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New results of Khalouta transform method for non-constant coefficients ordinary differential equations

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Abstract

The key objective of this manuscript is to present new results for solving non-constant coefficients ordinary differential equations using a new integral transform called Khalouta transform (KhT). This new integral transform was defined and developed by the author in 2023. The new results presented consist in finding general formulas of solutions for the non-constant coefficients ordinary differential equations. Various numerical examples are presented in order to show the efficiency of the obtained results. Numerical test results confirm that the KhT technique is very effective in obtaining the exact solution of linear differential problems.

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1. Introduction

Many problems that arise in applied mathematics, mathematical physics and engineering sciences are usually described by differential equations [3, 4, 8, 9, 11, 13, 15, 17]. The ordinary differential equations appear in many areas of mathematics, as well as science and engineering. Many mathematical techniques have been applied by many scientists and researchers in various fields of science and engineering to find solutions to

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these equations. Recently, integral transforms have been widely used in order to solve various kinds of differential equations.

Integral transforms have their origins in the 19^{th} century work of Joseph Fourier and Oliver Heaviside, then set up in a general framework in the 20^{th} century. The basic idea is to represent a function $f(t)$ in terms of a transform $I(s)$

$$I(s)=\int_{-\infty}^{+\infty}K(s,t)f(t)dt. \tag{1.1}$$

Here, $K(s,t)$ represents the kernel.

There are a number of important integral transforms like that of Laplace transform [7], Fourier transform [2], Sumudu transform [12], Elzaki transform [5], Aboodh transform [1], Shehu transform [10], ZZ-transform [16], they are defined by choosing different kernels.

Our aim in this manuscript is to present new results for solving non-constant coefficients ordinary differential equations using a new integral transform known as the KhT which is a generalization of the integral transforms mentioned above.

The remaining part of this manuscript is structured in five sections. In Sec. 2, we introduce the definition and properties of the KhT. In Sec. 3, we give our results in detail and study their application for solving non-constant coefficients ordinary differential equations under initial conditions (ICs) through illustrative numerical tests in Sec. 4. Lastly, Sec. 5 is devoted to the conclusions of the current research work.

2. Definition and properties of the KhT

Definition 2.1: [6] Let set $\mathcal{S}$ define as

$$\mathcal{S}=\{u(t):\exists K,\vartheta_1,\vartheta_2>0,|u(t)|<K\exp(\vartheta_j\,|t|),\text{if } t\in(-1)^j\times[0,\infty)\}. \tag{2.1}$$

The KhT of a function $u(t)\in\mathcal{S}$ is define as

$$\mathbb{KH}[u(t)]=\mathcal{K}(s,\gamma,\eta)=\frac{s}{\gamma\eta}\int_0^{\infty}\exp\left(-\frac{st}{\gamma\eta}\right)u(t)dt, \tag{2.2}$$

where $s>0,\gamma>0,\ \eta>0$ are the parameters of the KhT.

Property 1: (Linear property) The KhT is a linear operator

$$\mathbb{KH}[\lambda u(t)\pm\mu v(t)]=\lambda\mathbb{KH}[u(t)]\pm\mu\mathbb{KH}[v(t)], \tag{2.3}$$

where λ and μ are non-zero constant numbers.

Property 2: (Derivative property) If $u(t)$ is n^{th} differentiable, then

$$\mathbb{KH}[u^{(n)}(t)] = \frac{s^n}{\gamma^n \eta^n} \mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1} \left(\frac{s}{\gamma\eta} \right)^{n-k} u^{(k)}(0). \tag{2.4}$$

Property 3: (Inverse property) If $\mathbb{KH}[u(t)] = \mathcal{K}(s,\gamma,\eta)$ is the KhT of a function $u(t)$, then the inverse KhT of $u(t)$ is given by

$$\mathbb{KH}^{-1}[\mathcal{K}(s,\eta,\gamma)] = u(t), \text{for } t \geq 0. \tag{2.5}$$

Property 4: The inverse KhT of some basic functions

$$\begin{aligned}
\mathbb{KH}^{-1}[1] &= 1, \\
\mathbb{KH}^{-1}\left[\frac{\gamma\eta}{s} \right] &= t, \\
\mathbb{KH}^{-1}\left[\frac{\gamma^n \eta^n}{s^n} \right] &= \frac{t^n}{n!}, n \geq 0, \\
\mathbb{KH}^{-1}\left[\frac{s}{s - a\gamma\eta} \right] &= \exp(at), \\
\mathbb{KH}^{-1}\left[\frac{as\gamma\eta}{s^2 + a^2\gamma^2\eta^2} \right] &= \sin(at), \\
\mathbb{KH}^{-1}\left[\frac{as\gamma\eta}{s^2 - a^2\gamma^2\eta^2} \right] &= \sinh(at), \\
\mathbb{KH}^{-1}\left[\frac{s^2}{s^2 + a^2\gamma^2\eta^2} \right] &= \cos(at), \\
\mathbb{KH}^{-1}\left[\frac{s^2}{s^2 - a^2\gamma^2\eta^2} \right] &= \cosh(at),
\end{aligned} \tag{2.6}$$

where a is a constant number.

3. Fundamental results

Theorem 3.1: *The Khalouta transform of Bessel's function of zero order denoted by* $J_0(t)$ *is given by*

$$\mathbb{KH}[J_0(t)] = \frac{s}{\sqrt{s^2 + \gamma^2\eta^2}}. \tag{3.1}$$

Proof: Consider Bessel's function of order n, where $n \in \mathbb{N}$ is given by [14]

$$J_n(t)=\frac{t^n}{2^n n!}\left[1-\frac{t^2}{2.(2n+2)}+\frac{t^4}{2.4.(2n+2)(2n+4)}-\frac{t^6}{2.4.6.(2n+2)(2n+4)(2n+6)}+\ldots\right]. \tag{3.2}$$

When $n=0$, Bessel's function of zero order is given by

$$J_0(t)=1-\frac{t^2}{2^2}+\frac{t^4}{2^2.4^2}-\frac{t^6}{2^2.4^2.6^2}+\ldots \tag{3.3}$$

Applying the KhT and Property 1 to equation (3.3) gives

$$\begin{aligned}\mathbb{KH}[J_0(t)]&=\mathbb{KH}[1]-\frac{1}{2^2}\mathbb{KH}[t^2]+\frac{1}{2^2.4^2}\mathbb{KH}[t^4]-\frac{1}{2^2.4^2.6^2}\mathbb{KH}[t^6]+\ldots\\
&=1-\frac{1}{2^2}\frac{2!\gamma^2\eta^2}{s^2}+\frac{1}{2^2.4^2}\frac{4!\gamma^4\eta^4}{s^4}-\frac{1}{2^2.4^2.6^2}\frac{6!\gamma^6\eta^6}{s^6}+..\\
&=1-\frac{1}{2}\left(\frac{\gamma^2\eta^2}{s^2}\right)^1+\frac{1.3}{2.4}\left(\frac{\gamma^4\eta^4}{s^4}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{\gamma^6\eta^6}{s^6}\right)^3+\ldots.\\
&=\frac{1}{\sqrt{1+\frac{\gamma^2\eta^2}{s^2}}}\\
&=\frac{s}{\sqrt{s^2+\gamma^2\eta^2}}.\end{aligned} \tag{3.4}$$

Theorem 3.2: *Let the function* $t\exp(at)\in\mathcal{S}$, *then*

$$\mathbb{KH}[t\exp(at)]=\frac{s\gamma\eta}{(s-a\gamma\eta)^2}, \tag{3.5}$$

where a is a constant number.

Proof: Using equation (2.2), we get

$$\begin{aligned}\mathbb{KH}[t\exp(at)]&=\frac{s}{\gamma\eta}\int_0^\infty\exp\left(-\frac{st}{\gamma\eta}\right)t\exp(at)dt\\
&=\frac{s}{\gamma\eta}\int_0^\infty t\exp\left(-\left(\frac{s-a\gamma\eta}{\gamma\eta}\right)t\right)dt\\
&=\frac{s}{\gamma\eta}\left(-\frac{\gamma\eta}{s-a\gamma\eta}\left[t\exp\left(-\left(\frac{s-a\gamma\eta}{\gamma\eta}\right)t\right)\right]_0^\infty\right)+\frac{s}{\gamma\eta}\left(\frac{\gamma\eta}{s-a\gamma\eta}\int_0^\infty\exp\left(-\left(\frac{s-a\gamma\eta}{\gamma\eta}\right)t\right)dt\right)\\
&=\frac{s}{s-a\gamma\eta}\left[-\frac{\gamma\eta}{s-a\gamma\eta}\exp\left(-\left(\frac{s-a\gamma\eta}{\gamma\eta}\right)t\right)\right]_0^\infty\\
&=\frac{s\gamma\eta}{(s-a\gamma\eta)^2}.\end{aligned} \tag{3.6}$$

Theorem 3.3: *Let* $\mathbb{KH}[u(t)]=\mathcal{K}(s,\eta,\gamma)$ *be KhT of* $u(t)$, *then*

$$\mathbb{KH}[tu(t)] = \frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma) + \frac{\gamma\eta}{s}\mathcal{K}(s,\eta,\gamma). \tag{3.7}$$

Proof: From equation (2.2) we have

$$\mathbb{KH}[u(t)] = \mathcal{K}(s,\eta,\gamma) = \frac{s}{\gamma\eta}\int_0^\infty u(t)\exp\left(-\frac{st}{\gamma\eta}\right)dt. \tag{3.8}$$

By differentiating both sides of equation (3.8), we get

$$\begin{aligned}
\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma) &= \frac{\partial}{\partial\eta}\left(\frac{s}{\gamma\eta}\int_0^\infty u(t)\exp\left(-\frac{st}{\gamma\eta}\right)dt\right) \\
&= \int_0^\infty \frac{\partial}{\partial\eta}\left(\frac{s}{\gamma\eta}u(t)\exp\left(-\frac{st}{\gamma\eta}\right)\right)dt \\
&= \int_0^\infty \left(-\frac{s}{\gamma\eta^2}u(t)\exp\left(-\frac{st}{\gamma\eta}\right) + \frac{s}{\gamma\eta}\frac{s}{\gamma\eta^2}tu(t)\exp\left(-\frac{st}{\gamma\eta}\right)\right)dt \\
&= -\frac{1}{\eta}\left(\frac{s}{\gamma\eta}\int_0^\infty u(t)\exp\left(-\frac{st}{\gamma\eta}\right)\right) + \frac{s}{\gamma\eta^2}\left(\frac{s}{\gamma\eta}\int_0^\infty tu(t)\exp\left(-\frac{st}{\gamma\eta}\right)dt\right) \\
&= -\frac{1}{\eta}\mathcal{K}(s,\eta,\gamma) + \frac{s}{\gamma\eta^2}\mathbb{KH}[tu(t)].
\end{aligned} \tag{3.9}$$

Then equation (3.9) becomes

$$\mathbb{KH}[tu(t)] = \frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma) + \frac{\gamma\eta}{s}\mathcal{K}(s,\eta,\gamma). \tag{3.10}$$

Theorem 3.4: *Let* $\mathbb{KH}[u(t)] = \mathcal{K}(s,\eta,\gamma)$ *be KhT of* $u(t)$, *then*

$$\mathbb{KH}[tu'(t)] = \eta\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma). \tag{3.11}$$

Proof: Replacing $u(t)$ with $u'(t)$ into (3.7) and using Property 2 leads to

$$\begin{aligned}
\mathbb{KH}[tu'(t)] &= \frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\mathbb{KH}[u'(t)] + \frac{\gamma\eta}{s}\mathbb{KH}[u'(t)] \\
&= \frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\left(\frac{s}{\gamma\eta}\mathcal{K}(s,\gamma,\eta) - \frac{s}{\gamma\eta}u(0)\right) + \frac{\gamma\eta}{s}\left(\frac{s}{\gamma\eta}\mathcal{K}(s,\gamma,\eta) - \frac{s}{\gamma\eta}u(0)\right) \\
&= \frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\left(\frac{s}{\gamma\eta}\mathcal{K}(s,\gamma,\eta)\right) - \frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\left(\frac{s}{\gamma\eta}u(0)\right) + \mathcal{K}(s,\gamma,\eta) - u(0) \\
&= \frac{\gamma\eta^2}{s}\left(-\frac{s}{\gamma\eta^2}\mathcal{K}(s,\gamma,\eta) + \frac{s}{\gamma\eta}\frac{\partial}{\partial\eta}\mathcal{K}(s,\gamma,\eta)\right) - \frac{\gamma\eta^2}{s}\left(-\frac{s}{\gamma\eta^2}u(0)\right) \\
&\quad + \mathcal{K}(s,\gamma,\eta) - u(0).
\end{aligned} \tag{3.12}$$

Simplifying equation (3.12), we have

$$\mathbb{KH}\left[tu(t)\right]=\eta\frac{\partial}{\partial\eta}\mathcal{K}(s,\gamma,\eta). \tag{3.13}$$

Theorem 3.5: *Let* $\mathbb{KH}[u(t)]=\mathcal{K}(s,\eta,\gamma)$ *be KhT of* $u(t)$, *then*

$$\mathbb{KH}\left[tu''(t)\right]=\frac{s}{\gamma}\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)-\frac{s}{\gamma\eta}\mathcal{K}(s,\eta,\gamma)+\frac{s}{\gamma\eta}u(0). \tag{3.14}$$

Proof: Replacing $u(t)$ with $u''(t)$ into (3.7) and using Property 2 leads to

$$\begin{aligned}
\mathbb{KH}[tu''(t)]&=\frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\mathbb{KH}\left[u''(t)\right]+\frac{\gamma\eta}{s}\mathbb{KH}[u''(t)]\\
&=\frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\left(\frac{s^2}{\gamma^2\eta^2}\mathcal{K}(s,\gamma,\eta)-\frac{s^2}{\gamma^2\eta^2}u(0)-\frac{s}{\gamma\eta}u'(0)\right)\\
&\quad+\frac{\gamma\eta}{s}\left(\frac{s^2}{\gamma^2\eta^2}\mathcal{K}(s,\gamma,\eta)-\frac{s^2}{\gamma^2\eta^2}u(0)-\frac{s}{\gamma\eta}u'(0)\right)\\
&=\frac{\gamma\eta^2}{s}\left(-\frac{2s^2}{\gamma^2\eta^3}\mathcal{K}(s,\gamma,\eta)+\frac{s^2}{\gamma^2\eta^2}\frac{\partial}{\partial\eta}\mathcal{K}(s,\gamma,\eta)+\frac{2s^2}{\gamma^2\eta^3}u(0)+\frac{s}{\gamma\eta^2}u'(0)\right)\\
&\quad+\frac{s}{\gamma\eta}\mathcal{K}(s,\gamma,\eta)-\frac{s}{\gamma\eta}u(0)-u'(0)\\
&=-\frac{2s}{\gamma\eta}\mathcal{K}(s,\gamma,\eta)+\frac{s}{\gamma}\frac{\partial}{\partial\eta}\mathcal{K}(s,\gamma,\eta)+\frac{2s}{\gamma\eta}u(0)+u'(0)\\
&\quad+\frac{s}{\gamma\eta}\mathcal{K}(s,\gamma,\eta)-\frac{s}{\gamma\eta}u(0)-u'(0).
\end{aligned} \tag{3.15}$$

Simplifying equation (3.15), we have

$$\mathbb{KH}[tu(t)]=\frac{s}{\gamma}\frac{\partial}{\partial\eta}\mathcal{K}(s,\gamma,\eta)-\frac{s}{\gamma\eta}\mathcal{K}(s,\gamma,\eta)+\frac{s}{\gamma\eta}u(0). \tag{3.16}$$

4. Numerical tests

Example 4.1: Let us assume a non-constant coefficients ordinary differential equation as

$$u''(t)+tu'(t)-u(t)=0, \tag{4.1}$$

under ICs

$$u(0)=0, u'(0)=1. \tag{4.2}$$

Operating the KhT and Property 1 on equation (4.1) gives

$$\mathbb{KH}[u''(t)]+\mathbb{KH}[tu'(t)]-\mathbb{KH}[u(t)]=0. \tag{4.3}$$

Using Property 2 and Theorem 3.4, equation (4.3) becomes

$$\frac{s^2}{\gamma^2\eta^2}\mathcal{K}(s,\eta,\gamma)-\frac{s^2}{\gamma^2\eta^2}u(0)-\frac{s}{\gamma\eta}u'(0)+\eta\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)-\mathcal{K}(s,\eta,\gamma)=0. \tag{4.4}$$

Substituting the ICs (4.2) and simplifying equation (4.4), we get

$$\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)+\left(\frac{s^2}{\gamma^2\eta^3}-\frac{1}{\eta}\right)\mathcal{K}(s,\eta,\gamma)=\frac{s}{\gamma\eta^2}. \tag{4.5}$$

After a simple calculation, we obtain the solution to equation (4.5) as

$$\mathcal{K}(s,\eta,\gamma)=\frac{\gamma\eta}{s}+\psi\eta\exp\left(\frac{s^2}{2\gamma^2\eta^2}\right),\psi\in\mathbb{R}. \tag{4.6}$$

Since $u(0)=0,$ then $\psi=0,$ and so we arrive at

$$\mathcal{K}(s,\eta,\gamma)=\frac{\gamma\eta}{s}. \tag{4.7}$$

Applying the inverse KhT on (4.7) takes the form

$$u(t)=t. \tag{4.8}$$

Here, equation (4.8) represents the exact solution of equations (4.1)-(4.2).

Example 4.2: Let us assume a non-constant coefficients ordinary differential equation as

$$u''(t)+tu'(t)-2u(t)=6, \tag{4.9}$$

under ICs

$$u(0)=0,u'(0)=0. \tag{4.10}$$

Operating the KhT and Property 1 on equation (4.9) gives

$$\mathbb{KH}[u''(t)]+\mathbb{KH}[tu'(t)]-2\mathbb{KH}[u(t)]=\mathbb{KH}[6]. \tag{4.11}$$

Using Property 2 and Theorem 3.4, equation (4.11) becomes

$$\frac{s^2}{\gamma^2\eta^2}\mathcal{K}(s,\eta,\gamma)-\frac{s^2}{\gamma^2\eta^2}u(0)-\frac{s}{\gamma\eta}u'(0)+\eta\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)-2\mathcal{K}(s,\eta,\gamma)=6. \tag{4.12}$$

Substituting the ICs (4.10) and simplifying equation (4.12), we get

$$\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)+\left(\frac{s^2}{\gamma^2\eta^3}-\frac{2}{\eta}\right)\mathcal{K}(s,\eta,\gamma)=\frac{6}{\eta}. \tag{4.13}$$

After a simple calculation, we obtain the solution to equation (4.13) as

$$\mathcal{K}(s,\eta,\gamma)=6\frac{\gamma^2\eta^2}{s^2}+\psi\eta^2\exp\left(\frac{s^2}{2\gamma^2\eta^2}\right),\psi\in\mathbb{R}. \tag{4.14}$$

Since $u(0)=0,$ then $\psi=0,$ and so we arrive at

$$\mathcal{K}(s,\eta,\gamma)=6\frac{\gamma^2\eta^2}{s^2}. \tag{4.15}$$

Applying the inverse KhT on (4.15) takes the form

$$u(t)=3t^2. \tag{4.16}$$

Here, equation (4.16) represents the exact solution of equations (4.9)-(4.10).

Example 4.3: Let us assume a non-constant coefficients ordinary differential equation as

$$tu''(t)+(1-2t)u'(t)-2u(t)=0, \tag{4.17}$$

under ICs

$$u(0)=1,u'(0)=2. \tag{4.18}$$

Operating the KhT and Property 1 on equation (4.17) gives

$$\mathbb{KH}[tu''(t)]-2\mathbb{KH}[tu'(t)]+\mathbb{KH}[u'(t)]+\mathbb{KH}[u(t)]=0. \tag{4.19}$$

Using Theorems 3.4, 3.5 and Property 2, equation (4.19) becomes

$$\frac{s}{\gamma}\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)-\frac{s}{\gamma\eta}\mathcal{K}(s,\eta,\gamma)+\frac{s}{\gamma\eta}u(0)-2\eta\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)+$$
$$\frac{s}{\gamma\eta}\mathcal{K}(s,\eta,\gamma)-\frac{s}{\gamma\eta}u(0)-2\mathcal{K}(s,\eta,\gamma)=0. \tag{4.20}$$

Substituting the ICs (4.18) and simplifying equation (4.20), we get

$$\left(\frac{s}{\gamma}-2\eta\right)\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)-2\mathcal{K}(s,\eta,\gamma)=0. \tag{4.21}$$

After a simple calculation, we obtain the solution to equation (4.21) as

$$\mathcal{K}(s,\eta,\gamma)=\frac{\psi}{s-2\gamma\eta},\psi\in\mathbb{R}. \tag{4.22}$$

Applying the inverse KhT on (4.22) takes the form

$$u(t)=\psi\exp(2t). \tag{4.23}$$

Since $u(0)=1,$ then $\psi=1,$ and so we arrive at

$$u(t)=\exp(2t). \tag{4.24}$$

Here, equation (4.24) represents the exact solution of equations (4.17)-(4.18).

Example 4.4: Let us assume a non-constant coefficients ordinary differential equation as

$$tu''(t)+(t+1)u'(t)+2u(t)=\exp(-t), \tag{4.25}$$

under ICs

$$u(0)=0, u'(0)=1. \tag{4.26}$$

Operating the KhT and Property 1 on equation (4.25) gives

$$\mathbb{KH}[tu''(t)]+\mathbb{KH}[tu'(t)]+\mathbb{KH}[u'(t)]+2\mathbb{KH}[u(t)]=\mathbb{KH}[\exp(-t)]. \tag{4.27}$$

Using Theorems 3.4, 3.5 and Property 2, equation (4.27) becomes

$$\frac{s}{\gamma}\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)-\frac{s}{\gamma\eta}\mathcal{K}(s,\eta,\gamma)+\frac{s}{\gamma\eta}u(0)+\eta\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)$$
$$+\frac{s}{\gamma\eta}\mathcal{K}(s,\eta,\gamma)-\frac{s}{\gamma\eta}u(0)+2\mathcal{K}(s,\eta,\gamma)=\frac{s}{s+\gamma\eta}. \tag{4.28}$$

Substituting the ICs (4.26) and simplifying equation (4.28), we get

$$\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma)+\frac{2\gamma}{s+\gamma\eta}\mathcal{K}(s,\eta,\gamma)=\frac{s\gamma}{(s+\gamma\eta)^2}. \tag{4.29}$$

After a simple calculation, we obtain the solution to equation (4.29) as

$$\mathcal{K}(s,\eta,\gamma)=\frac{s\gamma\eta}{(s+\gamma\eta)^2}+\psi, \psi\in\mathbb{R}. \tag{4.30}$$

Applying the inverse KhT on (4.30) takes the form

$$u(t)=t\exp(-t)+\psi. \tag{4.31}$$

Since $u(0)=0,$ then $\psi=0,$ and so we arrive at

$$u(t)=t\exp(-t). \tag{4.32}$$

Here, equation (4.32) represents the exact solution of equations (4.25)-(4.26).

Example 4.5: Let us assume a non-constant coefficients ordinary differential equation as

$$tu''(t)+u'(t)+tu(t)=0, \tag{4.33}$$

under ICs

$$u(0)=1, u'(0)=0. \tag{4.34}$$

Operating the KhT and Property 1 on equation (4.33) gives

$$\mathbb{KH}[tu''(t)] + \mathbb{KH}[u'(t)] + \mathbb{KH}[tu(t)] = 0. \tag{4.35}$$

Using Theorems 3.3, 3.5 and Property 2, equation (4.35) becomes

$$\frac{s}{\gamma}\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma) - \frac{s}{\gamma\eta}\mathcal{K}(s,\eta,\gamma) + \frac{s}{\gamma\eta}u(0) + \frac{s}{\gamma\eta}\mathcal{K}(s,\eta,\gamma) - \frac{s}{\gamma\eta}u(0) + \frac{\gamma\eta^2}{s}\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma) + \frac{\gamma\eta}{s}\mathcal{K}(s,\eta,\gamma) = 0. \tag{4.36}$$

Substituting the ICs (4.34) and simplifying equation (4.36), we get

$$\left(\frac{s}{\gamma} + \frac{\gamma\eta^2}{s}\right)\frac{\partial}{\partial\eta}\mathcal{K}(s,\eta,\gamma) + \frac{\gamma\eta}{s}\mathcal{K}(s,\eta,\gamma) = 0. \tag{4.37}$$

After a simple calculation, we obtain the solution to equation (4.37) as

$$\mathcal{K}(s,\eta,\gamma) = \frac{\psi}{\sqrt{s^2+\gamma^2\eta^2}}, \tag{4.38}$$

Since $u(0)=1,$ then $\psi = s,$ and so we arrive at

$$\mathcal{K}(s,\eta,\gamma) = \frac{s}{\sqrt{s^2+\gamma^2\eta^2}}, \tag{4.39}$$

Applying the inverse KhT on (4.39) takes the form

$$u(t) = J_0(t). \tag{4.40}$$

Here, equation (4.40) represents the exact solution of equations (4.33)-(4.34).

5. Conclusion

In this manuscript, we have develop the KhT for solving non-constant coefficients ordinary differential equations by presenting new results. These results are reinforced with illustrative examples. The proposed examples have shown that the exact solution can be obtained with very little computational work and in a very short time. So, we conclude that the KhT technique is very powerful and effective in finding the solution to a large class of the proposed problems, and it also helps in solving many problems related to different kinds of linear and nonlinear differential equations.

References

[1] K. S. Aboodh, "The New Integral Transform 'Aboodh Transform'," *Global Journal of Pure and Applied Mathematics,* vol. 9, no. 1, pp. 35–43 (2013).

[2] R. N. Bracewell, *The Fourier Transform and Its Applications*, McGraw-Hill, New York (1999).

[3] W. E. Boyce and R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems, John Wiley & Sons, New York (2001).

[4] C. Chicone, Ordinary Differential Equations with Applications, Springer Verlag, New York (1999).

[5] T. M. Elzaki, "The New Integral Transform 'Elzaki Transform'," *Global Journal of Pure and Applied Mathematics*, vol. 7, no. 1, pp. 57–64 (2011).

[6] A. Khalouta, "A New Exponential Type Kernel Integral Transform: Khalouta Transform and Its Applications," *Mathematica Montisnigri*, vol. 57, pp. 5–23 (2023), DOI: 10.20948/mathmontis-2023-57-1.

[7] A. Kılıçman and H. E. Gadain, "On the Applications of Laplace and Sumudu Transforms," *Journal of the Franklin Institute*, vol. 347, no. 5, pp. 848–862 (2010).

[8] S. A. Mehdi, E. A. Kuffi, and J. A. Jasim, "Solving Ordinary Differential Equations Using a New General Complex Integral Transform," *Journal of Interdisciplinary Mathematics,* vol. 25, no. 6, pp. 1919–1932 (2022). [Online]. Available: https://doi.org/10.1080/09720502.2022.2089381.

[9] G. D. H. Panchaksharappa, Pharmaceutical Mathematics with Application to Pharmacy, Pharma Med Press, Hyderabad (2014).

[10] Z. D. Ridha and F. A. Mohommed, "Solving Linear Ordinary Differential Equations by Using Shehu Transform with Variable Coefficients," *Journal of Interdisciplinary Mathematics,* vol. 24, no. 8, pp. 2391–2400 (2021). [Online]. Available: https://doi.org/10.1080/09720502.2021.2000155.

[11] P. D. Ritger and N. J. Rose, Differential Equations with Applications, McGraw-Hill, New York (1968).

[12] M. Sahni, M. Parikh, and R. Sahni, "Sumudu Transform for Solving Ordinary Differential Equation in a Fuzzy Environment," *Journal of Interdisciplinary Mathematics,* vol. 24, no. 6, pp. 1565–1577 (2021). [Online]. Available: https://doi.org/10.1080/09720502.2020.1845468.

[13] L. Tabharit and Z. Dahmani, “Integro-Differential Equations of Arbitrary Orders Involving Convergent Series,” *Journal of Interdisciplinary Mathematics,* vol. 23, no. 5, pp. 935–953 (2020). [Online]. Available: https://doi.org/10.1080/09720502.2020.1711603.

[14] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge (1944).

[15] A. Zafar, Differential Equations and Their Applications, 3rd ed., Prentice Hall of India Pvt. Ltd, New Delhi (2016).

[16] Z. U. A. Zafar, “ZZ Transform Method,” *International Journal of Advanced Engineering and Global Technology*, vol. 4, no. 1, pp. 1605–1611 (2016).

[17] D. Zwillinger, Handbook of Differential Equations, Academic Press, Inc., New York (1989).

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