| g' \| = \| g \|^{-1}$.

Definition 2.7: [7] Let $A \in \mathcal{LR}(\mathcal{F},\mathcal{E})$. A is closed if its graph $G(A)$ is closed. The space of any closed linear relations from $\mathcal{F}$ to $\mathcal{E}$ is $\mathcal{CR}(\mathcal{F},\mathcal{E})$. $\mathcal{CR}(\mathcal{F},\mathcal{F}) = \mathcal{CR}(\mathcal{F})$ if $\mathcal{F} = \mathcal{E}$.

For further details on **n.A** linear relations, see [7].

3. Main Results

We get the next remark.

Remark 3.1: If $M \subset \mathcal{E}$ is a closed subspace. Hence

$$ind(Q_M^{\mathcal{E}}) = \dim M.$$

It is deduced by $N(Q_M^{\mathcal{E}}) = M$ and $R(Q_M^{\mathcal{E}}) = \mathcal{E} / M$. Then $\alpha(Q_M^{\mathcal{E}}) = \dim M$ and $\beta(Q_M^{\mathcal{E}}) = 0$.

Proposition 3.2: Presume that $|\mathbf{K}|$ is dense or $\| \mathcal{F} \| \subseteq |\mathbf{K}|$. If $A \in \mathcal{LR}(\mathcal{F})$. Hence the followings hold.

(i) A is continuous if and only if $\| A \| < \infty$;

(ii) $\gamma(A) > 0$ if and only if A is open.

In particular,

(iii) If $\dim R(A) < \infty$, then A is open;

(iv) If $\dim D(A) < \infty$, then A is continuous.

Proof: Identical to the demonstration of Proposition 2.3.2 of [5].

Definition 3.3: Let $C \in \mathcal{CR}(\mathcal{F}, \mathcal{E})$, C will have an index if $\alpha(C) = \dim N(C)$ and $\beta(C) = \dim(\mathcal{E} / R(C)) < \infty$. The index of C is determined by $ind(C) = \alpha(C) - \beta(C)$.

Definition 3.4: Let $C \in \mathcal{CR}(\mathcal{F})$. C is called superior semi-Fredholm if $R(C)$ is closed and $\alpha(C) < \infty$. The domain of any superior semi-Fredholm linear relations on $\mathcal{F}$ is $\Phi_+(\mathcal{F})$.

Definition 3.5: Let $C \in \mathcal{CR}(\mathcal{F})$. C inferior semi-Fredholm if $\beta(C) < \infty$. The domain of any inferior semi-Fredholm linear relations on $\mathcal{F}$ is $\Phi_-(\mathcal{F})$.

The domain $\Phi(\mathcal{F})$ of any Fredholm linear relations on $\mathcal{F}$ is $\Phi_+(\mathcal{F}) \cap \Phi_-(\mathcal{F})$. Let $C \in \Phi(\mathcal{F})$, we get $ind(C) = \alpha(C) - \beta(C)$.

Definition 3.6: Let $C \in \mathcal{LR}(\mathcal{F})$, C is **f.r** if $\dim R(C) < \infty$.

$\mathcal{FR}(\mathcal{F})$ will be the space of any **f.r** linear relations on $\mathcal{F}$.

Definition 3.7: Let $C \in \mathcal{LR}(\mathcal{F})$. C is said to be compact if $QC(B_{\mathcal{F}})$ is compactoid where $B_{\mathcal{F}} = \{w \in \mathcal{F} : \| w \| \leq 1\}$.

$\mathcal{KR}(\mathcal{F})$ denotes the space of any compact linear relations on $\mathcal{F}$. From [12], we get the following remark.

Remark 3.8: Presume that $\mathbf{K}$ is locally compact and $A \in \mathcal{LR}(\mathcal{F})$ is continuous. If A is compact, hence $QA(B_{\mathcal{F}})$ has a compact closure.

Proposition 3.9: Presume that $\mathbf{K}$ is locally compact and $A \in \mathcal{LR}(\mathcal{F})$ is continuous. If A is of finite-dimensional range, hence A is compact.

Proof: Identical to the argument of Proposition 5.2.5. of [13].

Example 3.10: Let $(x_n)_n \in (c_0(\mathbb{Q}_p))'$ and $(y_n)_n \in c_0(\mathbb{Q}_p)$ such that $\lim_{n\to\infty} \| x'_n \| \| y_n \| = 0$. Define the single-valued operator A on $c_0(\mathbb{Q}_p)$ by for all $x \in c_0(\mathbb{Q}_p)$,

$$Ax = \sum_{n=1}^{\infty} x'_n(x) y_n.$$

One can see that A is compact.

Example 3.11: Let $(\lambda_l)_l \in \mathbb{Q}_p$ with $\lim_{l\to\infty} |\lambda_l| = 0$. Define the single-valued operators A and $(A_n)_n$ on $c_0(\mathbb{Q}_p)$, respectively, by

for any $i \in \mathbb{N}, Ae_i = \lambda_i e_i$,

$$A_n e_i = \lambda_i e_i \text{ for } i = 0, \cdots, n \text{ and } A_n e_i = 0 \text{ whenever } n+1 \le i.$$

We get $(A_n)_n \in \mathcal{FR}(c_0(\mathbb{Q}_p))$ and $\lim_{n\to\infty} \| A_n - A \| = 0$. Hence $A \in \mathcal{KR}(c_0(\mathbb{Q}_p))$.

Lemma 3.12: *Presume that* $\mathbf{K} = \mathbb{Q}_p$. *If* $A \in \mathcal{CR}(\mathcal{E})$ *and* $K \in \mathcal{KR}(\mathcal{E})$ *such that* $D(A) \subset D(K)$ *and* $K(0) \subset A(0)$. *Hence*

(i) If $A \in \Phi_+(\mathcal{E})$, hence $A + K \in \Phi_+(\mathcal{E})$ with $ind(A+K) = ind(A)$;

(ii) If $A \in \Phi_-(\mathcal{E})$, hence $A + K \in \Phi_-(\mathcal{E})$ with $ind(A) = ind(A+K)$.

Proof: Identical to the argument of Lemma 3.7. of [1].

Proposition 3.13: Let $S, B \in \mathcal{LR}(\mathcal{E}, \mathcal{F})$ with $D(B) \subset D(S)$ and $S(0) \subset B(0)$. Then $B + S - S = B$.

Proof: If $(w, y) \in G\ (B+S-S)$, hence $w \in D\ (B+S-S) = D(B)$ with $y \in (B+S-S)\ w$. Thus $y \in Bw + S(0)$. From $S(0) \subset B(0)$, we get $y \in Bw + B(0)$. Hence $y \in Bw$ and $w \in D(B)$, we get $(w, y) \in G(B)$. Thus $G(B+S-S) \subset G(B)$. Reciprocally, if $w \in G(B)$, hence $w \in D(B)$ with $Bw = y + B(0)$. From $D(B) \subset D(S)$ and $S(0) \subset B(0)$, we get $(B+S-S)\ (0) = B(0)$ and $D(B) = D(B+S-S)$. Thus $w \in D\ (B+S-S)$ with $(B+S-S)\ w = y + (B+S-S)\ (0)$. Using ($i$) of Lemma 2.7 of [7], we get $y \in (B + S-S)w$ and $w \in D\ (B+S-S)$. Then $(w, y) \in G\ (B+S-S)$. Hence $G(B) \subset G\ (B+S-S)$. Thus $G(B) = G\ (B+S-S)$. Consequently $B = B+S-S$.

Proposition 3.14: Assume that $|\mathbf{K}|$ is dense or $\| \mathcal{F} \| \subseteq |\mathbf{K}|$. If $\mathcal{F}$ and $\mathcal{E}$ are two non-Archnormed spaces over $\mathbf{K}$ and $B, A \in \mathcal{LR}(\mathcal{F}, \mathcal{E})$ with $\overline{B(0)} \subset \overline{A(0)}$ and $D(A) \subset D(B)$, hence for each $x \in D(A)$,

$$\| (A-B)x \| \ge \| Ax \| - \| Bx \|. \tag{3.1}$$

Furthermore, if B and A are bounded, hence $A - B$ is bounded, and

$$\| A - B \| \ge \| A \| - \| B \|.$$

Proof: From $\overline{B(0)} \subset \overline{A(0)}$, one can see that $\overline{A(0)} = \overline{(A-B)(0)}$. From ($i$) of Proposition 2.13 of [7], we get for each $w \in D(A)$ and for each $y_1 \in Aw$ and $y_2 \in Bw$,

$$\begin{aligned} \|(A-B)w\| &= d(y_1-y_2,\overline{(A-B)(0)}) \\ &= d(y_1-y_2,\overline{A(0)}) \\ &\geq d(y_1,\overline{A(0)})-d(y_2,\overline{A(0)}) \\ &\geq d(y_1,\overline{A(0)})-d(y_2,\overline{B(0)}) \\ &\geq \|Aw\|-\|Bw\|, \end{aligned}$$

hence the equation (3.1) holds. In other ways, assume that A and B are bounded. Hence by Proposition 2.17 of [7] and Proposition 3.2, we have $A-B$ is bounded. It follows from (3.1) that for any $w\in D(A)$ such that $\|w\|\leq 1$, $\|Aw\| \leq \|(A-B)w\| + \|Bw\|$. From Proposition 2.17 of [7], $\|A\| \leq \|(A-B)\| + \|B\|$. Hence (3.1) holds.

Lemma 3.15: *Assume that* $\overline{|\mathbf{K}|}\supseteq\overline{\|\mathcal{F}\|}$ *and* $\mathbf{K}$ *is* **sc**. *If* $A,B\in\mathcal{LR}(\mathcal{F})$ *with* A *is one-to-one and open,* $\overline{B(0)}\subset\overline{A(0)}$ *and* $D(B)\supset D(A)$. *Hence* $\gamma(A-B)\geq\gamma(A)-\|B\|$. *Also, if* $\|B\|<\gamma(A)$, *hence* $A-B$ *is one-to-one and open.*

Proof: The argument is identical to the demonstration of Lemma 2.9.2 of [2].

Proposition 3.16: If $\mathbf{K}$ is **sc** and $A\in\mathcal{CR}(\mathcal{F})$, hence $N(A')=R(A)^{\perp}$.

Proof:

$$\begin{aligned} y'\in(A')^{-1}(0) \quad &\text{if and only if} \quad (0,y')\in G((A')^{-1}) \\ &\text{if and only if} \quad (y',0)\in G(A') \\ &\text{if and only if} \quad y'y=0 \text{ for any } y\in R(A) \\ &\text{if and only if} \quad y\in R(A)^{\perp}. \end{aligned}$$

The mapping theorem and closed graph theorem developed in the p-adic case as follows.

Theorem 3.17: *Presume that* $\mathbf{K}$ *is* **sc** *and* $\|\mathcal{F}\|\subseteq|\mathbf{K}|$. *Let* $A\in\mathcal{CR}(\mathcal{F})$. *Hence,*

(i) $D(A)$ is closed if and only if A is continuous;

(ii) $R(A)$ is closed if and only if A is open.

Proof: The argument is identical to the argument of Theorem 3.4.2 of [5].

4. Pseudospectra of non-Archimedean closed linear relations

In this part, we presume that $O, J \in \mathcal{CR}(\mathcal{F})$ and $N \in \mathcal{LR}(\mathcal{F})$ (continuous): $N(0) \subset J(0)$, $N(0) \subset O(0)$, $D(J) \subset D(N)$, $D(O) \subset D(N)$ and $\| N \| \neq 0$. We get:

Definition 4.1: The resolvent set $\rho(J,N)$ of a pencil $jN - J$ is

$$\rho(J,N) = \{j \in \mathbf{K} : (jN-J)^{-1} \in \mathcal{B}(\mathcal{F})\}.$$

The spectrum $\sigma(J,N)$ of a pencil $jN - J$ is $\sigma(J,N) = \mathbf{K} \setminus \rho(J,N)$.

Proposition 4.2: Presume that $\mathbf{K}$ is **sc** and $\| \mathcal{F} \| \subseteq |\mathbf{K}|$. Then $\rho(J,N)$ is open.

Proof: If $f \in \rho(J,N)$, hence $\gamma(fN-J) > 0$. If $\mu \in \mathbb{K}$ with $| f - \mu | < \frac{\gamma(fN-J)}{\|N\|}$, then by Corollary 2.21 of [7], $\mu N - J = (\mu - f)N + (fN - J)$ is one-to-one with dense range and open. Thus $\mu \in \rho(J,N)$, hence $\rho(J,N)$ is open set.

Proposition 4.3: If $J \in \mathcal{CR}(\mathcal{F})$ and $N \in \mathcal{BR}(\mathcal{F})$ with $N(0) \subset J(0)$. Hence for each λ, $\mu \in \rho(J,N)$, $R_N(\mu,J) - R_N(\lambda,J) = (\lambda - \mu)\, R_N(\mu,J)\, NR_N(\lambda,J)$ where $R_N(\lambda,J) = (\lambda N - J)^{-1}$.

Proof: From $R_N(\mu,J)(0) = \{0\}$ and using Corollary 2.8 of [7], we get

$$\begin{aligned} R_N(\mu,J)J(0) &= R_N(\mu,J)(\mu N - J)(0) = \\ &R_N(\mu,J)R_N(\mu,J)^{-1}(0) = R_N(\mu,J)(0) = \{0\}. \end{aligned}$$

Then,

$$\begin{aligned} (\lambda-\mu)R_N(\mu,J)NR_N(\lambda,J)x &= R_N(\mu,J)(\lambda-\mu)NR_N(\lambda,J)x \\ &= R_N(\mu,J)(\lambda N - J + J - \mu N)R_N(\lambda,J)x \\ &= R_N(\mu,J)R_N(\lambda,J)^{-1}R_N(\lambda,J)x \\ &\quad -R_N(\mu,J)R_N(\mu,J)^{-1}R_N(\lambda,N)x \\ &= R_N(\mu,J)(x + R_N(\lambda,J)^{-1}(0)) \\ &\quad -(R_N(\lambda,J)x + R_N(\mu,J)(0)) \\ &= R_N(\mu,J)x + R_N(\mu,J)J(0) - R_N(\lambda,J)x \\ &= R_N(\mu,J)x - R_N(\lambda,J)x. \end{aligned}$$

The following lemma is deduced from Proposition 4.3.

Corollary 4.4: *Let* $J \in \mathcal{CR}(\mathcal{F})$. *Then* $g, f \in \rho(L)$, $R(f,L)\ R(g,L) = R(g,L)$ $R(f,L)$ *where for each* $g \in \rho(L)$, $R(g,L) = (gI - L)^{-1}$.

Lemma 4.5: *Let* $\mathcal{F}$ *be a non-Archnormed space over* $\mathbf{K}$ *and* $\mathcal{E}$ *be a non-ArchBanach space over* $\mathbf{K}$. *If* $J \in \mathcal{CR}(\mathcal{F},\mathcal{E})$ *and* $B \in \mathcal{LR}(\mathcal{F},\mathcal{E})$ *(continuous) with* $B(0) \subset J(0)$ *and* $D(B) \supset D(J)$. *Hence* $J + B$ *is closed.*

Proof: The argument is identical to the demonstration of Lemma 3.6 of [1].

Definition 4.6: Let $\eta > 0$, the pseudo-spectrum $\sigma_\eta(O,N)$ of a pencil $O - fN$ is

$$\sigma_\eta(O,N) = \sigma(O,N) \cup \{f \in \mathbf{K} : \| (O - fN)^{-1} \| > \frac{1}{\eta}\},$$

with $\| (O - fN)^{-1} \| = \infty$ if $f \in \sigma(O,N)$. The pseudo-resolvent $\rho_\eta(O,N)$ of a pencil $O - fN$ is

$$\rho_\eta(O,N) = \rho(O,N) \cap \{f \in \mathbf{K} : \| (O - fN)^{-1} \| \leq \frac{1}{\eta}\}.$$

We get:

Proposition 4.7:

(i) $\sigma(O,N) = \bigcap_{\eta>0} \sigma_\eta(O,N)$;

(ii) $\sigma_{\eta_2}(O,N) \supset \sigma_{\eta_1}(J,N) \supset \sigma(O,N)$ *whenever* $0 < \eta_1 < \eta_2$.

Lemma 4.8: *Assume that* $\| \mathcal{F} \| \subseteq |\mathbf{K}|$. *Let* $\varepsilon > 0$ *and* $b \notin \sigma(J,N)$, *hence* $b \in \sigma_\varepsilon(J,N)$ *if, and only if, there is* $w \in \mathcal{F}$ *with* $\| w \| = 1$ *and*

$$\| (J - bN)w \| < \varepsilon.$$

Proof: If $b \in \sigma_\varepsilon(J,N) \setminus \sigma(J,N)$. Hence, there exists $w \in \mathcal{E}$ with

$$\| (J - bN)^{-1} w \| > \tfrac{\|w\|}{\varepsilon}. \tag{4.1}$$

Putting $x = (J - bN)^{-1} w$. Utilizing Corollary 2.8 of [7],

$$\begin{aligned}(J - bN)x &= (J - bN)(J - bN)^{-1} w \\ &= w + (J - bN)(0).\end{aligned}$$

Using Lemma 2.7 of [7], $w \in (J - bN)x$. From $N(0) \subset J(0)$, we get $(J - bN)\ (0) = J(0) - bN(0) = J(0)$. Then

$$\begin{aligned} d(w,(J-bN)(0)) &= \|(J-bN)x\| \\ &= d(w,J(0)) \\ &\le \|w\| = d(w,0). \end{aligned}$$

Hence

$$\|(J-bN)x\| \le \|w\|. \tag{4.2}$$

From (4.1) and (4.2),

$$\varepsilon\|x\| > \|w\| \ge \|(J-bN)x\|.$$

Utilizing $\|\mathcal{F}\| \subseteq |\mathbf{K}|$, there is $k \in \mathbb{K}$ with $\|x\| = |k|$. Posing $g = k^{-1}x$, we obtain $\|g\| = 1$ with

$$\|(J-bN)g\| < \varepsilon.$$

Reciprocally, we presume that there exists $w \in \mathcal{F}$ with $\|w\| = 1$ and

$$\|(J-bN)w\| < \varepsilon.$$

Utilizing $b \in \rho(J,N)$, we get $J-bN$ is one-to-one and open, thus

$$\gamma(J-bN)\|w\| \le \|(J-bN)w\| < \varepsilon$$

therefore $0 < \gamma(J-bN) < \varepsilon$. Utilizing Theorem 2.15 of [7], $\gamma(J-bN) = \|(J-bN)^{-1}\|^{-1}$. Hence,

$$\|(J-bN)^{-1}\| > \frac{1}{\varepsilon}.$$

So, $b \in \sigma_\varepsilon(J,N) \setminus \sigma(J,N)$.

The next result holds.

Theorem 4.9: *Assume that* $|\mathbf{K}| \supseteq \|\mathcal{F}\|$ *and* $\mathbf{K}$ *is* **sc**. *If* $\varepsilon > 0$. *Hence the following statements are equivalent.*

(i) $b \in \sigma_\varepsilon(J,N)$;

(ii) There is $C \in \mathcal{LR}(\mathcal{F})$ (continuous) with $D(J) \subset D(C)$, $C(0) \subset J(0)$ and $\|C\| < \varepsilon$ with $b \in \sigma(J+C,N)$;

(iii) Either $b \in \sigma(J,N)$ or $\|(J-bN)^{-1}\| > \frac{1}{\varepsilon}$.

Proof: $(i) \Rightarrow (ii)$ Suppose that $b \in \sigma_\varepsilon(J,N)$. If $b \in \sigma(J,N)$, we can set $C = 0$. If $b \in \sigma_\varepsilon(J,N)$. From Lemma 4.8, there exists $z \in \mathcal{F}$ with $\|z\| = 1$ and $\|(A-bN)z\| < \varepsilon$. From Theorem 2.6, there is $\varphi \in \mathcal{F}'$ with $\varphi(z) = 1$, $\|\varphi\| = \|z\|^{-1} = 1$. Define the relation C on $\mathcal{F}$ by for any $x \in \mathcal{F}$, $Cx = -\varphi(x)(J-bN)z$.

It is easy to see that C is single valued and everywhere defined (as $C(0)=0$). Moreover,

$$\| Cx \| \le \| x \| \| (J-bN)z \|.$$

For $x \neq 0$,

$$\frac{\|Cx\|}{\|x\|} \le \| (J-bN)z \|.$$

Thus,

$$\| C \| \le \| (J-bN)z \|.$$

Then $\| C \| < \varepsilon$. But,

$$\begin{aligned}(J+C-bN)z &= (J+C-bN)z\\ &= (J-bN)z+Cz\\ &= (J-bN)z-\varphi(z)(J-bN)z\\ &= (J-bN)(0)\\ &= (J-bN)(0)+C(0)\\ &= (J+C-bN)(0).\end{aligned}$$

Hence $0 \neq z \in N(J+C-bN)$, thus $(J+C-bN)z$ is not one-to-one, then $b \in \sigma(J+C,N)$.

$(ii) \Rightarrow (iii)$ We agrue by absurdity. If $b \not\in \sigma_{\varepsilon}(J,N)$, thus $b \in \rho(J,N)$ and $\| (J - bN)^{-1} \| \le \varepsilon^{-1}$. So, $b \in \rho(J,N)$ and $\gamma(J - bN) \ge \varepsilon$. Hence $J-bN$ is one-to-one with dense range and open. Since $J(0) \supset C(0) = (bN-J)(0)$ and $D(J-bN) = D(J) \subset D(C)$ and $\| C \| < \varepsilon \le \gamma(bN-J)$. By Corollary 2.21 of [7], $bN-J+C$ is open and one-to-one with dense range. So $b \in \rho(J+C,N)$ which is absurd. It is clear that (iii) implies (i).

From Theorem 4.9, we get.

Corollary 4.10: *Presume that* $|\mathbf{K}| \supseteq \| \mathcal{F} \|$ *and* $\mathbf{K}$ *is* **sc**. *If* $\varepsilon > 0$. *Hence*

$$\sigma_{\varepsilon}(O,N) = \bigcup_{C \in \mathcal{LR}(\mathcal{F}) : D(O) \subset D(C),\, O(0) \supset C(0),\, \|C\| < \varepsilon} \sigma(O+C,N).$$

Proposition 4.1: Presume that $|\mathbf{K}| \supseteq \| \mathcal{F} \|$ and $\mathbf{K}$ is **sc**. Let $J \in \mathcal{CR}(\mathcal{F})$ be one-to-one (dense range) and $N \in \mathcal{LR}(\mathcal{F})$ be continuous with $N(0) \subset J(0)$, $D(J) \subset D(N)$, $\| N \| \neq 0$ and $\varepsilon > 0$. Hence

$$\sigma_{\varepsilon}(J,N) \subset \left\{ f \in \mathbf{K} : | f | > -\frac{\varepsilon-\gamma(J)}{\|N\|} \right\}.$$

Proof: First case: If $\gamma(J) < \varepsilon$. Utilizing Theorem 2.15 of [7], $\| J^{-1} \| = \| (J-0N)^{-1} \| > \varepsilon^{-1}$. Hence $0 \in \sigma_\varepsilon(J,N)$ so there is no something to show.

Second case: If $\gamma(J) = \varepsilon$. Hence $J = J - 0N$ is open, then $0 \in \rho(J,N)$. From Theorem 2.15 of [7], $\| (J-0N)^{-1} \| = \varepsilon^{-1}$. Hence $0 \not\in \sigma_\varepsilon(J,N)$. Thus

$$\sigma_\varepsilon(J,N) \subset \{f \in \mathbf{K} : |f| > 0\}.$$

Third case: If $\gamma(J) > \varepsilon$, hence J is open. Let $f \in \mathbf{K} : |f| \le \frac{\gamma(J)-\varepsilon}{\|N\|}$. Hence $\| fN \| \le \gamma(J) - \varepsilon$, then $\| fN \| < \gamma(J)$. From Corollary 2.21 of [7], $fN - J$ is open and one-to-one with dense range i.e., $f \not\in \sigma(J,N)$. While $w \in D(J)$,

$$\| (fN - J)w \| \ge \| Jw \| - |f| \| Nw \| \ge (\gamma(J) - |f| \| N \|) \| w \|, \tag{4.3}$$

hence

$$\gamma(fN - J) \ge -|f| \| N \| + \gamma(J) \ge \varepsilon.$$

From Theorem 2.15 of [7], $\| (J - fN)^{-1} \| \le \frac{1}{\varepsilon}$. Thus $f \in \rho_\varepsilon(J,N)$. Hence

$$\sigma_\varepsilon(J,N) \subset \left\{ f \in \mathbf{K} : |f| > \frac{\gamma(J)-\varepsilon}{\|N\|} \right\}.$$

Theorem 4.12: *Presume that* $|\mathbf{K}| \supseteq \| \mathcal{F} \|$ *and* $\mathbf{K}$ *is* **sc**. *If* $\varepsilon > 0$. *Then* $\sigma_\varepsilon(J,N)$ *is open.*

Proof: If $f \in \overline{\rho_\varepsilon(J,N)}$, hence for $r \in]0, \frac{\varepsilon}{\|N\|}[$ with $B(f,r) \cap \rho_\varepsilon(J,N) \ne \varnothing$ where $B(f,r) = \{\alpha \in \mathbf{K} : r > |\alpha - f|\}$. If $\alpha \in B(f,r) \cap \rho_\varepsilon(J,N)$, hence $r > |\alpha - f|$ and $\alpha \in \rho_\varepsilon(J,N)$. Thus $\alpha \in \rho(J,N)$ and $\gamma(\alpha N - J) \ge \varepsilon$. Hence $\alpha N - J$ is open and one-to-one with dense range. From $|f - \alpha| < r < \varepsilon$ and Corollary 2.21 of [7], $fN - J$ is open and one-to-one with dense range. Then $f \in \rho(J,N)$. Let $w \in D(J)$,

$$\begin{aligned} \| (fN - J)w \| &= \| (J - \alpha N + \alpha N - fN)w \| \\ &\ge \| (J - \alpha N)w \| - |\alpha - f| \| Nw \| \\ &\ge \gamma(\alpha N - J) \| w \| - |\alpha - f| \| N \| \| w \|. \end{aligned}$$

Hence

$$\gamma(J - fN) \ge \gamma(J - \alpha N) - \| N \| |f - \alpha|.$$

Then

$$\gamma(fN - J) \ge r \| N \| + \varepsilon \text{ for all } 0 < r < \frac{\varepsilon}{\|N\|}.$$

Hence

$$\varepsilon \le \gamma(fN - J).$$

Consequently, $f \in \rho_\varepsilon(J,N)$. Then, $\rho_\varepsilon(J,N)$ is a closed set.

At the moment, we describe the essential pseudo-spectra of closed linear relations on non-ArchBanach spaces.

Definition 4.13: Assume that $\mathbf{K}$ is spherically complete. Let $\varepsilon > 0$, the essential pseudo-spectrum $\sigma_{e,\varepsilon}(J,N)$ of the pencil $J - fN$ is given by

$$\sigma_{e,\varepsilon}(J,N) = \bigcap_{K \in \mathcal{C}_J(\mathcal{F})} \sigma_\varepsilon(J+K,N),$$

where $\mathcal{C}_J(\mathcal{F}) = \{K \in \mathcal{KR}(\mathcal{F}) : D(K) \supset D(J),\ K(0) \subset J(0)\}$ and $\rho_{e,\varepsilon}(J,N)$ denotes the essential pseudoresolvent set of $J - fN$ given by $\rho_{e,\varepsilon}(J,N) = \mathbf{K} \setminus \sigma_{e,\varepsilon}(J,N)$.

$\mathcal{LR}_c(\mathcal{F})$ denotes the domain of any linear (continuous) relations on $\mathcal{F}$ and $\mathcal{LR}_s(\mathcal{F})$ is the domain of any linear (continuous) single valued relations on $\mathcal{F}$, we get:

Theorem 4.14: *Assume that* $\mathbf{K} = \mathbb{Q}_p$ *and* $\| \mathcal{F} \| \subseteq |\mathbb{Q}_p|$. *Hence the next are equivalent.*

(i) $f \notin \sigma_{e,\varepsilon}(J,N)$;

(ii) For any $C \in \mathcal{LR}_c(\mathcal{F})$ with $D(C) \supset D(J)$, $J(0) \supset C(0)$ and $\|C\| < \varepsilon$,
$$J + C - fN \in \Phi(\mathcal{F}) \text{ and } ind(-fN + J + C) = 0;$$

(iii) For any $B \in \mathcal{LR}_s(\mathcal{F})$ with $D(J) \subset D(B) : \varepsilon > \|B\|$,
$$J + B - fN \in \Phi(\mathcal{F}) \text{ and } ind(J + B - fN) = 0.$$

Proof: $(i) \Rightarrow (ii)$ If $f \notin \sigma_{e,\varepsilon}(J,N)$, hence there is $K \in \mathcal{C}_J(\mathcal{F})$ with $f \notin \sigma_\varepsilon (J+K,N)$. By Corollary 4.10, $f \in \rho\ (J+K+C,N)$ for each $C \in \mathcal{LR}(\mathcal{F})$ with $D(J) \subset D(K)$, $K(0) \subset J(0)$ (hence $D(J) = D(J) \cap D(K) = D(J+K)$), $\mathrm{D(C)} \supset D(J)$ and $\|C\| < \varepsilon$. We get

$$J + C + K - fN \in \Phi(\mathcal{F}) \text{ and } ind(J + C + K - fN) = 0. \tag{4.4}$$

By Lemma 3.12 , we get

$$J + C - fN \in \Phi(\mathcal{F}) \text{ and } ind(J + C - fN) = 0. \tag{4.5}$$

$(ii) \Rightarrow (iii)$ is simple.

$(iii) \Rightarrow (i)$ Presume that for any $B \in \mathcal{LR}_s(\mathcal{F})$ with $D(J) \subset D(B)$: $\varepsilon > \|B\|$,

$$J + B - fN \in \Phi(\mathcal{F}) \text{ and } ind(J + B - fN) = 0.$$

Let $f \in \mathbb{Q}_p$ with $J+B-fN \in \Phi(\mathcal{F})$ and $ind(J+B-fN)=0$. Set $n=\alpha(J+B-fN)=\beta(J+B-fN)$. Let $\{x_1,\cdots,x_n\}$ be the basis of $N(J+B-fN)$ and $\{y'_1,\cdots,y'_n\}$ be the basis of $R(J+B-fN)^{\perp}$. From Lemma 2.5, there are $x'_1,\cdots,x'_n \in \mathcal{F}'$ and $y_1,\cdots,y_n \in \mathcal{F}$ with

$$x'_k(x_l)=\delta_{k,l} \text{ and } y'_k(y_l)=\delta_{k,l},\ 1\le k,l\le n,$$

where $\delta_{k,l}=0$ if $k\neq l$ and $\delta_{k,l}=1$ if $k=l$. Consider the relation F on $\mathcal{F}$ by

$$\begin{array}{rccc} F: & \mathcal{F} & \to & \mathcal{F} \\ & u & \mapsto & \sum_{l=1}^{n} x'_l(u)y_l. \end{array}$$

Hence $F\in \mathcal{LR}_s(\mathcal{F})$. By, for each $x\in\mathcal{F}$,

$$\begin{aligned} \|Fx\| &= \|\sum_{l=1}^{n} x'_l(x)y_l\| \\ &\le \max_{1\le l\le n}\|x'_l(x)y_l\| \\ &\le \max_{1\le l\le n}(\|x'_l\|\|y_l\|)\|x\|. \end{aligned}$$

Then, $\dim R(F)$ is finite. Hence, $F\in\mathcal{FR}(\mathcal{F})$, thus $F\in\mathcal{KR}(\mathcal{F})$. Now, we will prove that

$$N(J+B-fN)\cap N(F)=\{0\}, \tag{4.6}$$

and

$$R(J+B-fN)\cap R(F)=\{0\} \tag{4.7}$$

for each $B\in\mathcal{LR}_s(\mathcal{F})$ with $D(J)\subset D(B)$. If $w\in N(J+B-fN)\cap N(F)$, thus

$$w=\sum_{l=1}^{n}\alpha_l x_l \text{ with } \alpha_1,\cdots,\alpha_n\in\mathbb{Q}_p.$$

Hence for any $1\le k\le n$, $x'_k(w)=\sum_{l=1}^{n}\alpha_l\delta_{k,l}=\alpha_k$. Moreover, if $w\in N(F)$, thus $Fw=0$, hence

$$\sum_{k=1}^{n}x'_k(w)y_k=0.$$

Therefore, we get for any $1\le k\le n$, $x'_k(w)=0$. So, $w=0$. Hence,

$$N(J+B-fN)\cap N(F)=\{0\}.$$

If $b\in R(B+J-fN)\cap R(F)$, hence $b\in R(J+B-fN)$ and $b\in R(F)$. Let $b\in R(F)$, we get

$$b = \sum_{j=1}^{n} \alpha_j y_j, \text{ with } \alpha_1, \cdots, \alpha_n \in \mathbb{Q}_p.$$

Hence for any $1 \le k \le n$, $y'_k(b) = \sum_{i=1}^{n} \alpha_i \delta_{k,i} = \alpha_k$. Furthermore, if $b \in R(A + B - fN)$, thus for each $1 \le k \le n$, $y'_k(b) = 0$. Hence $b = 0$. So,

$$R(J + B - fN) \cap R(F) = 0.$$

We get $F \in \mathcal{C}_J(\mathcal{F}) = \mathcal{C}_{J+B-fN}(\mathcal{F})$ since $D(J) = D(J + B - fN) \subset D(F)$, $F(0) \subset J(0) = (J + B - fN)(0)$. By Theorem 3.12, $J + B - fN + F \in \Phi(\mathcal{F})$ and $ind(J + B + F - fN) = 0$. Hence

$$\alpha(J + B + F - fN) = \beta(J + B + F - fN). \tag{4.8}$$

If $x \in N(J + B + F - fN)$, thus $0 \in (J + B + F - fN)x$, hence $-Fx \in (J + B - fN)x$, thus $-Fx \in R(J + B - fN) \cap R(F)$. Using (4.7), we get $Fx = 0$ and $0 \in (J + B - fN)x$. Thus $x \in N(J + B - fN) \cap N(F)$ and by (4.6), we obtain $x = 0$. Then $\alpha(J + B + F - fN) = 0$. By (4.8), $R(J + B + F - fN) = \mathcal{F}$. From $J + B + F - fN$ is closed and Theorem 3.17, $J + B + F - fN$ is open. Moreover, it is one-to-one with dense range. Consequently, $f \in \rho(J + B + F, N)$ for any $B \in \mathcal{LR}_s(\mathcal{F})$ with $D(J) \subset D(B)$ and $\|B\| < \varepsilon$. By Theorem 4.9, $f \not\in \sigma_\varepsilon(J + F, N)$. Then

$$f \not\in \sigma_{e,\varepsilon}(J, N) = \bigcap_{F \in \mathcal{C}_J(\mathcal{F})} \sigma_\varepsilon(J + F, N),$$

thus $f \not\in \sigma_{e,\varepsilon}(J, N)$.

Remark 4.15: Assume that $\mathbf{K} = \mathbb{Q}_p$ and $\|\mathcal{F}\| \subseteq |\mathbb{Q}_p|$. Let $\varepsilon > 0$. From Theorem 4.14,

$$\sigma_{e,\varepsilon}(J, N) = \bigcup_{C \in \mathcal{LR}(\mathcal{F}) : D(J) \subset D(C), C(0) \subset J(0), \|C\| < \varepsilon} \sigma_e(J + C, N) = \bigcup_{B : D(J) \subset D(B), \|B\| < \varepsilon} \sigma_e(J + B, N),$$

where $\sigma_e(J, N) = \{f \in \mathbf{K} : J - fN \text{ is not Fredholm of index } 0\}$

Proposition 4.16: Assume that $\mathbf{K} = \mathbb{Q}_p$, $\|\mathcal{E}\| \subseteq |\mathbb{Q}_p|$ and $\varepsilon > 0$. The followings hold.

(i) If $0 < \varepsilon_1 < \varepsilon_2$, hence $\sigma_e(J, N) \subset \sigma_{e,\varepsilon_1}(J, N) \subset \sigma_{e,\varepsilon_2}(J, N)$;
(ii) For any $\varepsilon > 0, \sigma_{e,\varepsilon}(J, N) \subset \sigma_\varepsilon(J, N)$;
(iii) $\sigma_e(J, N) \subset \bigcap_{\varepsilon > 0} \sigma_{e,\varepsilon}(J, N)$.

Proof:

(i) If $b \notin \sigma_{e,\varepsilon_2}(J,N)$, hence by Theorem 4.14, for each $C \in \mathcal{LR}_c(\mathcal{E})$ with $D(C) \supset D(J)$, $C(0) \subset J(0)$ and $\| C \| < \varepsilon_2$,

$$J + C - bN \in \Phi(\mathcal{E}) \text{ and } ind(J + C - bN) = 0.$$

Thus for each $C \in \mathcal{LR}_c(\mathcal{E})$ with $D(C) \supset D(J)$, $C(0) \subset J(0)$ and $\| C \| < \varepsilon_1$,

$$J + C - bN \in \Phi(\mathcal{E}) \text{ with } ind(J + C - bN) = 0.$$

Hence $b \notin \sigma_{e,\varepsilon_1}(J,N)$. If $b \notin \sigma_{e,\varepsilon_1}(J,N)$, thus from Theorem 4.14, for any $C \in \mathcal{LR}_c(\mathcal{E})$ with $D(C) \supset D(J)$, $C(0) \subset J(0)$ and $\| C \| < \varepsilon_1$,

$$J + C - bN \in \Phi(\mathcal{E}) \text{ with } ind(J + C - bN) = 0.$$

For $C = 0$, then $\lambda \notin \sigma_e(J,N)$.

(ii) For each $\varepsilon > 0$,

$$\sigma_{e,\varepsilon}(J,N) = \bigcap_{K \in \mathcal{C}_J(\mathcal{E})} \sigma_\varepsilon(J + K, N) \subset \sigma_\varepsilon(J + K, N) \text{ for each } K \in \mathcal{C}_J(\mathcal{E}).$$

In particular $K = 0$, we get the desired result.

(iii) By (i), for each $\varepsilon > 0$,

$$\sigma_e(J,N) \subset \sigma_{e,\varepsilon}(J,N).$$

Hence

$$\sigma_e(J,N) \subset \bigcap_{\varepsilon > 0} \sigma_{e,\varepsilon}(J,N).$$

5. Condition pseudospectrum of non-Archimedean bounded linear relations

In this section, we presume that $J \in \mathcal{BR}(\mathcal{F})$ is closed.

Definition 5.1: Let $O \in \mathcal{BR}(\mathcal{F})$ and $\eta > 0$. The condition pseudo-spectrum $\Lambda_\eta(O)$ of O on $\mathcal{F}$ is

$$\Lambda_\eta(O) = \sigma(O) \cup \{ f \in \mathbf{K} : \| (fI - O) \| \| (fI - O)^{-1} \| > \eta^{-1} \},$$

with $\| (O - fI) \| \| (fI - O)^{-1} \| = \infty$ if $f \in \sigma(O)$.

Utilizing Definition 5.1, get.

Proposition 5.2: If $O \in \mathcal{BR}(\mathcal{F})$, we get

(i) $\sigma(O) = \bigcap_{\eta>0} \Lambda_{\varepsilon}(O)$;

(ii) $\sigma(O) \subset \Lambda_{\eta_1}(O) \subset \Lambda_{\eta_2}(O)$ whenever $0 < \eta_1 \le \eta_2$.

Lemma 5.3: *Presume that* $|\mathbf{K}| \supseteq \|\mathcal{F}\|$ *and* $\mathbf{K}$ *is* **sc**. *Let* $J \in \mathcal{BR}(\mathcal{F})$, $\varepsilon > 0$ *and* $f \not\in \sigma(J)$. *Hence* $f \in \Lambda_{\varepsilon}(J) \setminus \sigma(J)$ *if, and only if, there is* $g \in \mathcal{F} \setminus \{0\}$ *with* $\| g \| = 1$ *and*

$$\| (J - fI)g \| < \varepsilon \| J - fI \|.$$

Proof: If $f \in \Lambda_{\varepsilon}(J) \setminus \sigma(J)$, thus

$$\| (J - fI) \| \| (fI - J)^{-1} \| > \varepsilon^{-1}.$$

Hence

$$\| (J - fI)^{-1} \| > \frac{1}{\varepsilon \|(fI - J)\|}.$$

Moreover,

$$\sup_{w \in \mathcal{F} \setminus \{0\}} \frac{\|(fI - J)^{-1} w\|}{\|w\|} > \frac{1}{\varepsilon \|(J - fI)\|}.$$

Thus there is $w \in \mathcal{F} \setminus \{0\}$ with

$$\| (J - fI)^{-1} w \| > \frac{\|w\|}{\varepsilon \|(J - fI)\|}. \tag{5.1}$$

Posing $x = (J - fI)^{-1} w$. Utilizing Corollary 2.8 of [7],

$$\begin{aligned} (J - fI)x &= (J - fI)(J - fI)^{-1} w \\ &= w + (J - fI)(0). \end{aligned}$$

By Lemma 2.7 of [7], $w \in (J - fI)x$. Using $(J - fI)\ (0) = J(0) - fI(0) = J(0)$, as a result

$$\| (J - fI)x \| \le \| w \|. \tag{5.2}$$

From (5.1) and (5.2),

$$\| x \| \| (A - fI) \| \varepsilon > \| y \| \ge \| (J - fI)x \|.$$

Utilizing $\| \mathcal{F} \| \subseteq |\mathbf{K}|$, there exists $k \in \mathbf{K}$ with $\| x \| = |k|$. Posing $w = k^{-1} x$, we get $\| w \| = 1$ and

$$\| (J - fI)w \| < \varepsilon \| (J - fI) \|.$$

Reciprocally, we presume that there exists $w \in \mathcal{E}$ with $\| w \| = 1$ and
$$\| (J - fI)w \| < \varepsilon \| (J - fI) \|.$$

From $f \in \rho(S)$, we get $S - fI$ is one-to-one and open, so
$$\gamma(J - fI) \| x \| \leq \| (J - fI)x \| < \varepsilon \| (J - fI) \|.$$

Hence $0 < \gamma\ (J - fI) < \varepsilon\ \| (J - fI) \|$. From Theorem 2.15 of [7], $\gamma(J - fI) = \| (J - fI)^{-1} \|^{-1}$. Thus,
$$\| (J - fI) \| \| (J - fI)^{-1} \| > \frac{1}{\varepsilon}.$$

Then, $f \in \sigma_{\varepsilon}(J) \setminus \sigma(J)$.

Theorem 5.4: *If* $O \in \mathcal{BR}(\mathcal{F}) : 0 \not\in \rho(O), \eta > 0$ *and* $j = \| O^{-1} \| \| O \|$, *hence*

(i) If $s \in \Lambda_{\eta}(O^{-1}) \setminus \{0\}$, so $\frac{1}{s} \in \Lambda_{\eta j}(O) \setminus \{0\}$;
(ii) If $\frac{1}{s} \in \Lambda_{\eta j}(O) \setminus \{0\}$, hence $s \in \Lambda_{\eta j^2}(O^{-1}) \setminus \{0\}$.

Proof: (i) Presume that $s \in \Lambda_{\eta}(O^{-1}) \setminus \{0\}$, so
$$\begin{aligned}
\frac{1}{\eta} < \| O^{-1} - sI \| \| (O^{-1} - sI)^{-1} \| &= \| s\left(\frac{I}{s} - O\right) O^{-1} \| \| s^{-1} O \left(\frac{I}{s} - O\right)^{-1} \| \\
&\leq |s| \| O^{-1} \| \| \left(\frac{I}{s} - O\right) \| \times \\
&\quad \| \left(\frac{I}{s} - O\right)^{-1} \| |s|^{-1} \| O \| \\
&= \| O^{-1} \| \| O \| \| \left(\frac{I}{s} - O\right) \| \times \| \left(\frac{I}{s} - O\right)^{-1} \|.
\end{aligned}$$

Hence, $\frac{1}{s} \in \Lambda_{\eta j}(O) \setminus \{0\}$.

(ii) Such as in (i), if $\frac{1}{s} \in \Lambda_{\eta j}(O) \setminus \{0\}$, hence $s \in \Lambda_{\eta j^2}(O^{-1}) \setminus \{0\}$.

Proposition 5.5: If $O \in \mathcal{BR}(\mathcal{F}), \eta > 0$ and $s \in \mathbf{K}$ with $\| O - sI \| \neq 0$. Hence

(i) $s \in \sigma_{\eta \| O - sI \|}(O) \iff s \in \Lambda_{\eta}(O)$;
(ii) $s \in \Lambda_{\frac{\eta}{\| O - sI \|}}(O) \iff s \in \sigma_{\eta}(O)$.

Proof:

(i) Presume that $s \in \Lambda_{\eta}(O)$, we get $s \in \sigma(O)$ or $\| (O - sI) \| \| (O - sI)^{-1} \| > \eta^{-1}$. Hence $s \in \sigma(O)$ or $\| (O - sI)^{-1} \| > \frac{1}{\eta \| (O - sI) \|}$. So $s \in \sigma_{\eta \| O - sI \|}(O)$. Analogously, if $s \in \sigma_{\eta \| O - sI \|}(O)$, so $s \in \Lambda_{\eta}(O)$.

(ii) Presume that $s \in \sigma_\eta(O)$, imply $s \in \sigma(O)$ or $\|(O-sI)^{-1}\| > \eta^{-1}$. Thus

$$s \in \sigma(O) \text{ or } \|(O-sI)\|\|(O-sI)^{-1}\| > \frac{\|(O-sI)\|}{\eta}.$$

So $s \in \sigma_{\frac{\eta}{\|O-sI\|}}(J)$. Analogously, if $s \in \sigma_\eta(O)$, hence $s \in \Lambda_{\frac{\eta}{\|O-sI\|}}(O)$.

Proposition 5.6: Presume that $O \in \mathcal{BR}(\mathcal{F})$, $\eta > 0$, $s, g \in \mathbf{K}$, $s \neq 0$, hence $g + s\Lambda_\eta(O) = \Lambda_\eta(sO + gI)$.

Proof: If $w \in \Lambda_\eta(sO+gI)$ and $s \neq 0$, hence $w \in \sigma(sO+gI)$ and

$$\begin{aligned}\frac{1}{\eta} &< \|sO+gI-wI\|\|(sO+gI-wI)^{-1}\| \\ &= \|(O-(w-g)s^{-1}I)\|\|(O-(w-g)s^{-1}I)^{-1}\|.\end{aligned}$$

So $(w-g)s^{-1} \in \Lambda_\eta(O)$, then $w \in g + s\Lambda_\eta(O)$. Similarly, we obtain $g + s\Lambda_\eta(O) \subseteq \Lambda_\eta(sO+gI)$.

Theorem 5.7: *Let $S, E \in \mathcal{BR}(\mathcal{F})$ and O be a closed and bounded operator with $0 \in \rho(O)$ and $j = \|O^{-1}\|\|O\|$. If $E = O^{-1}SO$, thus for any $\eta > 0$,*

$$\Lambda_{\frac{\eta}{j^2}}(E) \subseteq \Lambda_\eta(S) \subseteq \Lambda_{j^2\eta}(E).$$

Proof: Presume that $s \in \Lambda_{\frac{\eta}{j^2}}(E)$, hence $s \in \sigma(E) = \sigma(S)$ or

$$\begin{aligned}\frac{j^2}{\eta} < \|E-sI\|\|(E-sI)^{-1}\| &= \|O^{-1}(S-sI)O\|\|O^{-1}(S-sI)^{-1}O\| \\ &\leq (\|O\|\|O^{-1}\|)^2\|(S-sI)\|\|(S-sI)^{-1}\| \\ &\leq j^2\|(S-sI)\|\|(S-sI)^{-1}\|.\end{aligned}$$

Utilizing $j^2 > 0$, $s \in \Lambda_\eta(S)$. So $\Lambda_{\frac{\eta}{j^2}}(E) \subseteq \Lambda_\eta(S)$.
If $s \in \Lambda_\eta(S)$, thus $s \in \sigma(E) = \sigma(S)$ or

$$\begin{aligned}\frac{1}{\eta} < \|S-sI\|\|(S-sI)^{-1}\| &= \|O(E-sI)O^{-1}\|\|O(E-sI)^{-1}O^{-1}\| \\ &\leq (\|O\|\|O^{-1}\|)^2\|(E-sI)\|\|(E-sI)^{-1}\| \\ &\leq j^2\|(E-sI)\|\|(E-sI)^{-1}\|.\end{aligned}$$

Then $s \in \Lambda_{j^2\eta}(E)$. So $\Lambda_\eta(S) \subseteq \Lambda_{j^2\eta}(E)$.

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