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On graded weakly quasi n-absorbing submodule

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Abstract

In this paper, we present the concept of *Gr-W-Q-n*-AS. We provide several results regarding *Gr-W-Q-n*-AS.

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1. Introduction

In this paper, we assume that $\mathcal{B}$ is a commutative Ω-graded ring with identity, $\mathcal{V}$ is a unitary graded $\mathcal{B}$-Module and $n \in \mathbb{Z}^{+}$.

In [1], the authors presented and analyzed the notion of gr-primary ideal. The notion of gr-2-AI was defined and thoroughly analyzed in [2]. The notion of gr-prime submodule has been defined and investigated by various authors; see, for example, [3-5]. The notion of gr-2-AS, which

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generalizes the gr-prime submodule, was first defined in [6] and further investigated in [7, 8]. Subsequently, the authors in [9] introduced the notion of gr-*n*-AS. Here, we present the concept of *Gr*-*W*-*Q*-*n*-AS and discuss several of its properties.

From this point forward, we will use the same notations introduced in [7]. For foundational properties and additional details on graded modules, we direct readers to [10-13].

2. Results

By $P \leq_{\Omega}^{sub} \mathcal{V}$, we signify that *P* is a Ω-graded submodule of $\mathcal{V}$. Also, by $P <_{\Omega}^{sub} \mathcal{V}$, we signify that *P* is a proper Ω-graded submodule of $\mathcal{V}$ (see [12]).

Definition 2.1: Let $Q = \underset{g\in\Omega}{\oplus} Q_g \ \leq_{\Omega}^{sub} \mathcal{V}$, $g \in \Omega$ and $n \in \mathbb{Z}^+$.

(1) We say that P_g is a *g*-weakly quasi *n*-absorbing submodule (abbreviated as, *g*-*W*-*Q*-*n*-AS) of $\mathcal{V}_g$. If $P_g \neq \mathcal{V}_g$; and whenever $r_e \in \mathcal{B}_e$, $m \in \mathcal{V}_g$ and $n \in \mathbb{N}$ with $0 \neq r_e^n m \in P_g$, then either $r_e^{n-1} m \in P_g$ or $r_e^n \ (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$.

(2) We say that *P* is a graded weakly quasi *n*-absorbing submodule (abbreviated as, *Gr*-*W*-*Q*-*n*-AS) of $\mathcal{V}$. If $P \neq \mathcal{V}$; and whenever $0 \neq r^n m \in P$ where $r \in h(\mathcal{B})$, $m \in h(\mathcal{V})$ and $n \in \mathbb{N}$, then either $r^{n-1} m \in P$ or $r^n \in (P :_{\mathcal{B}} \mathcal{V})$.

Definition 2.2: Let $K = \underset{g\in\Omega}{\oplus} K_g \ \leq_{\Omega}^{sub} \mathcal{V}$, $g \in \Omega$ and *n* be a positive integer. Then *K* is called a graded quasi *n*-absorbing submodule (abbreviated as, *Gr*-*Q*-*n*-AS) of $\mathcal{V}$, if $r^n m \in K$, then either $r^{n-1} m \in K$ or $r^n \in (K :_{\mathcal{B}} \mathcal{V})$ where $r \in h(\mathcal{B})$, $m \in h(\mathcal{V})$ and $\mathbb{N}$.

It is clear that every *Gr*-*Q*-*n*-AS is a *Gr*-*W*-*Q*-*n*-AS, while the converse is not true.

Example 2.3: Let $\Omega = \mathbb{Z}_2$ and $\mathcal{B} = \mathbb{Z}$. Thus $\mathcal{B}$ is a graded Ω-ring with $\mathcal{B}_0 = \mathbb{Z}$ and $\mathcal{B}_1 = \{0\}$. Let $\mathcal{V} = \mathbb{Z}/12\mathbb{Z}$. Hence $\mathcal{V}$ is a graded $\mathcal{B}$-module with $\mathcal{V}_0 = \mathbb{Z}/12\mathbb{Z}$ and $\mathcal{V}_1 = \{0\}$. The graded submodule $P = \{0\}$ is a *Gr*-*W*-*Q*-n-AS of $\mathcal{V}$ which is not a *Gr*-*Q*-n-AS since $2^2.3 \in P$, $2^2 \notin (P :_{\mathcal{B}} \mathcal{V}) = 12\mathbb{Z}$ and $2.3 \notin P$.

Theorem 2.4: *Let $P = \underset{g\in\Omega}{\oplus} P_g \leq_{\Omega}^{sub} \mathcal{V}$ and $g \in \Omega$ with $P_g \neq \mathcal{V}_g$. If P is a Gr-W-Q-n-AS of $\mathcal{V}$, then P_g is a g-W-Q-n-AS of $\mathcal{V}_g$.*

Proof: Let P be a Gr-W-Q-n-AS of $\mathcal{V}$. Let $r_e \in \mathcal{B}_e$ and $m \in \mathcal{V}_g$ such that $0 \neq r_e^n m \in P_g \subseteq P$ and so either $r_e^{n-1} m \in P$ or $r_e^n \in (P :_{\mathcal{B}} \mathcal{V})$ and so either $r_e^{n-1} m \in P_g$ or $r_e^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ since $P_g = P \cap \mathcal{V}_g$ and $\mathcal{V}_g \subseteq \mathcal{V}$. Hence, P_g is a g-W-Q-n-AS of $\mathcal{V}_g$.

Definition 2.5: Let $\mathcal{V} = \underset{g\in\Omega}{\oplus} \mathcal{V}_g$ be a Ω-graded $\mathcal{B}$-module and $P = \underset{g\in\Omega}{\oplus} P_g$ $\leq_{\Omega}^{sub} \mathcal{V}$. Then.

(1) If P_g is a g-W-Q-n-AS of $\mathcal{V}_g$. Then $r_e \in \mathcal{B}_e$ is said to be gr-unbreakable element of P_g if there exists $m \in \mathcal{V}_g$ such that $r_e^n m = 0$ and neither $r_e^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ nor $r_e^{n-1} m \in P_g$.

(2) If P is a Gr-W-Q-n-AS of $\mathcal{V}$. Then $r \in h(\mathcal{B})$ is said to be gr-unbreakable element of P if there exists $m \in h(\mathcal{V})$ with $r^n m = 0$ and neither $r^n \in (P :_{\mathcal{B}} \mathcal{V})$ nor $r^{n-1} m \in P$.

Remark 2.6: For every Gr-W-Q-n-AS P of $\mathcal{V}$ which is not Gr-Q-n-AS of $\mathcal{V}$, there exists a gr-unbreakable element m of P.

Lemma 2.7: *Let $\mathcal{V} = \underset{g\in\Omega}{\oplus} \mathcal{V}_g$ be a graded $\mathcal{B}$-module, $P = \underset{g\in\Omega}{\oplus} P_g \leq_{\Omega}^{sub} \mathcal{V}$ and $g \in \Omega$. If P_g is a g-W-Q-n-AS of $\mathcal{V}_g$ and $r_e \in \mathcal{B}_e$ is a gr-unbreakable element of P_g. Then.*

(1) $r_e^n P_g = 0$.

(2) $(r_e + s_e)$ is gr-unbreakable element of P_g for every $s_e \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$.

Proof: Let P_g be a g-W-Q-n-AS of $\mathcal{V}_g$ and $r_e \in \mathcal{B}_e$ is a gr-unbreakable element of P_g.

(1) Let $r_e^n P_g \neq 0$. Then there exists $l \in P_g$ such that $0 \neq r_e^n l \in P_g$, so $r_e^{n-1} l \in P_g$ since P_g is a g-W-Q-n-AS of $\mathcal{V}_g$ and $r_e^n \notin (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. Since $r_e \in \mathcal{B}_e$ is a gr-unbreakable element of P_g, then there exists $m \in \mathcal{V}_g$ such that $r_e^n m = 0$ and neither $r_e^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ nor $r_e^{n-1} m \in P_g$. Which implies that $0 \neq r_e^n\ (m + l) \in P_g$, and so $r_e^{n-1}\ (m + l) \in P_g$ since P_g is a g-W-Q-n-AS of $\mathcal{V}_g$ and $r_e^n \notin (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. But $r_e^{n-1} l \in P_g$, so $r_e^{n-1} m \in P_g$, a contradiction. Thus, $r_e^n P_g = 0$.

(2) Since $r_e \in \mathcal{B}_e$ is a gr-unbreakable element of P_g, there exists $m \in \mathcal{V}_g$ such that $r_e^n m = 0$ and neither $r_e^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ nor $r_e^{n-1} m \in P_g$. Now, let

$s_e \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. If $0 \neq (r_e + s_e)^n m$, then $(r_e + s_e)^{n-1} m \in P_g$ or $(r_e + s_e)^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$, since P_g is a g-W-Q-n-AS of $\mathcal{V}_g$.

Case 1: If $(r_e + s_e)^{n-1}\ m \in P_g$, then $(r_e + s_e)^{n-1}\ m = (\sum_{j=0}^{n-1} \binom{n-1}{j} r_e^j s_e^{n-1-j})\ m = r_e^{n-1} m + \sum_{j=0}^{n-2} \binom{n-2}{j} r_e^j s_e^{n-2-j} m$ and $r_e^j s_e^{n-2-j}\ m \in P_g$ for all $j = 0, 1, \ldots, n-2$ and so $r_e^{n-1} m \in P_g$, a contradiction.

Case 2: If $(r_e + s_e)^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. Since $r_e^j s_e^{n-j} \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ for all $j = 0, 1, \ldots, n-1$, then $r_e^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$, a contradiction.

Thus, $0 = (r_e + s_e)^n m$ and neither $(r_e + s_e)^{n-1}\ m \in P_g$ nor $(r_e + s_e)^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ and hence $(r_e + s_e)$ is $gr-$unbreakable element of P_g for every $s_e \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$.

Theorem 2.8: *Let $P = \underset{g \in \Omega}{\oplus} P_g \leq_{\Omega}^{sub} \mathcal{V}$ and $g \in \Omega$. If $\mathcal{V}_g \neq P_g$ is a g-W-Q-n-AS of $\mathcal{V}_g$ which is not g-Q-n-AS of $\mathcal{V}_g$, then $(P_g :_{\mathcal{B}_e} \mathcal{V}_g) \subseteq \sqrt{Ann_{\mathcal{B}}(P_g)}$.*

Proof: Since P_g is a g-W-Q-n-AS of $\mathcal{V}_g$ which is not g-Q-n-AS of $\mathcal{V}_g$, there exists a gr-unbreakable m of P_g. Now, by Lemma 2.7 (2), for every $s_e \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ we have $(m + s_e)^n\ P_g = 0$ and so $(m + s_e) \in \sqrt{Ann_{\mathcal{B}}(P_g)}$ and by Lemma 2.7 (1), $m \in \sqrt{Ann_{\mathcal{B}}(P_g)}$ and so $s_e \in \sqrt{Ann_{\mathcal{B}}(P_g)}$. Hence $(P_g :_{\mathcal{B}_e} \mathcal{V}_g) \subseteq \sqrt{Ann_{\mathcal{B}}(P_g)}$.

Definition 2.9: Let $P = \underset{g \in \Omega}{\oplus} P_g \leq_{\Omega}^{sub} \mathcal{V}$ and $g \in \Omega$. Then P_g is called a g-nilpotent submodule of $\mathcal{V}_g$ if $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)^k\ P_g = 0$ for some positive integer k.

Corollary 2.10: *Let $\mathcal{B}$ be a Ω-graded Noetherian ring, $\mathcal{V}$ a graded $\mathcal{B}$-module and $\mathcal{V}_g \neq P_g$ be a g-W-Q-n-AS of $\mathcal{V}_g$ which is not g-Q-n-AS of $\mathcal{V}_g$ and $g \in \Omega$. Then P_g is a g-nilpotent of $\mathcal{V}_g$.*

Proof: Using Theorem 2.8, we have $(P_g :_{\mathcal{B}_e} \mathcal{V}_g) \subseteq \sqrt{Ann_{\mathcal{B}}(P_g)}$. Since $\mathcal{B}$ is a graded Noetherian, $\mathcal{B}_e$ is Noetherian by [14, Theorem 1.1] and so $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)^k \subseteq Ann_{\mathcal{B}}(P_g)$ and so $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)^k\ P_g = 0$. Hence, P_g is a g-nilpotent submodule of $\mathcal{V}_g$.

Corollary 2.11: *Let $\mathcal{B}$ be a Ω-graded von Neumann regular ring, $\mathcal{V} = \underset{g\in\Omega}{\oplus}\mathcal{V}_g$ a graded $\mathcal{B}$-module, $P = \underset{g\in\Omega}{\oplus}P_g \leq_{\Omega}^{sub} \mathcal{V}$ and $g \in \Omega$. If $\mathcal{V}_g \neq P_g$ is a g-W-Q-n-AS of $\mathcal{V}_g$ which is not g-Q-n-AS of $\mathcal{V}_g$. Then $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)P_g = 0$.*

Proof: Using Theorem 2.8, we have $(P_g :_{\mathcal{B}_e} \mathcal{V}_g) \subseteq \sqrt{Ann_{\mathcal{B}}(P_g)}$. Since $\mathcal{B}$ is a Ω-graded von Neumann regular rings, then $\mathcal{B}_e$ is a von Neumann regular and so $\sqrt{Ann_{\mathcal{B}}(P_g)} = Ann_{\mathcal{B}}(P_g)$, thus $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)P_g = 0$.

Corollary 2.12: *Let $\mathcal{B}$ be a Ω-graded von Neumann regular ring and $P = \underset{g\in\Omega}{\oplus}P_g \leq_{\Omega}^{sub} \mathcal{V}$. If $\mathcal{V}_g \neq P_g$ is a g-W-Q-n-AS of $\mathcal{V}_g$ which is not g-Q-n-AS of $\mathcal{V}_g$ for $g \in \Omega$ and $Ann_{\mathcal{B}}(\mathcal{V}_g) = 0$, then $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)^2 = 0$.*

Proof: By Corollary 2.11, we get $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)\, P_g = 0$ and since $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)^2 \subseteq ((P_g :_{\mathcal{B}_e} \mathcal{V}_g)\, P_g :_{\mathcal{B}_e} \mathcal{V}_g) = (0 :_{\mathcal{B}_e} \mathcal{V}_g) = Ann_{\mathcal{B}}(\mathcal{V}_g) = 0$, we have $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)^2 = 0$.

Theorem 2.13: *Let $\mathcal{V} = \underset{g\in\Omega}{\oplus}\mathcal{V}_g$ be a graded $\mathcal{B}$-module, 2 is a unit in $\mathcal{B}_e$ and $P = \underset{g\in\Omega}{\oplus}P_g \leq_{\Omega}^{sub} \mathcal{V}$. If P_g is a g-W-Q-n-AS of $\mathcal{V}_g$ which is not g-Q-n-AS of $\mathcal{V}_g$ for $g \in \Omega$, then $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)^2 P_g = 0$.*

Proof: Let P_g be a *g-W-Q-n*-AS of $\mathcal{V}_g$ which is not *g-Q-n*-AS of $\mathcal{V}_g$. By Lemma 2.7, for every $s_e \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$, $(r_e + s_e)^2\, P_g = (r_e - s_e)^2\, P_g = 0$ where r_e is a *gr*-unbreakable element of P_g. Which implies that $(r_e + s_e)^2\, P_g + (r_e - s_e)^2\, P_g = (r_e^2 + s_e^2)\, P_g = 2s_e^2 P_g = 0$ since r_e is a *gr*-unbreakable element of P_g. But 2 is a unit in $\mathcal{B}_e$ and so $s_e^2 P_g = 0$ for every $s_e \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. Now, let t_e, $s_e \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. Then $2t_e s_e P_g = ((t_e + s_e)^2 - t_e^2 - s_e^2)\, P_g = 0$ and since 2 is a unit in $\mathcal{B}_e$, we have $t_e s_e P_g = 0$. Hence $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)^2\, P_g = 0$.

Definition 2.14: Let $\mathcal{V} = \underset{g\in\Omega}{\oplus}\mathcal{V}_g$ be a graded $\mathcal{B}$-module and $P = \underset{g\in\Omega}{\oplus}P_g \leq_{\Omega}^{sub} \mathcal{V}$. Let $I = \underset{g\in\Omega}{\oplus}I_g \leq_{\Omega}^{id} \mathcal{B}$ and $K = \underset{g\in\Omega}{\oplus}K_g \leq_{\Omega}^{sub} \mathcal{V}$.

(i) We say that P_g is a *g*-weakly strongly quasi *n*-absorbing submodule (*g-W-S-Q-n*-AS) of $\mathcal{V}_g$. If $\mathcal{V}_g \neq P_g$ and $g \in \Omega$; and whenever $0 \neq I_e^n K_g \subseteq P_g$, then either $I_e^{n-1} K_g \subseteq P_g$ or $I_e^n \subseteq (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$.

(ii) We say that P is a graded weakly strongly quasi *n*-absorbing submodule (*Gr-W-S-Q-n*-AS) of $\mathcal{V}$. If $P \neq \mathcal{V}$ and $\lambda \in \Omega$; and whenever $0 \neq I_e^n K_\lambda \subseteq P$, then either $I_e^{n-1} K_\lambda \subseteq P$ or $I_e^n \subseteq (P :_{\mathcal{B}} \mathcal{V})$.

It is clear that every *Gr-W-S-Q-n*-AS is a Ω-*W-Q-n*-AS.

Theorem 2.15: *Let* $\mathcal{V} = \underset{g\in\Omega}{\oplus} \mathcal{V}_g$ *be a graded* $\mathcal{B}$*-module and* $P = \underset{g\in\Omega}{\oplus} P_g \leq_{\Omega}^{sub} \mathcal{V}$. *If* P *is a Gr-W-S-Q-n-AS of* $\mathcal{V}$ *and* $\mathcal{V}_g \neq P_g$ *where* $g \in \Omega$, *then* P_g *is a g-W-S-Q-n-AS of* $\mathcal{V}_g$.

Proof: Let P be a *Gr-W-S-Q-n*-AS of $\mathcal{V}$ and $g \in \Omega$ such that $\mathcal{V}_g \neq P_g$. Let $0 \neq I_e^n K_g \subseteq P_g$ for some ideal I_e of $\mathcal{B}_e$ and an $\mathcal{B}_e$-submodule K_g of $\mathcal{V}_g$. Then $0 \neq I_e^n K_g \subseteq P$ and since P is a Ω-*W-S-Q-n*-AS of $\mathcal{V}$, either $I_e^{n-1} K_g \subseteq P$ or $I_e^n \subseteq (P :_{\mathcal{B}} \mathcal{V})$ and so either $I_e^{n-1} K_g \subseteq P_g$ or $I_e^n \subseteq (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. Hence P_g is a *Gr-W-S-Q-n*-AS of $\mathcal{V}_g$.

Theorem 2.16: *Let* $\mathcal{B}$ *be a* Ω*-graded principal domain and* $P = \underset{g\in\Omega}{\oplus} P_g <_{\Omega}^{sub} \mathcal{V}$. *Then the following statemants are equivalent;*

(1) P is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

(2) P is a *Gr-W-S-Q-n*-AS of $\mathcal{V}$.

Proof: $(1) \Rightarrow (2)$. Let $g \in \Omega$ such that $0 \neq I_e^n K_g \subseteq P$ for some ideal I_e of $\mathcal{B}_e$ and $\mathcal{B}_e$-submodule K_g of $\mathcal{V}_g$. As $\mathcal{B}$ is a graded principal domain, $I_e = a\mathcal{B}_e$ and so $0 \neq a^n K_g \subseteq P$. Assume that $a^n \notin (P :_{\mathcal{B}} \mathcal{V})$. Claim : $a^{n-1} K_g \subseteq P$.

Let $x_g \in K_g$. If $0 \neq a^n x_g \in P$, then $a^{n-1} x_g \in P$ since P is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

If $0 = a^n x_g$. Since $0 \neq a^n K_g$, then there exists $y_g \in K_g$ such that $0 \neq a^n y_g = a^n(x_g + y_g) \in P$ so $a^{n-1}(x_g + y_g) \in P$ and hence $a^{n-1} x_g \in P$. Thus $a^{n-1} K_g \subseteq P$ and so $I_e^{n-1} K_g \subseteq P$. So P is a *Gr-W-S-Q-n*-AS of $\mathcal{V}$.

$(2) \Rightarrow (1)$ Straightforward.

Theorem 2.17: *Let* $P = \underset{g\in\Omega}{\oplus} P_g <_{\Omega}^{sub} \mathcal{V}$. *Then the following are equivalent;*

(1) P is a *Gr-W-S-Q-n*-AS of $\mathcal{V}$.

(2) If $0 \neq I_e^n x \subseteq P$ for some ideal I_e of $\mathcal{B}_e$ and $x \in h(\mathcal{V})$, then $I_e^n \subseteq (P :_{\mathcal{B}} \mathcal{V})$ or $I_e^{n-1} x \subseteq P$.

Proof: $(1) \Rightarrow (2)$ Straightforward.

(2) $\Rightarrow$ (1). Let $g \in \Omega$ such that $0 \neq I_e^n K_g \subseteq P$ for some ideal I_e of $\mathcal{B}_e$ and $\mathcal{B}_e$-submodule K_g of $\mathcal{V}_g$. Suppose that $I_e^n \nsubseteq (P :_{\mathcal{B}} \mathcal{V})$. We want to show that $I_e^{n-1} K_g \subseteq P$, by the way of contradiction; suppose that $I_e^{n-1} K_g \nsubseteq P$. Then $\exists x_g \in K_g$ with $I_e^{n-1} x_g \nsubseteq K$.

Case 1: If $0 \neq I_e^n x_g \subseteq P$, then by our assumption, $I_e^{n-1} x_g \subseteq P$ since $I_e^n \nsubseteq (P :_{\mathcal{B}} \mathcal{V})$, a contradiction.

Case 2: If $0 = I_e^n x_g$. Since $0 \neq I_e^n K_g \subseteq P$, then there exists $y_g \in K_g$ such that $0 \neq I_e^n y \subseteq P$, and so $0 \neq I_e^n (x_g + y_g) \subseteq P$. Which implies that $I_e^{n-1} (x_g + y_g) \subseteq P$ and so $I_e^{n-1} x_g \subseteq P$, a contradiction. So, $I_e^{n-1} K_g \subseteq P$. So P is a Ω-*W*-*S*-*Q*-*n*-AS of $\mathcal{V}$.

Theorem 2.18: *Let* $P \leq_{\Omega}^{sub} \mathcal{V}$ *and* $L \leq_{\Omega}^{sub} \mathcal{V}$ *with* $L \subseteq P$. *Then;*

(1) If P is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}$, then P/L is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}/L$.

(2) If L and P/L is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}$ and $\mathcal{V}/L$, then P is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}$.

Proof:

(1) Let P be a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}$. Then $P/L \neq \mathcal{B}/L$. Let $0 \neq a^n (x+L) \in P/L$ where $a \in h(\mathcal{B})$ and $x \in h(\mathcal{V})$. If $a^n \in (P :_{\mathcal{B}} \mathcal{V})$, then we are done. Now, assume that $a^n \notin (P :_{\mathcal{B}} \mathcal{V})$. Since $0 \neq a^n (x+L) \in P/L$, then $a^n x \in P$ and $a^n x \notin L$ and so $0 \neq a^n x \in P$. But P is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}$, so $a^{n-1} x \in P$. Therefore, $a^{n-1} (x+L) \in P/L$. So, P/L is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}/L$.

(2) Let L and P/L be *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}$ and $\mathcal{V}/L$. Let $0 \neq a^n x \in P$ for some $a \in h(\mathcal{B})$ and $x \in h(\mathcal{V})$. Then, $a^n (x+L) \in P/L$.

Case 1: If $a^n (x+L) = 0$, so $a^n x \in L$. Since L is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}$, then either $a^n \in (L :_{\mathcal{B}} \mathcal{V})$ or $a^{n-1} x \in L$ and since $L \subseteq P$, we have either $a^n \in (P :_{\mathcal{B}} \mathcal{V})$ or $a^{n-1} x \in P$.

Case 2: If $0 \neq a^n (x+L) \in P/L$. Since P/L is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}/L$, then either $a^n \in (P/L :_{\mathcal{B}/L} \mathcal{V}/L)$ or $a^{n-1} (x+L) \in P/L$ and so either $a^n \in (P :_{\mathcal{B}} \mathcal{V})$ or $a^{n-1} x \in P$.

So, P is a *Gr*-*W*-*Q*-*n*-AS of $\mathcal{V}$.

Definition 2.19: Let $\mathcal{B}$ be a Ω-graded ring and $I = \underset{g\in\Omega}{\oplus} I_g$ be a proper graded ideal of $\mathcal{B}$. Then

(1) we say that I is a graded weakly semi n-absorbing ideal (*Gr-W*-semi n-AI) of $\mathcal{B}$ if $0 \neq x^{n+1} \in I$ implies that $x^n \in I$ where $x \in h(\mathcal{B})$.

(2) We say that I is a graded Quasi n-absorbing ideal (*Gr-Q-n*-AI) of $\mathcal{B}$ if $r^n t \in I$ implies that $r^n \in I$ or $r^{n-1}t \in I$ where $r, t \in h(\mathcal{B})$.

(3) We say that $I_g \neq R_r$ is a g-weakly Quasi n-absorbing ideal (*g-W-Q-n*-AI) of $\mathcal{B}_e$ if $0 \neq r_e^n t_e \in I_g$ implies that $r_e^n \in I_g$ or $r_e^{n-1} t_e \in I_g$ where $r_e, t_e \in \mathcal{B}_e$.

(4) We say that I is graded weakly Quasi a n-absorbing ideal (*Gr-W-Q-n*-AI) of $\mathcal{B}$ if $0 \neq r^n t \in I$ implies that $r^n \in I$ or $r^{n-1}t \in I$ where $r, t \in h(\mathcal{B})$.

It is evident that every *Gr-Q-n*-AI is a *Gr-W-Q-n*-AI of $\mathcal{B}$ while the converse is not true and every *Gr-W-Q-n*-AI is a *Gr-W*-semi n-AI of $\mathcal{B}$.

Example 2.20: Let $\Omega = \mathbb{Z}_2$ and $\mathcal{B} = \mathbb{Z}_{18}$. Thus $\mathcal{B}$ is a Ω-graded ring with $\mathcal{B}_0 = \mathbb{Z}_{18}$ and $\mathcal{B}_1 = \{0\}$. So $I = \{0\}$ is a *Gr-W-Q-n*-AI of $\mathcal{B}$ which is not *Gr-Q-n*-AI since $3^2.2 \in I$, $3^2 \notin I$ and $3.2 \notin I$.

Theorem 2.21: *Let $I = \underset{g\in\Omega}{\oplus} I_g$ be a graded ideal of $\mathcal{B}$ and $g \in \Omega$ with $I_g \neq \mathcal{B}_g$. If I is a Gr-W-Q-n-AI of $\mathcal{B}$, then I_g is a g-W-Q-n-AI of $\mathcal{B}_g$.*

Proof: Let I be a *Gr-W-Q-n*-AI of $\mathcal{B}$. Let $r_e, t_e \in \mathcal{B}_e$ such that $0 \neq r_e^n t_e \in I_g \subseteq I$ and so either $r_e^n \in I$ or $r_e^{n-1} t_e \in I$ and so either $r_e^n \in I_e$ or $r_e^{n-1} t_e \in I_g$ since $I_g = I \cap \mathcal{B}_g$ and $\mathcal{B}_g \subseteq \mathcal{B}$. Hence, I_g is a *g-W-Q-n*-AI of $\mathcal{B}_g$.

Theorem 2.22: *Let $\underset{g\in\Omega}{\oplus} P_g <_{\Omega}^{sub} \mathcal{V}$.*

(1) If $(P :_{\mathcal{B}} t)$ is a *Gr-W*-semi $(n-1)$-AI of $\mathcal{B}$ for every $t \in h(\mathcal{V}) - P$, then P is a *Gr-W-Q-n*-AS of $\mathcal{V}$ for every $n \geq 3$.

(2) If P is a *Gr-W-Q-n*-AS of $\mathcal{V}$ and $Ann(t)$ is is a *Gr-W-Q-n*-AI of $\mathcal{B}$ for every $t \in h(\mathcal{V}) - P$, then $(P :_{\mathcal{B}} t)$ is a *Gr-W-Q-n*-AI of $\mathcal{B}$ for every $t \in h(\mathcal{V}) - P$.

Proof:

(1) Let $(P :_{\mathcal{B}} t)$ be a *Gr-W*-semi $(n-1)$-AI ideal of $\mathcal{B}$ for every $t \in h(\mathcal{V}) - P$. Let $0 \neq r^n m \in P$ where $r \in h(\mathcal{B})$ and $m \in h(\mathcal{V})$. If $m \in P$, then $r^{n-1}m \in P$ and so we are done. We may suppose that $m \in h(\mathcal{V}) - P$. If $r^n \in (P :_{\mathcal{B}} \mathcal{V})$, then we are done. Suppose that $r^n \notin (P :_{\mathcal{B}} \mathcal{V})$, so $r^n \neq 0$. By our assumption $(P :_{\mathcal{B}} m)$ is a *Gr-W*-semi $(n-1)$-AI of $\mathcal{B}$, and $0 \neq r^n \in (P :_{\mathcal{B}} m)$, then $r^{n-1} \in (P :_{\mathcal{B}} m)$ and so $r^{n-1}m \in P$. So, P is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

(2) Let P be a *Gr-W-Q-n*-AS of $\mathcal{V}$ and $Ann(t)$ is is a *Gr-W-Q-n*-AI of $\mathcal{B}$ for every $t \in h(\mathcal{V}) - P$. Let $t \in h(\mathcal{V}) - P$ and $r, s \in h(\mathcal{B})$ such that $0 \neq r^n s \in (P :_{\mathcal{B}} t)$ and $r^n \notin (P :_{\mathcal{B}} t)$. If $0 \neq r^n st \in P$, then $r^{n-1}st \in P$ and so $r^{n-1}s \in (P :_{\mathcal{B}} t)$ since P is a *Gr-W-Q-n*-AS of $\mathcal{V}$. If $0 = r^n st$, then $r^n s \in Ann(t)$ which is *Gr-Q-n*-AI of $\mathcal{B}$, so $r^{n-1}s \in Ann(t)$ and so $r^{n-1}s \in (P :_{\mathcal{B}} t)$. Hence, $(P :_{\mathcal{B}} t)$ is a *Gr-W-n*-AI of $\mathcal{B}$.

Lemma 2.23: *Let* $P = \bigoplus_{g \in \Omega} P_g <_{\Omega}^{sub} \mathcal{V}$. *Then the following are equivalent:*

(1) P is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

(2) For every $r \in h(\mathcal{B})$ and a submodule L_g of $\mathcal{V}_g$ with $0 \neq r^n L_g \subseteq P$, we have $r^{n-1}L_g \subseteq P$ or $r^n \in (P :_{\mathcal{B}} \mathcal{V})$.

Proof: $(1) \Rightarrow (2)$. Let $r \in h(\mathcal{B})$ and a submodule L_g of $\mathcal{V}_g$ with $0 \neq r^n L_g \subseteq P$ and $r^n \notin (P :_{\mathcal{B}} \mathcal{V})$. We want to show that $r^{n-1}L_g \subseteq P$. Let $m \in L_g$. If $0 \neq r^n m \in P$, then $r^{n-1}m \in P$ since P is a *Gr-W-Q-n*-AS of $\mathcal{V}$. If $0 = r^n m$. Since $0 \neq r^n L_g$, then there exists $l \in L_g$ such that $0 \neq r^n l = r^n (l + m) \in P$ so $r^{n-1}(l + m) \in P$ and so $r^{n-1}m \in P$. Which implies that $r^{n-1}L_g \subseteq P$.

$(2) \Rightarrow (1)$. Let $0 \neq r^n m \in P$ for some $r \in h(\mathcal{B})$ and $m \in h(\mathcal{V})$. Let $L_\lambda = m\mathcal{B}_e$ be a submodule of $\mathcal{V}_\lambda$ where $\lambda \in \Omega$. Then $0 \neq r^n L_\lambda \subseteq P$, by our assumption either $r^{n-1}L_\lambda \subseteq P$ or $r^n \in (P :_{\mathcal{B}} \mathcal{V})$. Thus, either $r^{n-1}m \in P$ or $r^n \in (P :_{\mathcal{B}} \mathcal{V})$. Hence, P is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

It is clear that the previous Lemma is true for P_g

Theorem 2.24: *Let* $\mathcal{V} = \bigoplus_{g \in \Omega} \mathcal{V}_g$ *be a faithful graded* $\mathcal{B}$*-module and* $P = \bigoplus_{g \in \Omega} P_g \leq_{\Omega}^{sub} \mathcal{V}$. *Let* P_g *be a proper submodule of a faithful* $\mathcal{V}_g$*.Then;*

(1) If P_g is a *g-W-Q-n*-AS of $\mathcal{V}_g$, then $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ is a *W-Q-n*-AI of $\mathcal{B}_e$.

(2) If $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ is a g-W-Q-n-AI of $\mathcal{B}_e$ and $\mathcal{V}_g$ is cyclic faithful $\mathcal{B}_e$ -module, then P_g is a W-Q-n-AS of $\mathcal{V}_g$.

Proof:

(1) Let P_g be a g-W-Q-n-AS of $\mathcal{V}_g$. Let $0 \neq r^n s \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ for some r, $s \in \mathcal{B}_e$. Since $\mathcal{V}_g$ is faithful $\mathcal{B}_e$-module, then $0 \neq r^n s\mathcal{V}_g$ and so set $L_g = s\mathcal{V}_g$ is a submodule of $\mathcal{V}_g$. Now, $0 \neq r^n L_g \subseteq P_g$ and since P_g is a g-W-Q-n-AS of $\mathcal{V}_g$, then by Lemma 2.23, we have either $r^{n-1}L_g \subseteq P_g$ or $r^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. Hence, $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ is a W-Q-n-AI of $\mathcal{B}_e$.

(2) Let $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ be a W-Q-n-AI of $\mathcal{B}_e$ and $\mathcal{V}_g = mR_e$ is cyclic faithful $\mathcal{B}_e$ -module. Let $0 \neq r^n t \in P_g$ for some $r \in \mathcal{B}_e$ and $t \in \mathcal{V}_g$, so there exists $b \in \mathcal{B}_e$ with $t = bm$ and hence $0 \neq r^n bm \in P_g$. Which implies that $0 \neq r^n b\ (P_g :_{\mathcal{B}_e} m) = (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. Since $(P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ is a W-Q-n-AI of $\mathcal{B}_e$, we have either $r^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ or $r^{n-1}b \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$. Thus, either $r^n \in (P_g :_{\mathcal{B}_e} \mathcal{V}_g)$ or $r^{n-1}bm = r^{n-1}t \in P_g$. Hence, P_g is a g-W-Q-n-AS of $\mathcal{V}_g$.

Theorem 2.25: *Let* $P = \underset{g\in\Omega}{\oplus} P_g <^{sub}_{\Omega} \mathcal{V}$. *If P is a Gr-W-semiprime submodule of* $\mathcal{V}$, *then P is a Gr-W-Q-n-AS of* $\mathcal{V}$ *for every positive integer* $n \geq 2$.

Proof: Let P be a Gr-W-semiprime submodule of $\mathcal{V}$. Let $0 \neq r^n m \in P$ for some $r \in h(\mathcal{B})$, $m \in h(\mathcal{V})$ and for some positive integer $n \geq 2$. Then $0 \neq r^2 (r^{n-2}m) \in P$. Since P is a Gr-W-semiprime submodule of $\mathcal{V}$, $r^{n-1}m \in P$. Hence, P is a Gr-W-Q-n-AS of $\mathcal{V}$.

Theorem 2.26: *Let* $\{P_i\}_{i\in I}$ *be a family of Gr-W-semiprime submodules of* $\mathcal{V}$. *Then* $P = \underset{i\in I}{\cap} P_i$ *is a Gr-W-Q-n-AS of* $\mathcal{V}$ *for all positive integer* $n \geq 2$.

Proof: Let $0 \neq r^n m \in P$ for some $r \in h(\mathcal{B})$, $m \in h(\mathcal{V})$ and for some positive integer $n \geq 2$. Then $0 \neq r^n m = r^2 (r^{n-2}m) \in P_i$ for all $i \in I$. Since P_i is a Gr-W-semiprime submodule of $\mathcal{V}$, $r^{n-1}m \in P_i$ for all $i \in I$ and so $r^{n-1}m \in P$. Hence, P is a Gr-W-Q-n-AS of $\mathcal{V}$.

Theorem 2.27: *Let* $\{P_\alpha\}_{\alpha\in J}$ *be a chain of Gr-W-Q-n-AS of* $\mathcal{V}$. *Then* $P = \underset{\alpha\in J}{\cap} P_\alpha$ *is a Gr-W-Q-n-AS of* $\mathcal{V}$.

Proof: Let $0 \neq r^n m \in P$ for some $r \in h(\mathcal{B})$ and $m \in h(\mathcal{V})$. Then $0 \neq r^n m \in P_\alpha$ for each $\alpha \in J$. Since P_α is a *Gr-W-Q-n*-absorbing submodules of $\mathcal{V}$, then we have two cases possible;

Case 1: If $r^n \in (P_\alpha :_{\mathcal{B}} \mathcal{V})$ for all $\alpha \in J$, then $r^n \in \bigcap_{\alpha \in J}(P_\alpha :_{\mathcal{B}} \mathcal{V}) = (\bigcap_{\alpha \in J} P_\alpha :_{\mathcal{B}} \mathcal{V}) = (P :_{\mathcal{B}} \mathcal{V})$.

Case 2: If $r^n \notin (P_{\alpha'} :_{\mathcal{B}} \mathcal{V})$ for some $\alpha' \in J$. Then $r^n \notin (P_\alpha :_{\mathcal{B}} \mathcal{V})$ for all $P_\alpha \subseteq P_{\alpha'}$. Since P_α is a *Gr-W-Q-n*-absorbing submodules of $\mathcal{V}$, then we have $r^{n-1} m \in P_\alpha$ for all $P_\alpha \subseteq P_{\alpha'}$. It follows that $r^{n-1} m \in P = \bigcap_{\alpha \in J} P_\alpha$. So, $P = \bigcap_{\alpha \in J} P_\alpha$ is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

Lemma 2.28: *Let $\mathcal{B}$ be a Ω-graded ring and $\mathcal{V}_1$, $\mathcal{V}_2$ be graded $\mathcal{B}$-modules with $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, $n \in \mathbb{Z}$ and P_1 (resp., P_2) be a proper Gr-W-Q-n-AS of $\mathcal{V}_1$ (resp., $\mathcal{V}_2$). Let $r \in h(\mathcal{B})$. Then the following statement are equivalent;*

(1) r is $gr-$unbreakable element of P_1 (resp., P_2).

(2) r is $gr-$unbreakable element of $P_1 \oplus \mathcal{V}_2$ (resp., $\mathcal{V}_1 \oplus P_2$).

Proof: $(1) \Rightarrow (2)$. Let r be $gr-$unbreakable element of P_1. Then there exists $m \in h(\mathcal{V}_1)$ with $r^n m = 0$ and neither $r^n \in (P_1 :_{\mathcal{B}} \mathcal{V}_1)$ nor $r^{n-1} m \in P_1$. Then $r^n(m,0) = (0,0)$ and neither $r^n \in (P_1 \oplus \mathcal{V}_2 :_{\mathcal{B}} \mathcal{V}_1 \oplus \mathcal{V}_2)$ nor $r^{n-1}(m,0) \in P_1 \oplus \mathcal{V}_2$. Hence r is $gr-$unbreakable element of $P_1 \oplus \mathcal{V}_2$.

$(2) \Rightarrow (1)$. Let r be $gr-$unbreakable element of $P_1 \oplus \mathcal{V}_2$. Then there exists $(x,y) \in h(\mathcal{V}_1 \oplus \mathcal{V}_2)$ with $r^n(x,y) = (0,0)$ and neither $r^n \in (P_1 \oplus \mathcal{V}_2 :_{\mathcal{B}} \mathcal{V}_1 \oplus \mathcal{V}_2)$ nor $r^{n-1}(x,y) \in P_1 \oplus \mathcal{V}_2$. Hence, $r^n x = 0$ for some $x \in h(\mathcal{V}_1)$ and neither $r^n \in (P_1 :_{\mathcal{B}} \mathcal{V}_1)$ nor $r^{n-1} x \in P_1$. Thus, r is $gr-$unbreakable element of P_1.

Similarly, we can show that r is $gr-$unbreakable element of P_2 if and only if r is $gr-$unbreakable element of $\mathcal{V}_1 \oplus P_2$.

Theorem 2.29: *Let $\mathcal{B}$ be a Ω-graded ring and $\mathcal{V}_1$, $\mathcal{V}_2$ be graded $\mathcal{B}$-modules and P_1 (resp., P_2) be a proper submodule of $\mathcal{V}_1$ (resp., $\mathcal{V}_2$). Consider the following assertions;*

(1) P_1 is a *Gr-W-Q-n*-AS of $\mathcal{V}_1$, P_2 is a *Gr-Q-n*-AS of $\mathcal{V}_2$ and $0 = r^n m$ whenever $r^n m \in P_2$ where $r \in h(\mathcal{B})$ and $m \in h(\mathcal{V}_2)$.

(2) P_2 is a *Gr-W-Q-n*-AS of $\mathcal{V}_2$, P_1 is a *Gr-Q-n*-AS of $\mathcal{V}_1$ and $0 = s^n l$ whenever $s^n l \in P_1$ where $s \in h(\mathcal{B})$ and $l \in h(\mathcal{V}_1)$.

If (1) or (2) holds, then $P_1 \oplus P_2$ is a *Gr-W-Q-*$(n+1)$-AS of $\mathcal{V}_1 \oplus \mathcal{V}_2$.

Proof: Assume that (1) holds. Let $0 \neq r^{n+1}(x, y) \in P_1 \oplus P_2$ for some $r \in h(\mathcal{B})$ and $(x, y) \in h(\mathcal{V}_1 \oplus \mathcal{V}_2)$. By our assumption, $r^n(ry) = 0$ and $0 \neq r^n(rx) \in P_1$. Since P_1 is a *Gr-W-Q-n*-AS of $\mathcal{V}_1$, then either $r^n x \in P_1$ or $r^n \in (P_1 :_{\mathcal{B}} \mathcal{V}_1)$ and so $r^n x \in P_1$. Since $0 = r^n(ry) \in P_2$ which is a *Gr-Q-n*-AS of $\mathcal{V}_2$, then $r^n y \in P_2$. Thus $r^n(x, y) \in P_1 \oplus P_2$. Hence, $P_1 \oplus P_2$ is a *Gr-W-Q-*$(n+1)$-AS of $\mathcal{V}_1 \oplus \mathcal{V}_2$. The same argument if assertion (2) holds.

Theorem 2.30: *Let $U \subseteq h(\mathcal{B})$ be a multiplication closed subset of $\mathcal{B}$. Then;*

(1) If P is a *Gr-W-Q-n*-AS of $\mathcal{V}$, then $U^{-1}P$ is a *Gr-W-Q-n*-AS of $U^{-1}\mathcal{V}$.

(2) If $U^{-1}P$ is a *Gr-W-Q-n*-AS of $U^{-1}\mathcal{V}$ and $U \cap \Omega - Zdv_{\mathcal{B}}(\mathcal{V}/P) = \phi$, then P is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

Proof:

(1) Let P be a *Gr-W-Q-n*-AS of $\mathcal{V}$. Let $\frac{0}{1} \neq (\frac{r}{s_1})^n(\frac{m}{s_2}) \in U^{-1}P$. Then $0 \neq r^n mu \in P$ for some $u \in U$. So, $0 \neq (ru)^n\, m \in P$ which is a *Gr-W-Q-n*-AS of $\mathcal{V}$ and so either $(ru)^{n-1} m \in P$ or $(ru)^n \in (P :_{\mathcal{B}} \mathcal{V})$. Therefore, $\frac{u^{n-1}r^{n-1}m}{u^{n-1}s_1^{n-1}s_2} = (\frac{r}{s_1})^{n-1}(\frac{m}{s_2}) \in U^{-1}P$ or $\frac{u^n r^n}{u^n s_1^n} = (\frac{r}{s})^n \in U^{-1}(P :_{\mathcal{B}} \mathcal{V}) \subseteq (U^{-1}P :_{U^{-1}\mathcal{B}} U^{-1}\mathcal{V})$. Hence, $U^{-1}P$ is a *Gr-W-Q-n*-AS of $U^{-1}\mathcal{V}$.

(2) Let $U^{-1}P$ be a *Gr-W-Q-n*-AS of $U^{-1}\mathcal{V}$ and $U \cap \Omega - Zdv_{\mathcal{B}}(\mathcal{V}/P) = \phi$. Let $r \in h(\mathcal{B})$ and $m \in h(\mathcal{V})$ with $0 \neq r^n m \in P$. Thus $\frac{0}{1} \neq (\frac{r}{1})^n(\frac{m}{1}) \in U^{-1}P$ and since $U^{-1}P$ is a *Gr-W-Q-n*-AS of $U^{-1}\mathcal{V}$, then either $(\frac{r}{1})^n \in (U^{-1}P :_{U^{-1}\mathcal{B}} U^{-1}\mathcal{V})$ or $(\frac{r}{1})^{n-1}(\frac{m}{1}) \in U^{-1}P$.

If $(\frac{r}{1})^{n-1}(\frac{m}{1}) \in U^{-1}P$, then $\exists s \in U$ with $r^{n-1}ms \in P$ and since $U \cap \Omega - Zdv_{\mathcal{B}}(\mathcal{V}/P) = \phi$, we get $r^{n-1}m \in P$.

If $(\frac{r}{1})^n \in (U^{-1}P :_{U^{-1}\mathcal{B}} U^{-1}\mathcal{V})$, then $r^n U^{-1}\mathcal{V} \subseteq U^{-1}P$. We want to show that $r^n \in (P :_{\mathcal{B}} \mathcal{V})$. Let $l \in h(\mathcal{V})$, then $\frac{r^n l}{1} \in r^n U^{-1}\mathcal{V} \subseteq U^{-1}P$ and so $\exists s \in U$ with $r^n ls \in P$ and since $U \cap \Omega - Zdv_{\mathcal{B}}(\mathcal{V}/P) = \phi$, we get $r^n l \in P$. Therefore, $r^n \in (P :_{\mathcal{B}} \mathcal{V})$.

Hence, P is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

Theorem 2.31: *Let* $g:\mathcal{V}\to\mathcal{V}'$ *be a graded homomorphism of graded* $\mathcal{B}-$ *modules.*

(1) Assume that g is a graded monomorphism. If P' is a *Gr-W-Q-n*-AS of $\mathcal{V}'$, then $g^{-1}(P')$ is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

(2) Assume that g is a graded epimorphism and $Ker(g)\subseteq P$. If P is a *Gr-W-Q-n*-AS of $\mathcal{V}$, then $g(P)$ is a *Gr-W-Q-n*-AS of $\mathcal{V}'$.

Proof:

(1) Let $0\neq r^n m\in g^{-1}(P')$ for some $r\in h(\mathcal{B})$ and $m\in h(\mathcal{V})$. Hence $0\neq r^n g(m)\in P'$ and since P' is a *Gr-W-Q-n*-AS of $\mathcal{V}'$, then either $r^n\in (P':_{\mathcal{B}}\mathcal{V}')$ or $r^{n-1}g(m)\in P'$. Which implies that $r^n\mathcal{V}'\subseteq P'$ or $g(r^{n-1}m)\in P'$, and so either $r^n\mathcal{V}\subseteq g^{-1}(P')$ or $r^{n-1}m\in g^{-1}(P')$. Thus, either $r^n\in(g^{-1}(P'):_{\mathcal{B}}\mathcal{V})$ or $r^{n-1}m\in g^{-1}(P')$. Hence, $g^{-1}(P')$ is a *Gr-W-Q-n*-AS of $\mathcal{V}$.

(2) Let $r\in h(\mathcal{B})$ and $m\in h(\mathcal{V}')$ with $0\neq r^n x\in g(P)$. Then $\exists m\in h(\mathcal{V})$ with $g(m)=x$. So, $0\neq r^n x = r^n g(m) = g(r^n m)\in g(P)$ and $Ker(g)\subseteq P$, then $0\neq r^n m\in P$ which is a *Gr-W-Q-n*-AS of $\mathcal{V}$. Therefore, $r^n\in (P:_{\mathcal{B}}\mathcal{V})$ or $r^{n-1}m\in P$ and so $r^n\mathcal{V}\subseteq P$ or $r^{n-1}m\in P$. It follows that $r^n\mathcal{V}'\subseteq g(P)$ or $r^{n-1}g(m) = r^{n-1}x\in P$. So, $g(P)$ is a *Gr-W-Q-n*-AS of $\mathcal{V}'$.

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