

Mixed multifractal dimensions of overlaps of self-similar sets and measures

Jihed Hattab

Department of Mathematics

Analysis, Probability and Fractals Laboratory LR18ES17

Faculty of Sciences of Monastir

University of Monastir

5000 Monastir

Tunisia

Abstract

In recent times, the mixed multifractal analysis has generated interest. The purpose of this paper is twofold. Firstly, we define the mixed multifractal Hewitt-Stromberg measures and introduce a multifractal formalism based on this type of measures. Secondly, we prove that the upper mixed multifractal box-dimension of the overlaps of self-similar sets and measures is strictly smaller than the mixed multifractal Hausdorff dimension of self-similar sets and measures. Finally, we demonstrate that this result is still valid for the mixed multifractal Hewitt-Stromberg dimensions.

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1. Introduction

During the last decades, many fields of physics aims to develop the notion of multifractal analysis because it has played a special role in the history of modern science [6, 27]. Many authors have focused on this analysis [15], [16] and its practical uses, particularly concerning data from the field of econophysics. [8].

Mixed multifractal analysis encompasses the study of both functions and measures. In certain cases, geometric sets are primarily known through measures which are depended on them, this means that take a measure ν and a set $\mathcal{X}$ implies that the quantity $\nu(\mathcal{X})$ represent the maximum value of $\nu(\mathcal{C})$ for all

E-mail: jihed.hattab@gmail.com

subsets $\mathcal{C} \subset \mathcal{X}$. In this approach, the focus is on the measure's properties rather than the geometric structure of $\mathcal{X}$.

The objective of this work is to develop a mixed multifractal analysis for a finite number of measures. Many natural fractal like objects, which are of interest, do not at all times arising from simultaneous functions, but instead derived from simultaneous measures. Thus, in [18, 19], a mixed multifractal formalism linked to the multifractal extensions of Hausdorff and packing measures and their corresponding dimensions is proven, founded on a generalization of the widely recognized vast deviation formalism.

Additionally, The rapid ascension of Hewitt-Stromberg measures, the measures which were established by Hewitt and Stromberg in their well-established textbook [13], has demonstrated over the intervening decades the success of this focused approach on the examination of Localized characteristics of fractals. Many serval authors have made a great deal of progress in that kind of measure, we can cite for exemple [2-4, 9, 24, 25]. This rapid progression motivate us to define the mixed multifractal Hewitt-Stromberg measures and to introduce a formalism that utilizes these measures within a multifractal framework.

Other important results have developed from the mixed multifractal analysis's study, which has shown its strength in characterizing the local behavior of measures, particularly those that are fractal (see [7, 18, 21, 26]). In [20], L. Olsen proposed a conjecture that compares the upper multifractal box-dimension of the overlaps of self-similar sets and measures and the multifractal Hausdorff dimension of self-similar sets and measures. Accordingly, we seek in this research to show that this result is still valid in the mixed case. However, to demonstrate the validity of this conjecture we need the concept of contracting similarities.

Given $\mathcal{S}_n: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $n = 1, \dots, \ell$ a contracting similarities, and $p_m = (p_{m,n})_{n=1, \dots, \ell}$ a probability vectors for $m = 1, \dots, k$. For every n , let identify the Lipschitz constant value of $\mathcal{S}_n$ by $\omega_n \in (0,1)$. Define $\mathcal{D}$ as a self-similar set and let $\mathbf{v} = (v_1, \dots, v_k)$ represent the vector-valued measure formed from self-similar mesures on $\mathbb{R}^d$. Suppose that the list $(\mathcal{S}_n)_n$ fulfills the Open Set Condition, denote by $\eta = \dim(\mathcal{D})$ the Hausdorff dimension of $\mathcal{D}$. It is commonly acknowledged that, the overlap $\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})$ possesses zero η -dimensional Hausdorff measure for all $n \neq m$, i.e.,

$$\mathcal{H}^\eta(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) = 0 \quad \text{for all } n \neq m.$$

In the sequel, we denote by $\mathbb{R}_-^k = \mathbb{R}_- \times \dots \times \mathbb{R}_-$, k -times and $\mathbb{R}_+^k = \mathbb{R}_+ \times \dots \times \mathbb{R}_+$, k -times.

The overlaps $\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})$ are likewise considered negligible viewed from a mixed multifractal framework. We establish that for all values of $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}_+^k$, the upper mixed multifractal box-dimension $\bar{\tau}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D}))$ of the overlaps is strictly inferior to the mixed multifractal Hausdorff dimension $\dim_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D})$ of $\mathcal{D}$, i.e.,

$$\bar{\tau}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) < \dim_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) \quad \text{for all } n \neq m.$$

This paper will be organized in the following manner. In the first step we introduce the mixed multifractal Hausdorff, packing and Hewitt-Stromberg measures and we propose a new multifractal formalism approach derived from this type of measures. The second point to check is studying the dimensions of overlaps of self-similar measures and sets.

2. Preliminaries and statements of results

Definition 1: [21] For $m = 1, \dots, k$, $\mathcal{D}$ is defined as a self-similar set associated to the list $(\mathcal{S}_1, \dots, \mathcal{S}_\ell, p_{m,1}, \dots, p_{m,\ell})$ if it is the unique non-empty compact subset of $\mathbb{R}^d$ satisfying

$$\mathcal{D} = \bigcup_n \mathcal{S}_n(\mathcal{D}).$$

Definition 2: [14] For $m = 1, \dots, k$, ν_m is defined as a self-similar measure associated to the list $(\mathcal{S}_1, \dots, \mathcal{S}_\ell, p_{m,1}, \dots, p_{m,\ell})$ if it is the unique Borel probability measure on $\mathbb{R}^d$ satisfying

$$\nu_m = \sum_n p_{m,n} \nu_m \circ \mathcal{S}_n^{-1}.$$

Definition 3: [5, 23] We define the list $(\mathcal{S}_1, \dots, \mathcal{S}_\ell)$ to fulfill the Open Set Condition (OSC) if there is an open non-empty and bounded subset W of $\mathbb{R}^d$ with

$$\bigcup_n \mathcal{S}_n(W) \subseteq W$$

and

$$\mathcal{S}_n(W) \cap \mathcal{S}_m(W) = \emptyset \quad \text{for all } n, m \text{ with } n \neq m.$$

Definition 4: We define the list $(\mathcal{S}_1, \dots, \mathcal{S}_\ell)$ to fulfill the Strong Set Condition (SSC) if there is an open non-empty and bounded subset W of $\mathbb{R}^d$ with

$$\bigcup_n \mathcal{S}_n(W) \subseteq W$$

and

$$\overline{\mathcal{S}_n(W)} \cap \overline{\mathcal{S}_m(W)} = \emptyset \quad \text{for all } n, m \text{ with } n \neq m.$$

Definition 5: [4] We assert that the vector-valued measure $\mathbf{v} = (\nu_1, \dots, \nu_k)$ composed of k Borel probability measures on $\mathbb{R}^d$ fulfill doubling condition if there is $b > 1$ such that, for every ball $\mathcal{B}(z, \omega)$ of center z and radius ω and for $m = 1, \dots, k$

$$\limsup_{\omega \rightarrow 0} \left(\sup_{z \in \text{supp } \nu_m} \frac{\nu_m(\mathcal{B}(z, b\omega))}{\nu_m(\mathcal{B}(z, \omega))} \right) < \infty.$$

2.1 Hausdorff and packing measures and dimensions

[11] For a subset $\mathcal{A} \subseteq \mathbb{R}^d$, let the diameter of $\mathcal{A}$ be defined as

$$\text{diam}(\mathcal{A}) = \sup\{|z - u| : z, u \in \mathcal{A}\}.$$

We say that the countable set $(V_n)_n$ is a δ -covering of $\mathcal{A}$ if

$$\mathcal{A} \subset \bigcup_n V_n \quad \text{and} \quad 0 < \text{diam}(V_n) \leq \delta.$$

For $\eta \geq 0$ and $\delta > 0$, the Hausdorff measure is expressed as

$$\mathcal{H}_\delta^\eta(\mathcal{A}) = \inf \{ \sum_n (\text{diam}(V_n))^\eta \mid (V_n)_n \text{ is a } \delta\text{-covering of } \mathcal{A} \}.$$

This leads us to give a definition of the η -dimensional Hausdorff measure of $\mathcal{A}$ by

$$\mathcal{H}^\eta(\mathcal{A}) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^\eta(\mathcal{A}) = \sup_{\delta > 0} \mathcal{H}_\delta^\eta(\mathcal{A}).$$

Next, we denote by $\dim_{\mathcal{H}}$, the Hausdorff dimension which is introduced by $\dim_{\mathcal{H}}(\mathcal{A}) = \inf\{\eta \geq 0 \mid \mathcal{H}^\eta(\mathcal{A}) = 0\} = \sup\{\eta \geq 0 \mid \mathcal{H}^\eta(\mathcal{A}) = +\infty\}$.

Now, for $\eta \geq 0$ and $\delta > 0$, we introduce the η -dimensional packing pre-measure as follows

$$\overline{\mathcal{P}}_\delta^\eta(\mathcal{A}) = \sup \{ \sum_n (2\omega_n)^\eta \mid (\mathcal{B}(z_n, \omega_n))_n \text{ is a } \delta\text{-packing of } \mathcal{A} \}$$

and

$$\overline{\mathcal{P}}^\eta(\mathcal{A}) = \lim_{\delta \rightarrow 0} \overline{\mathcal{P}}_\delta^\eta(\mathcal{A}) = \inf_{\delta > 0} \overline{\mathcal{P}}_\delta^\eta(\mathcal{A}).$$

This allows us to represent the η -dimensional measure of packing by

$$\mathcal{P}^\eta(\mathcal{A}) = \inf \{ \sum_n \overline{\mathcal{P}}^\eta(\mathcal{F}_n) \mid \mathcal{A} \subset \bigcup_n \mathcal{F}_n \}.$$

Similarly, we define the packing dimension $\dim_{\mathcal{P}}$ as follows

$$\dim_{\mathcal{P}}(\mathcal{A}) = \inf\{\eta \geq 0 \mid \mathcal{P}^\eta(\mathcal{A}) = 0\} = \sup\{\eta \geq 0 \mid \mathcal{P}^\eta(\mathcal{A}) = +\infty\}.$$

2.2 Mixed multifractal Hausdorff measure and dimension

[18] Consider $\mathbf{v} = (v_1, \dots, v_k)$ as a vector-valued measure composed of k probability measures on $\mathbb{R}^d$. The simultaneous scaling behavior of $\mathbf{v}$ is defined as follows

$$\lim_{\omega \rightarrow 0} \frac{\log \mathbf{v}(\mathcal{B}(z, \omega))}{\log \omega} = \left(\lim_{\omega \rightarrow 0} \frac{\log v_1(\mathcal{B}(z, \omega))}{\log \omega}, \dots, \lim_{\omega \rightarrow 0} \frac{\log v_k(\mathcal{B}(z, \omega))}{\log \omega} \right).$$

Given $\mathcal{A}$ a nonempty set of $\mathbb{R}^d$, $\delta > 0$ and let $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$ and $\eta \in \mathbb{R}$. Denote

$$\begin{aligned} \mathbf{v}(\mathcal{B}(z, \omega)) &= (v_1(\mathcal{B}(z, \omega)), \dots, v_k(\mathcal{B}(z, \omega))), \\ (\mathbf{v}(\mathcal{B}(z, \omega)))^{\mathbf{q}} &= (v_1(\mathcal{B}(z, \omega)))^{q_1} \dots (v_k(\mathcal{B}(z, \omega)))^{q_k}. \end{aligned}$$

We introduce the mixed multifractal Hausdorff pre-measure as

$$\overline{\mathcal{H}}_{\mathbf{v}, \delta}^{\mathbf{q}, \eta}(\mathcal{A}) = \inf \{ \sum_n \mathbf{v}(\mathcal{B}(z_n, \omega_n))^{\mathbf{q}} (2\omega_n)^\eta \mid (\mathcal{B}(z_n, \omega_n))_n \text{ is a centered } \delta\text{-covering of } \mathcal{A} \}.$$

The function $\overline{\mathcal{H}}_{\mathbf{v}}^{\mathbf{q}, \eta}$ is σ -subadditive but not increasing, then we propose the following modified definition

$$\overline{\mathcal{H}}_{\mathbf{v}}^{\mathbf{q}, \eta}(\mathcal{A}) = \lim_{\delta \rightarrow 0} \overline{\mathcal{H}}_{\mathbf{v}, \delta}^{\mathbf{q}, \eta}(\mathcal{A}) = \sup_{\delta > 0} \overline{\mathcal{H}}_{\mathbf{v}, \delta}^{\mathbf{q}, \eta}(\mathcal{A}),$$

and

$$\mathcal{H}_v^{q,\eta}(\mathcal{A}) = \sup_{F \subseteq \mathcal{A}} \overline{\mathcal{H}}_v^{q,\eta}(F).$$

The mixed multifractal Hausdorff measure $\mathcal{H}_v^{q,\eta}$ is an outer metric measure on $\mathbb{R}^d$.

There is a unique number $\dim_v^q(\mathcal{A}) \in [-\infty, +\infty]$ satisfying,

$$\mathcal{H}_v^{q,\eta}(\mathcal{A}) = \begin{cases} +\infty & \text{if } \eta < \dim_v^q(\mathcal{A}), \\ 0 & \text{if } \eta > \dim_v^q(\mathcal{A}). \end{cases}$$

2.3 Mixed multifractal packing measure and dimension

We use the same notations as previously, then we define the mixed multifractal packing pre-measure by

$$\overline{\mathcal{P}}_{v,\delta}^{q,\eta}(\mathcal{A}) = \sup \{ \sum_n \mathbf{v}(\mathcal{B}(z_n, \omega_n))^q (2\omega_n)^\eta \mid (\mathcal{B}(z_n, \omega_n))_n \text{ is a } \delta - \text{packing of } \mathcal{A} \}.$$

The function $\overline{\mathcal{P}}_v^{q,\eta}$ is increasing also not σ -subadditive, then we define

$$\overline{\mathcal{P}}_v^{q,\eta}(\mathcal{A}) = \lim_{\delta \rightarrow 0} \overline{\mathcal{P}}_{v,\delta}^{q,\eta}(\mathcal{A}) = \inf_{\delta > 0} \overline{\mathcal{P}}_{v,\delta}^{q,\eta}(\mathcal{A}),$$

and

$$\mathcal{P}_v^{q,\eta}(\mathcal{A}) = \inf_{\mathcal{A} \subseteq \bigcup_n \mathcal{A}_n} \sum_n \overline{\mathcal{P}}_v^{q,\eta}(\mathcal{A}_n).$$

The mixed multifractal packing measure $\mathcal{P}_v^{q,\eta}$ is an outer metric measure on $\mathbb{R}^d$.

There exists a unique number $\text{Dim}_v^q(\mathcal{A}) \in [-\infty, +\infty]$ such that,

$$\mathcal{P}_v^{q,\eta}(\mathcal{A}) = \begin{cases} +\infty & \text{if } \eta < \text{Dim}_v^q(\mathcal{A}), \\ 0 & \text{if } \eta > \text{Dim}_v^q(\mathcal{A}). \end{cases}$$

For more details see [18].

2.4 Mixed multifractal box-dimensions of Self-similar measures

Let $\mathcal{A} \subseteq \mathbb{R}^d$, $\omega > 0$.

A subset B of $\mathcal{A}$ is defined as an ω -separated subset of $\mathcal{A}$ if

$$|u - z| > 2\omega, \quad \text{for all } u, z \in B \text{ with } u \neq z.$$

For a probability measure $\mathbf{v} = (v_1, \dots, v_k)$ of $\mathbb{R}^d$ and $\mathbf{q} \in \mathbb{R}^k$, $\omega > 0$ we introduce the mixed packing moment scaling function of the measure $\mathbf{v}$ as in [21] by

$$\mathcal{M}_{v,\omega}^q(\mathcal{A}) = \sup \{ \sum_{z \in F} v_1(\mathcal{B}(z, \omega))^{q_1} \dots v_k(\mathcal{B}(z, \omega))^{q_k} \mid F \text{ is an } \omega - \text{separated subset of } \mathcal{A} \}.$$

It is also possible to define $\mathcal{M}_{v,\omega}^q(\mathcal{A})$ by

$$\mathcal{M}_{v,\omega}^q(\mathcal{A}) = \sup_{(\mathcal{B}(z_n, \omega))_n \text{ is a centered packing of } \mathcal{A}} \sum_n v_1(\mathcal{B}(z_n, \omega))^{q_1} \dots v_k(\mathcal{B}(z_n, \omega))^{q_k},$$

where $(\mathcal{B}(z_n, \omega))_n$ is a finite or countable family of closed balls such that $(\mathcal{B}(z_n, \omega)) \cap (\mathcal{B}(z_m, \omega)) = \emptyset$, for all $n \neq m$, in addition $z_n \in \mathcal{A}$.

Now, let us define the lower and the upper mixed multifractal box-dimensions by

$$\underline{\tau}_v^q(\mathcal{A}) = \liminf_{\omega \rightarrow 0} \frac{\log \mathcal{M}_{v, \omega}^q(\mathcal{A})}{-\log \omega}$$

and

$$\overline{\tau}_v^q(\mathcal{A}) = \limsup_{\omega \rightarrow 0} \frac{\log \mathcal{M}_{v, \omega}^q(\mathcal{A})}{-\log \omega}.$$

2.5 Hewitt-Stromberg measures and dimensions

Let $\eta > 0$ and $\omega > 0$. For $\mathcal{A} \subseteq \mathbb{R}^d$, we denote by $N_\omega(\mathcal{A})$ the following quantity

$$N_\omega(\mathcal{A}) = \inf\{\#\{I\} | (\mathcal{B}(z_n, \omega))_{n \in I} \text{ is a family of closed balls,} \\ \text{with } z_n \in \mathcal{A} \text{ and } \mathcal{A} \subseteq \bigcup_n \mathcal{B}(z_n, \omega)\}.$$

We introduce the Hewitt-Stromberg pre-measure of $\mathcal{A}$ by the following expression

$$L^\eta(\mathcal{A}) = \liminf_{\omega \rightarrow 0} N_\omega(\mathcal{A})(2\omega)^\eta$$

and

$$\overline{H}^s(\mathcal{A}) = \sup_{\mathcal{F} \subseteq \mathcal{A}} L^\eta(\mathcal{F}).$$

Next, let us define the lower η -dimensional Hewitt-Stromberg measure $H^\eta(\mathcal{A})$ by

$$H^\eta(\mathcal{A}) = \inf_{\mathcal{A} \subseteq \bigcup_n \mathcal{F}_n} \sum_n \overline{H}^\eta(\mathcal{F}_n).$$

It is possible to define as follows the lower Hewitt-Stromberg dimension which is denoted by $\underline{\dim}_{MB}(\mathcal{A})$

$$\underline{\dim}_{MB}(\mathcal{A}) = \sup\{\eta \geq 0 | H^\eta(\mathcal{A}) = +\infty\} = \inf\{\eta \geq 0 | H^\eta(\mathcal{A}) = 0\}.$$

Analogously, we introduce

$$M_\omega(\mathcal{A}) = \sup\{\#\{I\} | (\mathcal{B}(z_n, \omega))_{n \in I} \text{ is a family of closed balls with } z_n \\ \in \mathcal{A} \text{ and } \mathcal{B}(z_n, \omega) \cap \mathcal{B}(z_m, \omega) = \emptyset \text{ for } n \neq m\},$$

for more details (see [10] and [11]).

Moreover, the Hewitt-Stromberg pre-measure of $\mathcal{A}$ is defined by the following expression

$$\overline{P}^\eta(\mathcal{A}) = \limsup_{\omega \rightarrow 0} M_\omega(\mathcal{A})(2\omega)^\eta.$$

The upper η -dimensional Hewitt-Stromberg measure, which we denote by $P^\eta(\mathcal{A})$ is defined by

$$P^\eta(\mathcal{A}) = \inf_{\mathcal{A} \subseteq \bigcup_n \mathcal{F}_n} \sum_n \overline{P}^\eta(\mathcal{F}_n).$$

Also, we denote by $\overline{\dim}_{MB}(\mathcal{A})$ the upper Hewitt-Stromberg dimension which is defined as follows

$$\overline{\dim}_{MB}(\mathcal{A}) = \sup\{\eta \geq 0 \mid P^\eta(\mathcal{A}) = +\infty\} = \inf\{\eta \geq 0 \mid P^\eta(\mathcal{A}) = 0\}.$$

2.6 Mixed multifractal Hewitt-Stromberg measures and dimension

Given $q \in \mathbb{R}^k$, $\omega > 0$. For $\mathcal{A} \subseteq \mathbb{R}^d$, we denote by $N_{v,\omega}^q(\mathcal{A})$ the following quantity

$$N_{v,\omega}^q(\mathcal{A}) = \inf\{\sum_n \nu(\mathcal{B}(z_n, \omega))^q \mid (\mathcal{B}(z_n, \omega))_n \text{ is a centered covering of } \mathcal{A}\}.$$

Then, for $\eta \in \mathbb{R}$ we define the pre-measure of $\mathcal{A}$ by

$$L_v^{q,\eta}(\mathcal{A}) = \liminf_{\omega \rightarrow 0} N_{v,\omega}^q(\mathcal{A})(2\omega)^\eta.$$

$L_v^{q,\eta}$ is not only not increasing but also not countably sub-additive, so it is necessary to apply a standard modification to obtain an outer measure. Thus, we consider

$$\overline{H}_v^{q,\eta}(\mathcal{A}) = \inf_{\mathcal{A} \subseteq \bigcup_n \mathcal{A}_n} \sum_n L_v^{q,\eta}(\mathcal{A}_n),$$

and

$$H_v^{q,\eta}(\mathcal{A}) = \sup_{\mathcal{F} \subseteq \mathcal{A}} \overline{H}_v^{q,\eta}(\mathcal{F}).$$

There is a unique number $b_v^q(\mathcal{A}) \in [-\infty, +\infty]$ satisfying,

$$H_v^{q,\eta}(\mathcal{A}) = \begin{cases} +\infty & \text{if } \eta < b_v^q(\mathcal{A}), \\ 0 & \text{if } \eta > b_v^q(\mathcal{A}). \end{cases}$$

In the same manner, we define

$$M_{v,\omega}^q(\mathcal{A}) = \sup\{\sum_n \nu(\mathcal{B}(z_n, \omega))^q \mid (\mathcal{B}(z_n, \omega))_n \text{ is a centered packing of } \mathcal{A}\}.$$

We express the pre-measure of $\mathcal{A}$ as

$$\overline{P}_v^{q,\eta}(\mathcal{A}) = \limsup_{\omega \rightarrow 0} M_{v,\omega}^q(\mathcal{A})(2\omega)^\eta.$$

The function $\overline{P}_v^{q,\eta}$ is increasing but not σ -additive, so we obtain an outer measure as follows

$$P_v^{q,\eta}(\mathcal{A}) = \inf_{\mathcal{A} \subseteq \bigcup_n \mathcal{F}_n} \sum_n \overline{P}_v^{q,\eta}(\mathcal{F}_n).$$

There is a unique number $B_v^q(\mathcal{A}) \in [-\infty, +\infty]$ satisfying,

$$P_v^{q,\eta}(\mathcal{A}) = \begin{cases} +\infty & \text{if } \eta < B_v^q(\mathcal{A}), \\ 0 & \text{if } \eta > B_v^q(\mathcal{A}). \end{cases}$$

Similarly, there is a unique number $\Delta_v^q(\mathcal{A}) \in [-\infty, +\infty]$ satisfying,

$$\overline{P}_{\mathbf{v}}^{q,\eta}(\mathcal{A}) = \begin{cases} +\infty & \text{if } \eta < \Delta_{\mathbf{v}}^q(\mathcal{A}), \\ 0 & \text{if } \eta > \Delta_{\mathbf{v}}^q(\mathcal{A}). \end{cases}$$

Remark 1: It is obvious for $\eta > 0$, we obtain

$$\mathcal{H}_{\mathbf{v}}^{0,\eta} = \mathcal{H}^{\eta}, \quad \mathcal{P}_{\mathbf{v}}^{0,\eta} = \mathcal{P}^{\eta}$$

and

$$H_{\mathbf{v}}^{0,\eta} = H^{\eta}, \quad P_{\mathbf{v}}^{0,\eta} = P^{\eta},$$

where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^k$.

The following result outlines key properties of the mixed multifractal Hewitt-Stromberg measures and introduces fundamental inequalities that the mixed multifractal Hewitt-Stromberg measures, the mixed multifractal Hausdorff measure and the mixed multifractal packing measure satisfies.

Theorem 1: Let $\mathbf{q} \in \mathbb{R}^k$, $\eta \in \mathbb{R}$ and $\mathcal{A} \subseteq \mathbb{R}^d$. Consider the vector-valued measure $\mathbf{v} = (v_1, \dots, v_k)$ composed of k Borel probability measures on $\mathbb{R}^d$. We have

- (1) $H_{\mathbf{v}}^{q,\eta}$ and $P_{\mathbf{v}}^{q,\eta}$ are outer measures.
- (2) For every $\mathbf{q} \in \mathbb{R}_+^k \setminus \{\mathbf{0}\}$ and $\mathbf{v}$ satisfying the doubling condition, we have $H_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq P_{\mathbf{v}}^{q,\eta}(\mathcal{A})$.
- (3) For every $\mathbf{q} \in \mathbb{R}_-^k$, $H_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq P_{\mathbf{v}}^{q,\eta}(\mathcal{A})$.
- (4) There is an integer $\xi \in \mathbb{N}$, such that $H_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq \xi P_{\mathbf{v}}^{q,\eta}(\mathcal{A})$.
- (5) When $\mathbf{q} \in \mathbb{R}_-^k$ or $\mathbf{q} \in \mathbb{R}_+^k \setminus \{\mathbf{0}\}$ and $\mathbf{v}$ satisfying the doubling condition, we have

$$\mathcal{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq H_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq P_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq \mathcal{P}_{\mathbf{v}}^{q,\eta}(\mathcal{A}).$$

Proof:

(1) It is straightforward to show that $H_{\mathbf{v}}^{q,\eta}$ is increasing and $H_{\mathbf{v}}^{q,\eta}(\emptyset) = 0$. If $\{\mathcal{A}_{\ell}\}$ is any countable collection of sets where $\mathcal{A} = \bigcup_{\ell} \mathcal{A}_{\ell}$. We can suppose without loss of generality that $\sum_{\ell} \overline{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}_{\ell}) < \infty$. Fix $\omega > 0$, $\varepsilon > 0$, and $\ell \in \mathbb{N}$, let $(\mathcal{B}(z_{\ell n}, \omega))_n$ be a centered covering of $\mathcal{A}_{\ell} \cap \mathcal{A}$ satisfying the following inequality

$$\sum_n \mathbf{v}(\mathcal{B}(z_{\ell n}, \omega))^q (2\omega)^{\eta} \leq \overline{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}_{\ell} \cap \mathcal{A}) + \frac{\varepsilon}{2^{\ell}}.$$

Then,

$$\begin{aligned} \overline{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}) &= \overline{H}_{\mathbf{v}}^{q,\eta}(\bigcup_{\ell} (\mathcal{A}_{\ell} \cap \mathcal{A})) \leq \sum_{\ell} \sum_n \mathbf{v}(\mathcal{B}(z_{\ell n}, \omega))^q (2\omega)^{\eta} \\ &\leq \sum_{\ell} (\overline{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}_{\ell} \cap \mathcal{A}) + \frac{\varepsilon}{2^{\ell}}) \leq \sum_{\ell} \overline{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}_{\ell} \cap \mathcal{A}) + \varepsilon \leq \sum_{\ell} \overline{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}_{\ell}) + \varepsilon. \end{aligned}$$

Since ε was arbitrary and $\mathcal{A} \subseteq \bigcup_{\ell} \mathcal{A}_{\ell}$ we obtain that

$$H_{\mathbf{v}}^{q,\eta}(\bigcup_{\ell} \mathcal{A}_{\ell}) \leq \sum_n H_{\mathbf{v}}^{q,\eta}(\mathcal{A}_{\ell}).$$

In the same manner we conclude that $P_v^{q,\eta}$ is an outer measure.

(2) Let $\mathcal{A} \subseteq \mathbb{R}^d$. For $a \in \mathbb{N} \setminus \{0\}$, write

$$\mathcal{A}_a = \left\{ z \in \mathcal{A} \mid \frac{\nu_m(\mathcal{B}(z, 5\omega))}{\nu_m(\mathcal{B}(z, \omega))} < a \text{ for } 0 < \omega < \frac{1}{a}, \forall 1 \leq m \leq k \right\}.$$

We may clearly assume that $\bar{P}_v^{q,\eta}(\mathcal{A}) < \infty$. Fix $a \in \mathbb{N}^*$ and let $V = \{\mathcal{B}(z, \omega) \mid z \in \mathcal{A}_a\}$. Then V is a Vitali covering (see [17]) of $\mathcal{A}_a$. By using Vitali's Lemma, there exists a countable or finite subfamily $(\mathcal{B}(z_n, \omega))_n \subseteq V$ such that $\mathcal{B}(z_n, \omega) \cap \mathcal{B}(z_m, \omega) = \emptyset$, for $n \neq m$ and

$$\mathcal{A}_a \setminus \bigcup_{n=1}^k \mathcal{B}_n \subseteq \bigcup_{n=k+1}^\infty \mathcal{B}(z_n, 5\omega), \forall k.$$

Since $z_n \in \mathcal{A}_a$, then for $\varepsilon > 0$, we have

$$\begin{aligned} \sum_n \nu(\mathcal{B}(z_n, 5\omega))^q (10\omega)^\eta &\leq a^q 5^\eta \sum_n (\nu(\mathcal{B}(z_n, \omega)))^q (2\omega)^\eta \\ &\leq a^q 5^\eta M_{v,\omega}^q(\mathcal{A}_a) (2\omega)^\eta \\ &\leq a^q 5^\eta (\bar{P}_v^{q,\eta}(\mathcal{A}) + \frac{\varepsilon}{2}) < +\infty. \end{aligned}$$

Therefore, we may choose $P \in \mathbb{N}$ such that

$$\sum_{n=P+1}^{+\infty} \nu(\mathcal{B}(z_n, 5\omega))^q (10\omega)^\eta \leq \frac{\varepsilon}{2}.$$

Consequently,

$$\begin{aligned} N_{v,\omega}^q(\mathcal{A}_a) (2\omega)^\eta &\leq \sum_{n=1}^P \nu(\mathcal{B}(z_n, \omega))^q (2\omega)^\eta \\ &\quad + \sum_{n=P+1}^{+\infty} \nu(\mathcal{B}(z_n, 5\omega))^q (10\omega)^\eta \\ &\leq \sum_{n=1}^P \nu(\mathcal{B}(z_n, \omega))^q (2\omega)^\eta + \frac{\varepsilon}{2} \\ &\leq M_{v,\omega}^q(\mathcal{A}_a) (2\omega)^\eta + \frac{\varepsilon}{2} \\ &\leq \bar{P}_v^{q,\eta}(\mathcal{A}_a) + \varepsilon. \end{aligned}$$

Making $\omega \rightarrow 0$, since ε was arbitrary and $\mathcal{A} = \bigcup_a \mathcal{A}_a$ we obtain that $H_v^{q,\eta}(\mathcal{A}) \leq P_v^{q,\eta}(\mathcal{A})$.

(3) To establish that $H_v^{q,\eta}(\mathcal{A}) \leq P_v^{q,\eta}(\mathcal{A})$, when $q \in \mathbb{R}_+^k$ we use the same arguments as in statement (2).

(4) For $\mathcal{F} \subseteq \mathcal{A}$, by using the theorem of Besicovitch's covering (see [22]), we can affirm that there exists $\xi \in \mathbb{N}$ and ξ -sub-families $\{\mathcal{B}(z_{nm}, \omega)_m, 1 \leq n \leq \xi\}$ such that $\mathcal{F} \subset \bigcup_{n=1}^\xi \bigcup_m (\mathcal{B}(z_{nm}, \omega))$ and $(\mathcal{B}(z_{nm}, \omega))_m$ is a centered packing of $\mathcal{F}$ for all n . Then

$$\begin{aligned} N_{v,\omega}^q(\mathcal{F}) &\leq \sum_{n=1}^\xi \sum_m \nu(\mathcal{B}(z_{nm}, \omega))^q (2\omega)^\eta \\ &\leq \sum_{n=1}^\xi M_{v,\omega}^q(\mathcal{F}) (2\omega)^\eta \\ &\leq \xi M_{v,\omega}^q(\mathcal{F}) (2\omega)^\eta. \end{aligned}$$

Therefore,

$$L_{\mathbf{v}}^{q,\eta}(\mathcal{F}) \leq \xi \bar{P}_{\mathbf{v}}^{q,\eta}(\mathcal{F}),$$

also

$$\bar{H}_{\mathbf{v}}^{q,\eta}(\mathcal{F}) \leq \xi P_{\mathbf{v}}^{q,\eta}(\mathcal{F}) \leq \xi P_{\mathbf{v}}^{q,\eta}(\mathcal{A}).$$

Finally, we get

$$H_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq \xi P_{\mathbf{v}}^{q,\eta}(\mathcal{A}).$$

(5) Let $\mathcal{F} \subset \mathcal{A}$, consider $(\mathcal{B}(z_n, \omega))_n$ as a centered covering of $\mathcal{F}$. As a result,

$$\bar{\mathcal{H}}_{\mathbf{v},\omega}^{q,\eta}(\mathcal{F}) \leq N_{\mathbf{v},\omega}^q(\mathcal{F})(2\omega)^\eta.$$

It is evident that from the definitions $\bar{\mathcal{H}}_{\mathbf{v}}^{q,\eta}(\mathcal{F}) \leq L_{\mathbf{v}}^{q,\eta}(\mathcal{F})$. Given $(\mathcal{A}_n)_n$ as a countable collection of subsets of $\mathbb{R}^d$ with the condition that $\mathcal{F} \subseteq \bigcup_n \mathcal{A}_n$, then

$$\bar{\mathcal{H}}_{\mathbf{v}}^{q,\eta}(\mathcal{F}) \leq \sum_n L_{\mathbf{v}}^{q,\eta}(\mathcal{A}_n) \quad \text{and} \quad \bar{\mathcal{H}}_{\mathbf{v}}^{q,\eta}(\mathcal{F}) \leq \bar{H}_{\mathbf{v}}^{q,\eta}(\mathcal{F}).$$

At last, we establish that $\mathcal{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq H_{\mathbf{v}}^{q,\eta}(\mathcal{A})$.

Alternatively, we have $\bar{\mathcal{P}}_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \geq \bar{P}_{\mathbf{v}}^{q,\eta}(\mathcal{A})$ and

$$\mathcal{P}_{\mathbf{v}}^{q,\eta}(\mathcal{A}) = \inf_{\mathcal{A} \subseteq \bigcup_n \mathcal{F}_n} \left\{ \sum_i \bar{\mathcal{P}}_{\mathbf{v}}^{q,\eta}(\mathcal{F}_n) \right\} \geq \inf_{\mathcal{A} \subseteq \bigcup_n \mathcal{F}_n} \left\{ \sum_n \bar{P}_{\mathbf{v}}^{q,\eta}(\mathcal{F}_n) \right\} = P_{\mathbf{v}}^{q,\eta}(\mathcal{A}),$$

which prove the desired result.

Now, we list the relationship between the lower and upper mixed multifractal Hewitt-Stromberg functions and the mixed multifractal box dimension and the multifractal pre-Hewitt-Stromberg dimension. At the outset, it is worth mentioning that there is a unique number $d_{\mathbf{v}}^q(\mathcal{A}) \in [-\infty, +\infty]$ satisfying,

$$\begin{cases} L_{\mathbf{v}}^{q,\eta}(\mathcal{A}) = +\infty & \text{if } \eta < d_{\mathbf{v}}^q(\mathcal{A}), \\ L_{\mathbf{v}}^{q,\eta}(\mathcal{A}) = 0 & \text{if } \eta > d_{\mathbf{v}}^q(\mathcal{A}). \end{cases}$$

Proposition 1 Let $\mathcal{A} \subset \text{supp } \mathbf{v} = \mathcal{D}$

(1) For $\mathbf{q} \in \mathbb{R}_+^k \setminus \{\mathbf{0}\}$ and $\mathbf{v}$ satisfying the doubling condition, we have

$$b_{\mathbf{v}}^{\mathbf{q}}(\mathcal{A}) = \sup_{\mathcal{F} \subseteq \mathcal{A}} \left\{ \inf \left\{ \sup_n d_{\mathbf{v}}^{\mathbf{q}}(\mathcal{F}_n) \mid \mathcal{F} \subseteq \bigcup_n \mathcal{F}_n, \mathcal{F}_n \text{ are bounded in } \mathbb{R}^d \right\} \right\}.$$

(2) For $\mathbf{q} \in \mathbb{R}^k$, we have

$$B_{\mathbf{v}}^{\mathbf{q}}(\mathcal{A}) = \inf \left\{ \sup_n \bar{\tau}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{A}_n) \mid \mathcal{A} \subseteq \bigcup_n \mathcal{A}_n, \mathcal{A}_n \text{ are bounded in } \mathbb{R}^d \right\}.$$

Proof:

(1) We first prove that $d_{\mathbf{v}}^{\mathbf{q}}(\mathcal{A}) = \liminf_{\omega \rightarrow 0} \frac{\log N_{\mathbf{v},\omega}^{\mathbf{q}}(\mathcal{A})}{-\log \omega}$. Suppose for some $\varepsilon > 0$ that

$$\liminf_{\omega \rightarrow 0} \frac{\log N_{\mathbf{v}, \omega}^q(\mathcal{A})}{-\log \omega} > d_{\mathbf{v}}^q(\mathcal{A}) + \varepsilon.$$

Thus, we can find $\beta > 0$ such that for any $\omega \leq \beta$,

$$N_{\mathbf{v}, \omega}^q(\mathcal{A}) \omega^{d_{\mathbf{v}}^q(\mathcal{A}) + \varepsilon} > 1.$$

Then,

$$L_{\mathbf{v}}^{q, d_{\mathbf{v}}^q(\mathcal{A}) + \varepsilon}(\mathcal{A}) \geq 2^{d_{\mathbf{v}}^q(\mathcal{A}) + \varepsilon}$$

which leads to a contradiction, Finally we deduce that

$$\liminf_{\omega \rightarrow 0} \frac{\log N_{\mathbf{v}, \omega}^q(\mathcal{A})}{-\log \omega} \leq d_{\mathbf{v}}^q(\mathcal{A}) + \varepsilon, \text{ for any } \varepsilon > 0,$$

for more details see [1].

Next, denote

$$\sigma = \sup_{\mathcal{F} \subseteq \mathcal{A}} \left\{ \inf_n \left\{ \sup_n d_{\mathbf{v}}^q(\mathcal{F}_n) \mid \mathcal{F} \subseteq \bigcup_n \mathcal{F}_n, \mathcal{F}_n \text{ are bounded in } \mathbb{R}^d \right\} \right\}.$$

Suppose that $\sigma < b_{\mathbf{v}}^q(\mathcal{A})$ and choose $\gamma \in (\sigma, b_{\mathbf{v}}^q(\mathcal{A}))$. For $\mathcal{F} \subseteq \mathcal{A}$, there is $\{\mathcal{A}_n\}$ as a family of bounded subset of $\mathcal{F}$ in order that $\mathcal{F} \subseteq \bigcup_n \mathcal{A}_n$ and $\sup_n d_{\mathbf{v}}^q(\mathcal{A}_n) < \gamma$. It is clear that $L_{\mathbf{v}}^{q, \gamma}(\mathcal{A}_n) = 0$, then we conclude that $\overline{H}_{\mathbf{v}}^{q, \gamma}(\mathcal{F}) = 0$ which implies that $H_{\mathbf{v}}^{q, \gamma}(\mathcal{A}) = 0$. It is a contradiction. On the other hand, we suppose that $\sigma > b_{\mathbf{v}}^q(\mathcal{A})$. For $\gamma \in (b_{\mathbf{v}}^q(\mathcal{A}), \sigma)$, we have $H_{\mathbf{v}}^{q, \gamma}(\mathcal{A}) = 0$. This leads to the conclusion that $\overline{H}_{\mathbf{v}}^{q, \gamma}(\mathcal{F}) = 0, \forall \mathcal{F} \subseteq \mathcal{A}$. Consequently this suggests the existence of $\{\mathcal{A}_n\}$ as a family of bounded subset of $\mathcal{F}$ with the property that $\mathcal{F} \subseteq \bigcup_n \mathcal{A}_n$ and $\sup_n L_{\mathbf{v}}^{q, \gamma}(\mathcal{A}_n) < \infty$. We deduce that, $\sup_n d_{\mathbf{v}}^q(\mathcal{A}_n) \leq \gamma$, and it is also a contradiction.

(2) The second statement's proof is analogous to the first one then we use the same arguments, the proof is therefore omitted.

Notations: For $\mathcal{A} \subseteq \text{supp } \mathbf{v} =: \mathcal{D}$, denote

$$b_{\mathbf{v}}(\mathbf{q}) = b_{\mathbf{v}}^q(\text{supp } \mathbf{v}),$$

$$B_{\mathbf{v}}(\mathbf{q}) = B_{\mathbf{v}}^q(\text{supp } \mathbf{v}),$$

$$\Lambda_{\mathbf{v}}(\mathbf{q}) = \Delta_{\mathbf{v}}^q(\text{supp } \mathbf{v}).$$

Proposition 2:

- (1) The functions $\mathbf{q} \mapsto \Lambda_{\mathbf{v}}(\mathbf{q})$ and $\mathbf{q} \mapsto B_{\mathbf{v}}(\mathbf{q})$ are both convex.
- (2) For $i = 1, 2, \dots, k$, the following functions: $q_i \mapsto b_{\mathbf{v}}(\mathbf{q})$, $q_i \mapsto B_{\mathbf{v}}(\mathbf{q})$ moreover $q_i \mapsto \Lambda_{\mathbf{v}}(\mathbf{q})$ are decreasing.

Proof:

(1) Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^k$, $\sigma \in [0, 1]$, $\eta > \Lambda_{\mathbf{v}}(\mathbf{p})$ and $t > \Lambda_{\mathbf{v}}(\mathbf{q})$, denote $\sigma \mathbf{q} = (\sigma q_1, \dots, \sigma q_k)$. Let now define $(\mathcal{B}(z_i, \omega))_i$ as a centered δ -packing of set $\mathcal{A}$. Applying Hölder's inequality, we have

$$\overline{P}_v^{\sigma q+(1-\sigma)p, \sigma t+(1-\sigma)\eta}(\mathcal{A}) \leq (\overline{P}_v^{q,t}(\mathcal{A}))^\sigma (\overline{P}_v^{p,\eta}(\mathcal{A}))^{1-\sigma}.$$

Then

$$\overline{P}_v^{\sigma q+(1-\sigma)p, \sigma t+(1-\sigma)\eta}(\mathcal{A}) = 0, \quad \forall \eta > \Lambda_v(\mathbf{p}) \quad \text{and} \quad t > \Lambda_v(\mathbf{q}).$$

Consequently,

$$\Lambda_v(\sigma \mathbf{q} + (1-\sigma)\mathbf{p}) \leq \sigma \Lambda_v(\mathbf{q}) + (1-\sigma)\Lambda_v(\mathbf{p}).$$

Next, we demonstrate the convexity of B_v . Let $\varepsilon > 0$, $t = B_v(\mathbf{q})$ and $\eta = B_v(\mathbf{p})$, we obtain

$$P_v^{q,t+\varepsilon}(\mathcal{A}) = P_v^{p,\eta+\varepsilon}(\mathcal{A}) = 0.$$

Then, It is possible to find coverings $(K_i)_i$ and $(H_i)_i$ of set $\mathcal{A}$ satisfying

$$\sum_i \overline{P}_v^{q,t+\varepsilon}(H_i) \leq 1 \quad \text{and} \quad \sum_i \overline{P}_v^{p,\eta+\varepsilon}(K_i) \leq 1.$$

For $n \in \mathbb{N}$, let $\mathcal{A}_n = \bigcup_{1 \leq i, j \leq n} (H_i \cap K_j)$. Hence, $(\mathcal{A}_n)_n$ is a covering of $\mathcal{A}$, we get

$$\begin{aligned} P_v^{\sigma q+(1-\sigma)p, \sigma t+(1-\sigma)\eta+\varepsilon}(\mathcal{A}_n) &\leq P_v^{\sigma q+(1-\sigma)p, \sigma t+(1-\sigma)\eta+\varepsilon}\left(\bigcup_{i,j=1}^n (H_i \cap K_j)\right) \\ &\leq \sum_{i,j=1}^n P_v^{\sigma q+(1-\sigma)p, \sigma t+(1-\sigma)\eta+\varepsilon}(H_i \cap K_j) \\ &\leq \sum_{i,j=1}^n \overline{P}_v^{\sigma q+(1-\sigma)p, \sigma t+(1-\sigma)\eta+\varepsilon}(H_i \cap K_j). \end{aligned}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned} P_v^{\sigma q+(1-\sigma)p, \sigma t+(1-\sigma)\eta+\varepsilon}(\mathcal{A}_n) &\leq \left(\sum_{i,j=1}^n \overline{P}_v^{q,t+\varepsilon}(H_i \cap K_j)\right)^\sigma \left(\sum_{i,j=1}^n \overline{P}_v^{p,\eta+\varepsilon}(H_i \cap K_j)\right)^{1-\sigma} \\ &\leq \left(\sum_{i,j=1}^n \overline{P}_v^{q,t+\varepsilon}(H_i)\right)^\sigma \left(\sum_{i,j=1}^n \overline{P}_v^{p,\eta+\varepsilon}(K_j)\right)^{1-\sigma} \\ &\leq \left(n \sum_{i=1}^n \overline{P}_v^{q,t+\varepsilon}(H_i)\right)^\sigma \left(n \sum_{j=1}^n \overline{P}_v^{p,\eta+\varepsilon}(K_j)\right)^{1-\sigma} \\ &\leq n^\sigma n^{1-\sigma} = n < \infty. \end{aligned}$$

Since ε was arbitrary, we obtain

$$B_v(\sigma \mathbf{q} + (1-\sigma)\mathbf{p}) \leq \sigma B_v(\mathbf{q}) + (1-\sigma)B_v(\mathbf{p}).$$

(2) For $i = 1, \dots, k$, let $\mathbf{p} = (q_1, \dots, q_{i-1}, p_i, q_{i+1}, \dots, q_k)$, $\mathbf{q} = (q_1, \dots, q_{i-1}, q_i, q_{i+1}, \dots, q_k)$ with $p_i \leq q_i$ reel numbers. Let $(\mathcal{B}(z_i, \omega))_i$ be a centered covering of $\mathcal{F}$ for $\mathcal{F} \subseteq \mathcal{A}$, we can find a σ -family $\{\mathcal{F}_i\}$ of bounded subset of $\mathcal{F}$ in such a way that $\mathcal{F} \subseteq \bigcup_i \mathcal{F}_i$. We have

$$L_v^{q,\eta}(\mathcal{F}_i) \leq L_v^{p,\eta}(\mathcal{F}_i).$$

Then,

$$\overline{H}_v^{q,\eta}(\mathcal{F}) \leq \overline{H}_v^{p,\eta}(\mathcal{F}).$$

We therefore conclude that

$$H_v^{q,\eta}(\mathcal{A}) = \sup_{\mathcal{F} \subseteq \mathcal{A}} \overline{H}_v^{q,\eta}(\mathcal{F}) \leq \sup_{\mathcal{F} \subseteq \mathcal{A}} \overline{H}_v^{p,\eta}(\mathcal{F}) = H_v^{p,\eta}(\mathcal{A}).$$

Hence,

$$H_v^{q,\eta}(\mathcal{A}) = 0, \quad \forall \eta > b_v(\mathbf{p}).$$

Consequently,

$$b_v(\mathbf{q}) < \eta, \quad \forall \eta > b_v(\mathbf{p}).$$

Finally,

$$b_v(\mathbf{q}) \leq b_v(\mathbf{p}).$$

Similarly, we prove the monotony of B_v and Λ_v .

3. Results

We begin this section with some notations to make our work easier. For $\ell, t \in \mathbb{N}$, let

$$\Omega_t = \{1, \dots, \ell\}^t, \quad \Omega = \bigcup_t \Omega_t,$$

signifying that Ω_t represents the family of sequences of length t and Ω refers to all finite sequences.

If $\mathbf{n} \in \Omega_t$, we can write $|\mathbf{n}| = t$. Additionally if $\mathbf{n} = (n_1, \dots, n_t) \in \Omega_t$, we will denote

$$\mathcal{S}_{\mathbf{n}} = S_{n_1} \circ \dots \circ S_{n_t},$$

$$\omega_{\mathbf{n}} = \omega_{n_1} \dots \omega_{n_t},$$

$$p_{m,\mathbf{n}} = p_{m,n_1} \dots p_{m,n_t}.$$

Also the truncation of $\mathbf{n}$ to the t -th place is defined by $\mathbf{n}|t = (n_1, \dots, n_t)$, for $\mathbf{n} \in \Omega$. Therefore, we define $\pi: \Omega_t \rightarrow \mathcal{D}$ by $\{\pi(\mathbf{n})\} = \bigcap_t \mathcal{D}_{\mathbf{n}|t}$, where $\mathcal{D}_{\mathbf{n}} = S_{n_1} \circ \dots \circ S_{n_t}(\mathcal{D})$.

In addition, the concatenation of $\mathbf{n}$ and $\mathbf{m}$ is defined by $\mathbf{nm} = (n_1, \dots, i_{|\mathbf{n}|}, m_1, \dots, m_{|\mathbf{m}|})$, for $\mathbf{n}, \mathbf{m} \in \Omega$.

If $\mathbf{l}, \mathbf{m} \in \Omega$, we write $\mathbf{l} \preceq \mathbf{m}$ if and only if $\mathbf{m} \neq \mathbf{ulw}$, for all $\mathbf{u}, \mathbf{w} \in \Omega$.

If $\Phi \subseteq \Omega$ and $\mathbf{m} \in \Omega$, we write $\Phi \preceq \mathbf{m}$ if and only if $\mathbf{l} \preceq \mathbf{m}$, for all $\mathbf{l} \in \Phi$.

Since $(S_1, \dots, S_\ell)$ satisfies the (OSC), we have $W \cap \mathcal{D} \neq \emptyset$ then we can select $\mathbf{l} \in \Omega$ such that $\mathcal{S}_{\mathbf{l}}(\mathcal{D}) \subseteq W$. So, there exists $t \in \mathbb{N}$ such that

$$\Phi_t = \{\mathbf{l} \in \Omega_t \mid \mathcal{S}_{\mathbf{l}}(\mathcal{D}) \subseteq W\} \neq \emptyset.$$

For $\mathbf{q} \in \mathbb{R}^k$ and $t \in \mathbb{N}$ with $\Phi_t \neq \emptyset$, we write

$$E_t(\eta) = \sum_{\substack{|\mathbf{n}|=t \\ \mathbf{n} \in \Phi_t}} p_{1,\mathbf{n}}^{q_1} \dots p_{k,\mathbf{n}}^{q_k} \omega_{\mathbf{n}}^\eta, \quad \text{for } \eta \in \mathbb{R}.$$

The function $E_t: \mathbb{R} \rightarrow \mathbb{R}$ is both strictly not increasing and continuous with $\lim_{\eta \rightarrow -\infty} E_t(\eta) = +\infty$ and $\lim_{\eta \rightarrow +\infty} E_t(\eta) = 0$, by using the intermediate value theorem there exists a unique $\gamma_t(\mathbf{q}) \in \mathbb{R}$ such that

$$\sum_{\substack{|\mathbf{n}|=t \\ \mathbf{n} \in \Phi_t}} p_{1,\mathbf{n}}^{q_1} \dots p_{k,\mathbf{n}}^{q_k} \omega_{\mathbf{n}}^{\gamma_t(\mathbf{q})} = 1. \quad (3.1)$$

Furthermore, we define $\beta: \mathbb{R}^k \rightarrow \mathbb{R}$, the unique function satisfying

$$\sum_{\mathbf{n}} p_{1,\mathbf{n}}^{q_1} \dots p_{k,\mathbf{n}}^{q_k} \omega_{\mathbf{n}}^{\beta(\mathbf{q})} = 1.$$

Observe that since

$$\sum_{\substack{|\mathbf{n}|=t \\ \mathbf{n} \in \Phi_t}} p_{1,\mathbf{n}}^{q_1} \dots p_{k,\mathbf{n}}^{q_k} \omega_{\mathbf{n}}^{\gamma_t(\mathbf{q})} = 1 = \sum_{|\mathbf{n}|=t} p_{1,\mathbf{n}}^{q_1} \dots p_{k,\mathbf{n}}^{q_k} \omega_{\mathbf{n}}^{\beta(\mathbf{q})}, \text{ then}$$

$$\gamma_t(\mathbf{q}) < \beta(\mathbf{q}).$$

Now, let

$$\gamma(\mathbf{q}) = \inf_{\Phi_t \neq \emptyset} \gamma_t(\mathbf{q}).$$

The fact that $\mathcal{S}_l(\mathcal{D})$ is compact for $\mathbf{l}$ for all Ω leads to the conclusion that

$$c_t = \min_{l \in \Phi_t} \text{dist}(\mathcal{S}_l(\mathcal{D}), \mathbb{R}^d \setminus W) > 0, \text{ for all } t \in \mathbb{N} \text{ with } \Phi_t \neq \emptyset. \quad (3.2)$$

Also, we clearly have

$$\mathcal{S}_{\mathbf{n}}(\mathcal{D}) \subseteq \overline{\mathcal{S}_{\mathbf{n}}(W)} \quad \text{for all } \mathbf{n} \in \Omega,$$

and

$$\mathcal{S}_{\mathbf{n}}(\mathcal{D}) \cap \mathcal{S}_{\mathbf{m}}(W) = \emptyset, \text{ for all } \mathbf{n}, \mathbf{m} \in \Omega \text{ with } |\mathbf{n}| = |\mathbf{m}| \text{ and } \mathbf{n} \neq \mathbf{m}.$$

The following main result proves that, for a vector-valued measure composed of self-similar measures, all the dimensions which we have mentioned of the self-similar set $\mathcal{D}$ coincide and equal to $\beta(\mathbf{q})$.

Theorem 2: *In the case that the (SSC) is satisfied, we have*

$$\dim_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) = b_{\mathbf{v}}(\mathbf{q}) = B_{\mathbf{v}}(\mathbf{q}) = \text{Dim}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) = \underline{\tau}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) = \overline{\tau}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) = \beta(\mathbf{q}), \quad \text{for } \mathbf{q} \in \mathbb{R}^k.$$

Proof: It is easy to see from [21] that, for all $\mathbf{q} \in \mathbb{R}^k$

$$\underline{\tau}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) = \overline{\tau}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) = \beta(\mathbf{q}). \quad (3.3)$$

On the other hand, there exists an integer $\xi \in \mathbb{N}$ (see Theorem 1) for which the following inequality holds

$$\mathcal{H}_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq H_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq \xi P_{\mathbf{v}}^{q,\eta}(\mathcal{A}) \leq \xi \mathcal{P}_{\mathbf{v}}^{q,\eta}(\mathcal{A}),$$

and this implies that

$$\dim_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) \leq b_{\mathbf{v}}(\mathbf{q}) \leq B_{\mathbf{v}}(\mathbf{q}) \leq \text{Dim}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}). \quad (3.4)$$

In addition, according to L. Olsen results ([21]) It is possible to define a probability measure $\mathcal{V}_{\mathbf{q}}$ on Ω_t such that $\mathcal{V}_{\mathbf{q}}([\mathbf{n}]) = p_{1,\mathbf{n}}^{q_1} \dots p_{k,\mathbf{n}}^{q_k} \omega_{\mathbf{n}}^{\beta(\mathbf{q})}$ for all $\mathbf{n} \in \Omega$. Next, we introduce the probability measure $\nu_{\mathbf{q}}$ on $\mathcal{D}$ by $\nu_{\mathbf{q}} = \mathcal{V}_{\mathbf{q}} \circ \pi^{-1}$. It shows immediately that $\nu_{\mathbf{q}}(\mathcal{D}_{\mathbf{n}}) = p_{1,\mathbf{n}}^{q_1} \dots p_{k,\mathbf{n}}^{q_k} \omega_{\mathbf{n}}^{\beta(\mathbf{q})}$, for all $\mathbf{n} \in \Omega$. After, we use simple calculations to obtain that $\mathcal{H}_{\mathbf{v}}^{q,\eta}(\mathcal{D}) > 0$, $\forall \mathbf{q} \in \mathbb{R}^k$. Then

$$\beta(\mathbf{q}) \leq \dim_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}). \quad (3.5)$$

It is generally acknowledged that a self-similar measure is a doubling measure. Since $\mathbf{v}$ is a vector-valued measure composed of a self-similar measures, we use the same arguments as in [22] but in the mixed case to obtain

$$\text{Dim}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}) \leq \overline{\tau}_{\mathbf{v}}^{\mathbf{q}}(\mathcal{D}). \quad (3.6)$$

Combining inequalities (3.3), (3.4), (3.5) and (3.6) we show the desired result.

The next point to deal with in this paper is proving that the upper mixed multifractal box-dimension of the overlaps is strictly inferior to the mixed multifractal Hausdorff dimension.

Theorem 3: Suppose that $(\mathcal{S}_1, \dots, \mathcal{S}_\ell)$ fulfills the (OSC). Let $\mathbf{q} \in \mathbb{R}_+^k$.

(1) For $\mathbf{n}, \mathbf{m} \in \Omega$ with $|\mathbf{n}| = |\mathbf{m}|$ and $\mathbf{n} \neq \mathbf{m}$, we obtain

$$\bar{\tau}_v^q(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) \leq \gamma(\mathbf{q}) < \beta(\mathbf{q}) = \bar{\tau}_v^q(\mathcal{D}).$$

(2) $\text{Dim}_v^q \left(\bigcup_{\substack{|\mathbf{n}|=|\mathbf{m}|=t \\ \mathbf{n} \neq \mathbf{m}}} \mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D}) \right) \leq \gamma(\mathbf{q}) < \beta(\mathbf{q}) = \text{Dim}_v^q(\mathcal{D}).$

(3) $\text{dim}_v^q \left(\bigcup_{\substack{|\mathbf{n}|=|\mathbf{m}|=t \\ \mathbf{n} \neq \mathbf{m}}} \mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D}) \right) \leq \gamma(\mathbf{q}) < \beta(\mathbf{q}) = \text{dim}_v^q(\mathcal{D}).$

(4) $b_v^q \left(\bigcup_{\substack{|\mathbf{n}|=|\mathbf{m}|=t \\ \mathbf{n} \neq \mathbf{m}}} \mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D}) \right) \leq \gamma(\mathbf{q}) < \beta(\mathbf{q}) = b_v(\mathbf{q}).$

(5) $B_v^q \left(\bigcup_{\substack{|\mathbf{n}|=|\mathbf{m}|=t \\ \mathbf{n} \neq \mathbf{m}}} \mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D}) \right) \leq \gamma(\mathbf{q}) < \beta(\mathbf{q}) = B_v(\mathbf{q}).$

4. Proof of Theorem 3

In order to establish our main result Theorem 3, we need some technical lemmas.

Lemma 1: [20] Suppose that $(\mathcal{S}_1, \dots, \mathcal{S}_\ell)$ fulfills the (OSC). Let $t \in \mathbb{N}$ such that $\Phi_t \neq \emptyset$ and define c_t as stated in (3.2). If $\mathbf{n} = (n_1, \dots, n_{|\mathbf{n}|}), \mathbf{m} = (m_1, \dots, m_{|\mathbf{m}|}) \in \Omega$ with $n_1 \neq m_1$ and $\text{dist}(\mathcal{S}_n(\mathcal{D}), \mathcal{S}_m(\mathcal{D})) \leq c_t \omega_n$, then

$$\Phi_t \not\subseteq (n_2, \dots, n_{|\mathbf{n}|}).$$

Lemma 2: [14] Let $\omega, a, b > 0$. Consider the collection of subsets $(U_\alpha)_\alpha$ of $\mathbb{R}^d$. Suppose that every set U_α includes $\mathcal{B}_\alpha$ which is a closed ball of radius $a\omega$ also is included within a closed ball of radius $b\omega$ under the condition that $\mathcal{B}_\alpha \cap \mathcal{B}_m = \emptyset$, $\alpha \neq m$. As a result, for all $h > 0$ and $z \in \mathbb{R}^d$,

$$|\{\alpha | U_\alpha \cap \mathcal{B}(z, h\omega) \neq \emptyset\}| \leq (h + 2b)^d a^{-d}.$$

Lemma 3: [12] Suppose that $(\mathcal{S}_1, \dots, \mathcal{S}_\ell)$ fulfills the (OSC). Then

$$v_m(\mathcal{S}_n(\mathcal{D})) = p_{m,n} \quad \text{for all } \mathbf{n} \in \Omega \text{ and } m = 1, \dots, k.$$

For $\mathbf{q} \in \mathbb{R}_+^k$ and $\mathbf{n}, \mathbf{m} \in \Omega$ with $|\mathbf{n}| = |\mathbf{m}|$ and $\omega > 0$, we define

$$\mathcal{Q}_{\mathbf{n}, \mathbf{m}}^q(\omega) = \mathcal{M}_{v, \omega}^q(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{B}(\mathcal{S}_m(\mathcal{D}), \omega)),$$

where $\mathcal{B}(\mathcal{A}, \omega) = \{z | \text{dist}(z, \mathcal{A}) \leq \omega\}$ with $\mathcal{A} \subseteq \mathbb{R}^d$ and $\omega > 0$.

In addition, for $\omega > 0$ write

$$\Omega(\omega) = \{\mathbf{n} \in \Omega | \omega_n \text{diam}(\mathcal{D}) < \omega \leq \omega_{|\mathbf{n}|-1} \text{diam}(\mathcal{D})\}.$$

Proposition 3: Suppose that $(\mathcal{S}_1, \dots, \mathcal{S}_\ell)$ satisfies the (OSC). Fix $\mathbf{q} \in \mathbb{R}_+^k$. Consider $t \in \mathbb{N}$ where $\Phi_t \neq \emptyset$. Let $\mathbf{n}, \mathbf{m} \in \Omega$ with $\mathbf{n} \neq \mathbf{m}$ and $|\mathbf{n}| = |\mathbf{m}| = t$. Then there is a constant $\mathcal{K} > 0$, such that

$$Q_{n,m}^q(\omega) \leq \mathcal{K} \sum_{\substack{\omega/diam(\mathcal{D}) \leq \omega_\lambda \\ \Phi_t \not\subseteq \lambda}} p_{1,\lambda}^{q_1} \dots p_{k,\lambda}^{q_k}$$

Proof: Let $T = diam(\mathcal{D})$, $\omega_{min} = \min_{i=1,\dots,\ell} \omega_i$ and $\omega_{max} = \max_{i=1,\dots,\ell} \omega_i$. Let $\omega > 0$ and define $\mathcal{F}$ as an ω -separated subset of $\mathcal{S}_n(\mathcal{D}) \cap \mathcal{B}(\mathcal{S}_m(\mathcal{D}), \omega)$. For each $z \in \mathcal{F}$ we can select $\lambda(z) \in \Omega(\omega)$ that satisfies $z \in \mathcal{S}_{\lambda(z)}(\mathcal{D})$, which implies that

$$\mathcal{S}_{\lambda(z)}(\mathcal{D}) \subseteq \mathcal{B}(z, \omega) \cap \mathcal{D} \subseteq \bigcup_{\substack{\lambda \in \Omega(r) \\ dist(z, \mathcal{S}_\lambda(\mathcal{D})) \leq \omega}} \mathcal{S}_\lambda(\mathcal{D}),$$

for all $z \in \mathcal{F}$, thus

$$\begin{aligned} v_1(\mathcal{B}(z, \omega))^{q_1} \dots v_k(\mathcal{B}(z, \omega))^{q_k} &\leq \\ v_1(\mathcal{S}_{\lambda(z)}(\mathcal{D}))^{q_1} \dots v_k(\mathcal{S}_{\lambda(z)}(\mathcal{D}))^{q_k}, &\quad \text{for } \mathbf{q} \in \mathbb{R}_+^k. \end{aligned} \quad (4.1)$$

Let c_t be as in (3.2), now take $N \in \mathbb{N}$ such that $\omega_{max}^{-(N-1)} \geq 2T c_t^{-1} \omega_{min}^{-(t-1)}$. We clearly have by using [20],

If $z \in \mathcal{F}$, $\lambda = (\lambda_1, \dots, \lambda_{|\lambda|}) \in \Omega(\omega)$ and

$$dist(z, \mathcal{S}_\lambda(\mathcal{D})) \leq \omega, \text{ then } \Phi_t \not\subseteq (\lambda_{t+1}, \dots, \lambda_{|\lambda|-N}). \quad (4.2)$$

Due to the fact that $\mathcal{F}$ is ω -separated and W contains a ball of radius ρ_1 , then this suggests the existence of $(\mathcal{B}_y)_{y \in \mathcal{F}}$ as a family of balls that are pairwise disjoint and with radii equal to $\min((\omega_{min}/T)\rho_1, 1)\omega$ such that $\bigcup_{\substack{\lambda \in \Omega(r) \\ dist(y, \mathcal{S}_\lambda(\mathcal{D})) \leq \omega}} \overline{\mathcal{S}_\lambda(W)}$ contains $\mathcal{B}_y$.

Moreover, $\bigcup_{\substack{\lambda \in \Omega(r) \\ dist(y, \mathcal{S}_\lambda(\mathcal{D})) \leq \omega}} \overline{\mathcal{S}_\lambda(W)}$ is enclosed within a ball of radius $\omega + \max_{\lambda \in \Omega(r)} diam(\overline{\mathcal{S}_\lambda(W)}) \leq (1 + \rho_2/T)\omega$. Since $\mathcal{S}_\lambda(\mathcal{D}) \subseteq \overline{\mathcal{S}_\lambda(W)}$, by making use Lemma 2 we obtain

$$\begin{aligned} &\left| \left\{ y \in \mathcal{F} \mid \left(\bigcup_{\substack{\lambda \in \Omega(\omega) \\ dist(y, \mathcal{S}_\lambda(\mathcal{D})) \leq \omega}} \mathcal{S}_\lambda(\mathcal{D}) \right) \cap \left(\bigcup_{\substack{\lambda \in \Omega(r) \\ dist(z, \mathcal{S}_\lambda(\mathcal{D})) \leq \omega}} \mathcal{S}_\lambda(\mathcal{D}) \right) \neq \emptyset \right\} \right| \\ &\leq \left| \left\{ y \in \mathcal{F} \mid \left(\bigcup_{\substack{\lambda \in \Omega(\omega) \\ dist(y, \mathcal{S}_\lambda(\mathcal{D})) \leq \omega}} \overline{\mathcal{S}_\lambda(W)} \right) \cap \mathcal{B}\left(z, \left(1 + \frac{\rho_2}{T}\right)\omega\right) \neq \emptyset \right\} \right| \\ &\leq \mathcal{C}_3, \end{aligned}$$

for all $z \in \mathcal{F}$ with $\mathcal{C}_3 = (((1 + \rho_2/T) + 2(1 + \rho_2/T))/\min(\omega_{min}/T\rho_1, 1))^d$.

Whence by (4.2)

$$\begin{aligned} &\sum_{z \in \mathcal{F}} \sum_{\substack{\lambda \in \Omega(\omega) \\ dist(z, \mathcal{S}_\lambda(\mathcal{D})) \leq \omega}} v_1(\mathcal{S}_\lambda(\mathcal{D}))^{q_1} \dots v_k(\mathcal{S}_\lambda(\mathcal{D}))^{q_k} \leq \\ &\mathcal{C}_3 \sum_{\substack{\lambda \in \Omega(\omega) \\ \Phi_t \not\subseteq (\lambda_{t+1}, \dots, \lambda_{|\lambda|-N})}} v_1(\mathcal{S}_\lambda(\mathcal{D}))^{q_1} \dots v_k(\mathcal{S}_\lambda(\mathcal{D}))^{q_k}, \end{aligned} \quad (4.3)$$

for $\mathbf{q} \in \mathbb{R}_+^k$.

Combining (4.1), (4.3) and Lemma 2, we obtain

$$\sum_{z \in \mathcal{F}} v_1(\mathcal{B}(z, \omega))^{q_1} \dots v_k(\mathcal{B}(z, \omega))^{q_k}$$

$$\begin{aligned}
&\leq \sum_{z \in \mathcal{F}} \nu_1(\mathcal{S}_{\lambda(z)}(\mathcal{D}))^{q_1} \dots \nu_k(\mathcal{S}_{\lambda(z)}(\mathcal{D}))^{q_k} \\
&\leq \sum_{z \in \mathcal{F}} \sum_{\substack{\lambda \in \Omega(\omega) \\ \text{dist}(z, \mathcal{S}_{\lambda}(\mathcal{D})) \leq \omega}} \nu_1(\mathcal{S}_{\lambda}(\mathcal{D}))^{q_1} \dots \nu_k(\mathcal{S}_{\lambda}(\mathcal{D}))^{q_k} \\
&\leq \mathcal{C}_3 \sum_{\substack{\lambda \in \Omega(\omega) \\ \Phi_t \not\subseteq (\lambda_{t+1}, \dots, \lambda_{|\lambda|-N})}} \nu_1(\mathcal{S}_{\lambda}(\mathcal{D}))^{q_1} \dots \nu_k(\mathcal{S}_{\lambda}(\mathcal{D}))^{q_k} \\
&\leq \mathcal{C}_3 \sum_{\substack{\omega / \text{diam}(\mathcal{D}) \leq \omega_{\lambda_{t+1}, \dots, \lambda_{|\lambda|-N}} \\ \Phi_t \not\subseteq (\lambda_{t+1}, \dots, \lambda_{|\lambda|-N})}} p_{1,\lambda}^{q_1} \dots p_{k,\lambda}^{q_k} \\
&= \mathcal{C}_3 \sum_{|u|=t} \sum_{|v|=N} \sum_{\substack{\omega / \text{diam}(\mathcal{D}) \leq \omega_h \\ \Phi_t \not\subseteq h}} p_{1,uhv}^{q_1} \dots p_{k,uhv}^{q_k} \\
&\leq \mathcal{C}_3 n^{t+N} \left(\max_{n=1, \dots, \ell} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} \right)^{t+N} \sum_{\substack{\omega / \text{diam}(\mathcal{D}) \leq \omega_h \\ \Phi_t \not\subseteq h}} p_{1,h}^{q_1} \dots p_{k,h}^{q_k},
\end{aligned}$$

and the result yields.

Proposition 4: Suppose that $(S_1, \dots, S_\ell)$ satisfies the (OSC). Let $\mathbf{q} \in \mathbb{R}_+^k$ and $t \in \mathbb{N}$ such that $\Phi_t \neq \emptyset$. Then there is $\mathcal{K} > 0$, such that

$$\sum_{\substack{\omega \leq \omega_n \\ \Phi_t \not\subseteq n}} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} \leq \mathcal{K} \omega^{-\gamma_t(\mathbf{q})}, \text{ for all } \omega > 0.$$

Proof: Define the functions $f, g : (0, \infty) \rightarrow [0, \infty)$ by

$$f(\omega) = \sum_{\substack{\omega \leq \omega_n \\ \Phi_t \not\subseteq n}} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} \quad \text{and} \quad g(\omega) = \omega^{\gamma_t(\mathbf{q})} f(\omega).$$

Write $\delta = (\min_{n=1, \dots, \ell} \omega_n)^t$. If $0 < \omega < \delta$, then

$$\begin{aligned}
f(\omega) &= \sum_{\substack{|n|=t \\ n \notin \Phi_t}} \sum_{\substack{\omega \leq \omega_{it} \\ \Phi_t \not\subseteq nt}} p_{1,nt}^{q_1} \dots p_{k,nt}^{q_k} \\
&\leq \sum_{\substack{|n|=t \\ n \notin \Phi_t}} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} \sum_{\substack{\omega_n^{-1} \omega \leq \omega_t \\ \Phi_t \not\subseteq t}} p_{1,t}^{q_1} \dots p_{k,t}^{q_k} \\
&= \sum_{\substack{|n|=t \\ n \notin \Phi_t}} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} f(\omega_n^{-1} \omega).
\end{aligned}$$

Then, for all $0 < \omega < \delta$ we obtain

$$\begin{aligned}
g(\omega) &\leq \sum_{\substack{|n|=t \\ n \notin \Phi_t}} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} \omega_n^{\gamma_t(\mathbf{q})} (\omega_n^{-1} \omega)^{\gamma_t(\mathbf{q})} f(\omega_n^{-1} \omega) \\
&= \sum_{\substack{|n|=t \\ n \notin \Phi_t}} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} \omega_n^{\gamma_t(\mathbf{q})} g(\omega_n^{-1} \omega).
\end{aligned}$$

Let $\Delta = (\max_{n=1, \dots, \ell} \omega_n)^t$, by making use the previous inequality also the definition of $\gamma_t(\mathbf{q})$ it follows that, if

$0 < b < \delta$, consequently

$$\begin{aligned}
\sup_{b\Delta \leq \omega < \delta} g(\omega) &\leq \sup_{b\Delta \leq \omega < \delta} \sum_{\substack{|n|=t \\ n \notin \Phi_t}} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} \omega_n^{\gamma_t(\mathbf{q})} g(\omega_n^{-1} \omega) \\
&\leq \sum_{\substack{|n|=t \\ n \notin \Phi_t}} p_{1,n}^{q_1} \dots p_{k,n}^{q_k} \omega_n^{\gamma_t(\mathbf{q})} \sup_{b \leq \eta} g(\eta) \\
&= \sup_{b \leq \eta} g(\eta).
\end{aligned}$$

Finally, we get

$$\sup_{b\Delta \leq \omega} g(\omega) \leq \sup_{b \leq \eta} g(\eta).$$

This suggests the existence of $\mathcal{C} > 0$ for which

$$\omega^{\gamma_t(\mathbf{q})} f(\omega) = g(\omega) \leq \mathcal{C}, \quad \forall \omega > 0.$$

Return now towards the demonstration of Theorem 3.

Proof: Let $t \in \mathbb{N}$ such that $\Phi_t \neq \emptyset$. Fix $\mathbf{n}, \mathbf{m} \in \Omega$ with $|\mathbf{n}| = |\mathbf{m}| = t$ and $\mathbf{n} \neq \mathbf{m}$, it can be inferred from propositions 3 and 4 that there is a constant $\mathcal{K} > 0$ satisfying

$$Q_{\mathbf{n}, \mathbf{m}}^q(\omega) \leq \mathcal{K} \omega^{-\gamma_t(\mathbf{q})}, \quad \forall \omega > 0. \quad (4.4)$$

Since $M_{\mathbf{v}, \omega}^q(\mathcal{S}_{\mathbf{n}}(\mathcal{D}) \cap \mathcal{S}_{\mathbf{m}}(\mathcal{D})) \leq Q_{\mathbf{n}, \mathbf{m}}^q(\omega)$ for all $\omega > 0$, by using (4.4) we have

$$\begin{aligned} \bar{\tau}_{\mathbf{v}}^q(\mathcal{S}_{\mathbf{n}}(\mathcal{D}) \cap \mathcal{S}_{\mathbf{m}}(\mathcal{D})) &\leq \limsup_{\omega \rightarrow 0} \frac{\log Q_{\mathbf{n}, \mathbf{m}}^q(\omega)}{-\log \omega} \\ &\leq \limsup_{\omega \rightarrow 0} \frac{\log(\mathcal{K} \omega^{-\gamma_t(\mathbf{q})})}{-\log \omega} \\ &= \gamma_t(\mathbf{q}). \end{aligned}$$

Let $\zeta \in \mathbb{N}$ and $\rho > 0$ be as in Besicovitch's Covering Theorem [20]. Let $\varepsilon > 0$. Let $0 < \delta < \rho$ and consider $(\mathcal{B}(z_n, \omega_n))_n$ as a centered δ -packing of $\mathcal{S}_{\mathbf{n}}(\mathcal{D}) \cap \mathcal{S}_{\mathbf{m}}(\mathcal{D})$. Write $\rho_l = \frac{\rho}{2^{l-1}}$, for $l \in \mathbb{N}$. If $n, l \in \mathbb{N}$ such that $\rho_{l+1} \leq \omega_n < \rho_l$, it can be concluded from the definition of sequence $(\rho_l)_l$ that

$$(2\omega_n)^{\gamma_t(\mathbf{q})+\varepsilon} \leq \mathcal{C}_0 p_l^{\gamma_t(\mathbf{q})+\varepsilon} \quad \text{and} \quad (2\omega_n)^{\gamma_t(\mathbf{q})+\varepsilon} \leq \mathcal{C}_0 p_{l+1}^{\gamma_t(\mathbf{q})+\varepsilon}, \quad (4.5)$$

where $\mathcal{C}_0 = 2^{\gamma_t(\mathbf{q})+\varepsilon} \max(1, 2^{\pm(\gamma_t(\mathbf{q})+\varepsilon)})$.

Next, let $l \in \mathbb{N}$, since $|z_n - z_m| > \omega_n + \omega_m \geq 2\rho_{l+1} = \rho_l$, $\forall n \neq m$ with $\rho_{l+1} \leq \omega_n < \rho_l$ and $\rho_{l+1} \leq \omega_m < \rho_l$. The use of Besicovitch's Covering Theorem (see [22]) on the collection $\{\mathcal{B}(z_n, \rho_l) \mid \rho_{l+1} \leq \omega_n < \rho_l\}$ allows us the existence of ζ subfamilies $I_{l_1}, \dots, I_{l_\zeta}$ of $\{n \mid \rho_{l+1} \leq \omega_n < \rho_l\}$ such that $\{n \mid \rho_{l+1} \leq \omega_n < \rho_l\} = \bigcup_{\eta} I_{l_\eta}$ and $\mathcal{B}(z_n, \rho_l) \cap \mathcal{B}(z_m, \rho_l) = \emptyset$, for all $n, m \in I_{l_\eta}$ with $n \neq m$. We conclude that

$$\begin{aligned} \sum_{\rho_{l+1} \leq \omega_n < \rho_l} v_1(\mathcal{B}(z_n, \rho_l))^{q_1} \dots v_k(\mathcal{B}(z_n, \rho_l))^{q_k} \\ \leq \sum_{\eta=1}^{\zeta} \sum_{n \in I_{l_\eta}} v_1(\mathcal{B}(z_n, \rho_l))^{q_1} \dots v_k(\mathcal{B}(z_n, \rho_l))^{q_k} \\ \leq \sum_{\eta=1}^{\zeta} \mathcal{M}_{\mathbf{v}, \rho_l}^q(\mathcal{S}_{\mathbf{n}}(\mathcal{D}) \cap \mathcal{B}(\mathcal{S}_{\mathbf{m}}(\mathcal{D}), \rho_l)) \\ = \zeta \mathcal{M}_{\mathbf{v}, \rho_l}^q(\mathcal{S}_{\mathbf{n}}(\mathcal{D}) \cap \mathcal{B}(\mathcal{S}_{\mathbf{m}}(\mathcal{D}), \rho_l)), \end{aligned}$$

for all $l \in \mathbb{N}$ and $\mathbf{q} \in \mathbb{R}_+^k$.

It now follows from (4.4), (4.5) and the previous inequality that

$$\begin{aligned} \sum_n v_1(\mathcal{B}(z_n, \omega_n))^{q_1} \dots v_l(\mathcal{B}(z_n, \omega_n))^{q_k} (2\omega_n)^{\gamma_t(\mathbf{q})+\varepsilon} \\ = \sum_l \sum_{\rho_{l+1} \leq \omega_n < \rho_l} v_1(\mathcal{B}(z_n, \omega_n))^{q_1} \dots v_k(\mathcal{B}(z_n, \omega_n))^{q_k} (2\omega_n)^{\gamma_t(\mathbf{q})+\varepsilon} \\ \leq \mathcal{C}_0 \sum_l \sum_{\rho_{l+1} \leq \omega_n < \rho_l} v_1(\mathcal{B}(z_n, \rho_{l+1}))^{q_1} \dots v_k(\mathcal{B}(z_n, \rho_{l+1}))^{q_k} \rho_{l+1}^{\gamma_t(\mathbf{q})+\varepsilon} \\ \leq \mathcal{C}_0 \sum_l \mathcal{M}_{\mathbf{v}, \rho_{l+1}}^q(\mathcal{S}_{\mathbf{n}}(\mathcal{D}) \cap \mathcal{B}(\mathcal{S}_{\mathbf{m}}(\mathcal{D}), \rho_{l+1})) \rho_{l+1}^{\gamma_t(\mathbf{q})+\varepsilon} \end{aligned}$$

$$\begin{aligned}
 &\leq \mathcal{C}_0 \zeta \sum_l \rho_l^{\gamma_t(q)+\varepsilon} Q_{n,m}^q(\rho_l) \\
 &\leq \mathcal{C}_0 \zeta \mathcal{K} \sum_l \rho_l^{\gamma_t(q)+\varepsilon} \rho_l^{-\gamma(q)} \\
 &= \mathcal{C}_0 \zeta \mathcal{K} \sum_l \frac{\delta^\varepsilon}{2^{\varepsilon(l-1)}} \\
 &= \mathcal{C}_0 \zeta \mathcal{K} \frac{2^\varepsilon}{2^{\varepsilon-1}} \delta^\varepsilon \\
 &= \mathcal{A} \delta^\varepsilon,
 \end{aligned}$$

where $\mathcal{A} = \mathcal{C}_0 \zeta \mathcal{K} \frac{2^\varepsilon}{2^{\varepsilon-1}} < \infty$.

We therefore deduce that

$$\overline{\mathcal{P}}_{v,\delta}^{q,\gamma_t(q)+\varepsilon}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) \leq \mathcal{A} \delta^\varepsilon,$$

and

$$0 \leq \mathcal{H}_v^{q,\gamma_t(q)+\varepsilon}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) \leq \mathcal{P}_v^{q,\gamma_t(q)+\varepsilon}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) = 0.$$

In addition, for all $q \in \mathbb{R}_+^k$, we have

$$\begin{aligned}
 \mathcal{H}_v^{q,\gamma_t(q)+\varepsilon}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) &\leq H_v^{q,\gamma_t(q)+\varepsilon}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) \\
 &\leq P_v^{q,\gamma_t(q)+\varepsilon}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})) \\
 &\leq \mathcal{P}_v^{q,\gamma_t(q)+\varepsilon}(\mathcal{S}_n(\mathcal{D}) \cap \mathcal{S}_m(\mathcal{D})).
 \end{aligned}$$

This achieves the proof of Theorem 3.

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