Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 7, pp. 2613–2621
DOI : 10.47974/JIM-2194

FS-convergent in norm (Fsc_n^1–Convergent) in a fuzzy normed Riesz spaces

Shatha Abdul-Hussein Kadhim *
Shaimaa Abdul-Hussein Kadhim †
Mohammed Yahya Abed §
Department of Mathematics
Collage of Education for Pure Sciences
University of Kerbala
Kerbala 56001
Iraq

Abstract

In this paper, we introduce the concept of fuzzy statistical convergent in norm (Fsc_n^1-Convergent) and (Fsc_n^1-Cauchy) for sequence in fuzzy normed Riesz spaces (FNRS) and study some outcomes related to these notions and we have established a number of its properties. also we see that a sequence be a FS-convergent in norm if and only if there exist a subsequence convergent to x_0. So we proved that a fuzzy σ-complete Riesz space $\mathfrak{S}$ with the Fsc_n^1-Norm is order continuous it's equivalent to every increasing with bounded above is Fsc_n^1-convergent. And note every Fsc_n^1-Cauchy sequence is Convergent to an element $T_0 \in \mathfrak{S}$. Thus, we verified that $\mathfrak{S}$ is a (BFRS).

Subject Classification (2010): *34K28.*

Keywords: *Banach fuzzy riesz space (BFRS), Fuzzy statistical convergent in norm (Fsc_n^1-Convergent), Fsc_n^1-Cauchy.*

* *ORCID-ID:* **0000-0002-6249-2819**
† *ORCID-ID:* **0000-0001-8440-9134**
§ *ORCID-ID:* **0000-0003-4345-1146**

* *E-mail:* **shatha.abdulhussein@uokerbala.edu.iq (Corresponding Author)**
† *E-mail:* **shaimaa.a@uokerbala.edu.iq**
§ *E-mail:* **mohammed.y@uokerbala.edu.iq**

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1. Introduction

In 1951 through the works of Fast [1], Steinhaus [2], and Schoenberg [3]. Since then, the idea has been adapted to several settings-such as locally convex spaces [4], probabilistic normed spaces [5,6], intuitionistic fuzzy normed spaces (including the developments of Albayrak and Pehlivan) [7,8], and locally solid Riesz spaces (LSRS) [9]. A Riesz space, introduced by Riesz in 1928 [10], is a lattice-ordered vector space and underpins applications in measure theory, operator theory, optimization, and economics [11]. Its fuzzy counterpart-often called a fuzzy Riesz space-has been analyzed and characterized in a number of studies [12,15-22]. In this paper, we propose the notion of FS-convergence in norm within F-Riesz spaces, establish foundational properties, and derive several related results. We also show that a sequence is FS-convergent in norm if and only if it possesses a subsequence that converges in norm to the same limit. T_0. So we proved that a F-σ-complete Riesz space $\mathfrak{S}$ with the Fsc_n^1-Norm is order continuous it's equivalent to every increasing with bounded above is Fsc_n^1-convergent. And note everyFsc_n^1-Cauchy sequence isFsc_n^1-Convergent to an element $T_0 \in \mathfrak{S}$. Thus, we verified that $\mathfrak{S}$ is a BFRS.

Definition 1 [12]: An F-order on a crisp set $\mathfrak{S}$is a fuzzy $\sqsubseteq$ of $\mathfrak{S} \times \mathfrak{S}$ With the following properties achieved.

(i) $\forall \ \ x\epsilon\mathfrak{S}, \dot{Q}(T,T) = 1$ reflexivity,

(ii) $\forall\, T, \acute{W}\epsilon\mathfrak{S}, \dot{Q}(T,\acute{W}) + \dot{Q}(\acute{W},T) > 1 \ implies, T = \acute{W}$ anti-symmetry,

(iii) $\forall\, T, \acute{W}, z\epsilon\mathfrak{S}, \dot{Q}(T,\acute{W}) \geq \bigvee_{\acute{W}\epsilon X}[\dot{Q}(T,\acute{W}) \wedge \dot{Q}(\acute{W},z)]$,where $\dot{Q}: \mathfrak{S} \times \mathfrak{S} \to [0,1]$, is a function of the fuzzy $\sqsubseteq$ of $\mathfrak{S} \times \mathfrak{S}$. A set with a specific F- order is called a fuzzy ordered set(fos).

Proposition 1 **[12]:** If $\mathcal{H} \subseteq \mathfrak{S}$.

(i) $sup\mathcal{H}$, if it$\exists sup\mathcal{H}$, it is unique.

(ii) $inf\mathcal{H}$, if it $\exists\ inf\mathcal{H}$, it is unique.

Notation 1 [11]: $T \vee \acute{W} = sup\{T, \acute{T}\}\ and\ T \wedge \acute{W} = inf\{T, \acute{W}\}$.

Definition 2 [13]: An(real)linear space $\mathfrak{S}$ is said to be an FOLS if$\mathfrak{S}$ meets the following terms.

(i) If $T, \acute{W}\epsilon\,\mathfrak{S}$ STthe $\dot{Q}(T,\acute{W}) > \frac{1}{2}$ then $\dot{Q}(T,\acute{W}) \leq \dot{Q}(T+\acute{T}, \acute{W}+\acute{T}), for\ all \acute{T}\epsilon\,\mathfrak{S}$,

(ii) If T , $\acute{W}\epsilon\,\mathfrak{S}$ STthe $\dot{Q}(T,\acute{W}) > \frac{1}{2}$ then $\dot{Q}(T,\acute{W}) \leq \dot{Q}(\alpha T, \alpha\acute{W}), for\ all\ 0 \leq \alpha\epsilon\,\mathbb{R}$,

Definition 3 [14]: A positive part of element $T\epsilon\,\mathfrak{S}$ is $T_+ = T\vee 0$, the negative part$T_- = (-T)\vee 0$ and the absolute of T is $|T| = T \vee -T$.

Theorem 1 [14]: *In a FRS, the following properties:*

(i) $\dot{Q}\left(|T+\acute{W}|,|T|+|\acute{W}|\right)>\frac{1}{2}$,

(ii) $|\alpha T|=|\alpha||T|$,

(iii) $\dot{Q}\left(\left||T|-|\acute{W}|\right|,|T-\acute{W}|\right)>\frac{1}{2}$,

(iv) $|T-\acute{W}|=(T\vee\acute{W})-(T\wedge\acute{W})$.

2. Main Result

This section establishes fundamental theorems on the behavior of fuzzy statistical convergence in norm within fuzzy normed Riesz spaces (FNRS). In addition, we demonstrate that a fuzzy Riesz space qualifies as a BFRS.

Definition 5 [16]: Let $\mathfrak{S}$ be a real FRS and $\|.\|:\mathfrak{S}\to\mathfrak{S}$. Then the F-Riesz norm satisfying the following condition for every $T,W\in\mathbb{R},\dot{Q}(|T|,|W|)>\frac{1}{2}$ implies $\dot{Q}(\|T\|,\|W\|)>\frac{1}{2}$. Binary $(\mathfrak{S},\|\cdot\|)$ is called a FN-Riesz space(for short, FNRS). A NRS which is also a B-space its termed a BF-Riesz space BFRS.

Definition 6: Let $(\mathfrak{S},\|.\|)$ be a FNRS. A sequence $(T_j)_{j\in\mathbb{N}}$ in$\mathfrak{S}$ is said to be FS-convergent in norm to $T_0\in\mathfrak{S}$ (in short Fsc_n^1-convergent). Provided that each $\in>0$, the set $\left\{j\in\mathbb{N}:\dot{Q}\left(\epsilon,\left\|T_{\mathrm{j}}-T_0\right\|\right)>\frac{1}{2}\right\}$, i.e., for each$\in>0$, $\lim_{n\to\infty}\frac{1}{n}\left|\left\{j\in\mathbb{N}:\dot{Q}\left(\epsilon,\left\|T_{\mathrm{j}}-T_0\right\|\right)>\frac{1}{2}\right\}\right|=0$,has natural density zero.And thus we will get $x_j\xrightarrow{Fsc_n^1}x_0$.

Definition 7: The sequence $(T_j)_{j\in\mathbb{N}}$ of elements of FRS $\mathfrak{S}$ is said to fuzzy statistically Cauchy (in short, Fsc_n^1-Cauchy) if for every $\in>0$, there exist positive integer$\mathfrak{N}$ so that each j, k $\geq\mathfrak{N}$ the set $\left\{\mathrm{j},k\in\mathbb{N}:\dot{Q}\left(\epsilon,\left\|T_{\mathrm{j}}-T_k\right\|\right)>\frac{1}{2}\right\}$ has natural density zero.

Definition 8 **[13]:**

(i) A for $\mathfrak{S}$ is called F-lattice(F-lattice) if $\forall$ the finite F $\subseteq\mathfrak{S}$ have supremums and infimums.

(ii) A F-lattice $\mathfrak{S}$ is said that complete if $\forall F\subseteq$ of $\mathfrak{S}$ has sup and inf.

(iii) An F-lattice $\mathfrak{S}$ is said that σ-complete if each countable subset has sup and inf.

Theorem 2: *Let $\mathfrak{S}$ be a FNRS, $(T_{\mathrm{j}})_{j\in\mathbb{N}}$ and $(y_{\mathrm{j}})_{j\in\mathbb{N}}$ be two sequences of points in$\mathfrak{S}$. Then*

(i) If $x_j\xrightarrow{Fsc_n^1}x_0$ and$W_j\xrightarrow{Fsc_n^1}W_0$, then $T_j\vee W_j\xrightarrow{Fsc_n^1}T_0\vee W_0$ and $T_j\wedge W_j\xrightarrow{Fsc_n^1}T_0\wedge W_0$.

(ii) If $x_j \xrightarrow{Fsc_n^1} x_0$, then $x_j^+ \xrightarrow{Fsc_n^1} x_0^+$, $x_j^- \xrightarrow{Fsc_n^1} x_0^-$ and $|x_j| \xrightarrow{Fsc_n^1} |x_0|$.

(iii) If $x_j \xrightarrow{Fsc_n^1} x_0$, $W_j \xrightarrow{Fsc_n^1} W_0$ and $T_j \perp W_j$ then $T_0 \perp W_0$.

(iv) If $x_j \xrightarrow{Fsc_n^1} x_0$ and $T_j \xrightarrow{Fsc_n^1} W_0$, then $T_0 = W_0$.

Proof:

(i) Suppose that $x_j \xrightarrow{Fsc_n^1} x_0$ and $W_j \xrightarrow{Fsc_n^1} W_0$. Then for every $\in > 0$ the sets, $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j - T_0\|\right) > \frac{1}{2}\right\}$ and $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|W_j - W_0\|\right) > \frac{1}{2}\right\}$ have natural density zero (respectively). we have for every $\in > 0$ the set $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j - T_0\| \vee \|W_j - W\|\right) > \frac{1}{2}\right\}$. Accordingly, we can write for every $\in > 0$, $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j \vee W_j - T_0 \vee W_0\|\right) > \frac{1}{2}\right\} \subset \left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j - T_0\| \vee \|W_j - W_0\|\right) > \frac{1}{2}\right\}$, since the set on the right-hand has natural density zero. Therefore the set $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j \vee W_j - T_0 \vee W_0\|\right) > \frac{1}{2}\right\}$ has natural density zero. Hence we have $T_j \vee W_j \xrightarrow{Fsc_n^1} T_0 \vee W_0$.

Similarly, $T_j \wedge W_j \xrightarrow{Fsc_n^1} T_0 \wedge W_0$.

(ii) Assume that $x_j \xrightarrow{Fsc_n^1} x_0$, then for every $\in > 0$ the set $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j - T_0\|\right) > \frac{1}{2}\right\}$ has natural density zero. To prove the first statement, $x_j^+ \xrightarrow{Fsc_n^1} x_0^+$. Since it permitted that the set $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j^+ - T_0^+\|\right) > \frac{1}{2}\right\}$ be a subset of $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j - T_0\|\right) > \frac{1}{2}\right\}$, for every $\in > 0$. From this we have that $\delta\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j^+ - T_0^+\|\right) > \frac{1}{2}\right\} = 0$. Hence $x_j^+ \xrightarrow{Fsc_n^1} x_0^+$.

Similarly, we can obtained assertion $x_j^- \xrightarrow{Fsc_n^1} x_0^-$. On the other hand, to prove the assertion $|x_j| \xrightarrow{Fsc_n^1} |x_0|$. Now from the relation $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|\,|T_j| - |T_0|\,\|\right) > \frac{1}{2}\right\} \subset \left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|T_j - T_0\|\right) > \frac{1}{2}\right\}$ where $\in > 0$, implies the set $\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \|\,|T_j| - |T_0|\,\|\right) > \frac{1}{2}\right\}$ has natural density zero. Hence the desired result follows now.

(iii) Let $x_j \xrightarrow{Fsc_n^1} x_0$, $W_j \xrightarrow{Fsc_n^1} W_0$ and $x_j \perp W_j$. Then by using (ii), we have that $|x_j| \xrightarrow{Fsc_n^1} |x_0|$ and $|W_j| \xrightarrow{Fsc_n^1} |W_0|$ respectively. Also by using the second part of hypothesis (i), we have $|x_j| \wedge |W_j| \xrightarrow{Fsc_n^1} |x_0| \wedge |W_0|$. Since $|T_j| \wedge |W_j| = 0$. Then, we get $x_0 \perp W_0$.

(iv) Let $x_j \xrightarrow{Fsc_n^1} x_0$, $T_{\mathrm{j}} \xrightarrow{Fsc_n^1} W_0$. Then for every $\in > 0$, we have the sets $\left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{\mathrm{j}} - T_0\right\|\right) > \frac{1}{2}\right\}$ and $\left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{\mathrm{j}} - W_0\right\|\right) > \frac{1}{2}\right\}$. Then we can write $\left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(0, \left\|T_{j\mathrm{j}} - T_0\right\| - \left\|T_{\mathrm{j}} - W_0\right\|\right) > \frac{1}{2}\right\}$. Also we know $\left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\left|\left\|T_{\mathrm{j}} - T_0\right\| - \left\|T_{\mathrm{j}} - W_0\right\|\right|, \left\|T_0 - T_j + T_j - W_0\right\|\right) > \frac{1}{2}\right\}$. It follows that $\dot{Q}(0, \|T_0 - W_0\|) > \frac{1}{2}$. Hence $T_0 = W_0$.

Theorem 3: *Assume that* $\mathfrak{S}$ *be a FNRS,* $(T_{\mathrm{j}})_{j\in\mathbb{N}}$, $(W_j)_{\mathrm{j}\in\mathbb{N}}$ *and* $(z_j)_{\mathrm{j}\in\mathbb{N}}$ *be three sequences of points in* $\mathfrak{S}$ $ST\dot{Q}\left(T_{\mathrm{j}}, W_j\right) > \frac{1}{2}, \dot{Q}\left(W_{\mathrm{j}}, z_j\right) > \frac{1}{2}\ \forall\ \mathrm{j} \in \mathbb{N}\ \ x_j \xrightarrow{Fsc_n^1} x_0$ and $\mathrm{z}_j \xrightarrow{Fsc_n^1} x_0$ then $W_{\mathrm{j}} \xrightarrow{Fsc_n^1} T_0$.

Proof: Since $x_j \xrightarrow{Fsc_n^1} x_0$ and $\mathrm{z}_j \xrightarrow{Fsc_n^1} x_0$, then $\forall\ \mathrm{y} \in > 0$, we have the sets $A = \left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\left\|T_{\mathrm{j}} - T_0\right\|, \frac{\epsilon}{2}\right) > \frac{1}{2}\right\}$ and $B = \left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\left\|z_{\mathrm{j}} - T_0\right\|, \frac{\epsilon}{2}\right) > \frac{1}{2}\right\}$ has natural density 1. Consequently, $A + B = \left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\left\|T_{\mathrm{j}} - T_0\right\| + \left\|z_{\mathrm{j}} - T_0\right\|, \frac{\epsilon}{2}\right) > \frac{1}{2}\right\}$. Also from condition (i) of the Definition 2 it follows that if $\dot{Q}\left(T_{\mathrm{j}}, W_j\right) > \frac{1}{2} and\ \dot{Q}\left(W_{\mathrm{j}}, z_j\right) > \frac{1}{2}$ then we have $\dot{Q}(T_j - T_0, W_j - T_0) > \frac{1}{2}$ and $\dot{Q}(W_j - T_0, z_j - T_0) > \frac{1}{2}$. Thereby we can write the set $\left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\left\|W_{\mathrm{j}} - T_0\right\|, \left\|T_j - T_0\right\| + \left\|z_{\mathrm{j}} - T_0\right\|\right) > \frac{1}{2}\right\}$. From this set and the set $A + B$ implies that for every $\in > 0$, the set $\left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\left\|W_j - T_0\right\|, \epsilon\right) > \frac{1}{2}\right\}$ has natural density 1. Thus $W_{\mathrm{j}} \xrightarrow{Fsc_n^1} T_0$.

Theorem 5: *Assume that* $\mathfrak{S}$ *be a FNRS. A sequence* $(T_j)_{\mathrm{j}\in\mathbb{N}}$ *be a FS-convergent in norm if and only if the following condition is satisfied:*

For every $\in > 0$, the set $\left\{j, i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{\mathrm{j}} - T_{\mathrm{j_i}}\right\|\right) > \frac{1}{2}\right\}$ has natural density zero,where $(T_{j_{\mathrm{i}}})_{\mathrm{i}\in\mathbb{N}}$ is a subsequence of $(T_j)_{j\in\mathbb{N}}$ such that $\lim_{i\to\infty} x_{j_i} = x_0$, i.e., $\left\|T_{j_{\mathrm{i}}} - T_0\right\| = 0, \mathrm{i} \to \infty$.

Proof: Assume that $(T_j)_{j\in\mathbb{N}}$ be a FS-convergent in norm. To prove for every $\epsilon > 0$, the set $\left\{j, i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_{\mathrm{j_i}}\right\|\right) > \frac{1}{2}\right\}$ has natural density zero. Now by using Definition 6 of a sequence $(T_j)_{j\in\mathbb{N}}$ to an element $\mathcal{X}_0$, we have that for every $\in > 0$, the set

$$\left\{\mathrm{j} \in \mathbb{N}: \dot{\mu}\left(\epsilon, \left\|T_{\mathrm{j}} - T_0\right\|\right) > \frac{1}{2}\right\} \tag{2}$$

has natural density zero. Thereby, we can write

$$\left\{j, i \in \mathbb{N}: \dot{Q}\left(2\epsilon, \left\|T_j - T_{j_i}\right\|\right) > \frac{1}{2}\right\} = \left\{j_i \in \mathbb{N}: \dot{Q}\left(2\epsilon, \left\|T_j - T_{j_i} + T_0 - T_0\right\|\right) > \frac{1}{2}\right\}$$
$$= \left\{j_{,i} \in \mathbb{N}: \dot{Q}\left(2\epsilon, \left\|\left(T_j - T_0\right) + \left(T_0 - T_{j_i}\right)\right\|\right) > \frac{1}{2}\right\}$$
$$\leq \left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_0\right\|\right) > \frac{1}{2}\right\} + \left\{i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_0 - T_{j_i}\right\|\right) > \frac{1}{2}\right\}$$
$$\leq max\left[\left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_0\right\|\right) > \frac{1}{2}\right\}, \left\{i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_0 - T_{j_i}\right\|\right) > \frac{1}{2}\right\}\right]$$

by using(2)we have,

$$\leq \max\left[0, \left\{i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{j_i} - T_0\right\|\right) > \frac{1}{2}\right\}\right] \quad (3)$$

From this we have that $\lim_{i\to\infty} x_{j_i} = x_0$. Now $(T_{j_i})_{i\in\mathbb{N}}$ is convergent, also can be show that $(T_{j_i})_{i\in\mathbb{N}}$ is FS-convergent in norm. It further implies that for every $\in > 0$, the set

$$\left\{i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{j_i} - T_0\right\|\right) > \frac{1}{2}\right\} \quad (4)$$

has natural density zero. Then from (3) and (4) we have for every $\in > 0$, the set $\left\{j, i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_{j_i}\right\|\right) > \frac{1}{2}\right\}$ has natural density zero.

Conversely, suppose for every $\in > 0$, the set

$$\left\{j, i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_{j_i}\right\|\right) > \frac{1}{2}\right\} \quad (5)$$

has natural density zero.

Now to prove that the sequence $(T_{j_i})_{i\in\mathbb{N}}$ be a FS-convergent in norm. Let us consider $\in > 0$, the set

$$\left\{j, i \in \mathbb{N}: \dot{Q}\left(2\epsilon, \left\|T_j - T_0\right\|\right) > \frac{1}{2}\right\} = \left\{j, i \in \mathbb{N}: \dot{Q}\left(2\epsilon, \left\|T_j - T_{j_i} + T_{j_i} - T_0\right\|\right) > \frac{1}{2}\right\}$$
$$\leq \left\{j, i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_{j_i}\right\|\right) > \frac{1}{2}\right\} + \left\{i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{j_i} - T_0\right\|\right) > \frac{1}{2}\right\}$$
$$\leq max\left[\left\{j, i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_{j_i}\right\|\right) > \frac{1}{2}\right\}, \left\{i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{j_i} - T_0\right\|\right) > \frac{1}{2}\right\}\right]$$

by using(5)we have,

$\leq max\left[0, \left\{i \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{j_i} - T_0\right\|\right) > \frac{1}{2}\right\}\right]$ by using implies for every$\in > 0$, the set $\left\{j \in \mathbb{N}: \dot{Q}\left(2\epsilon, \left\|T_j - T_0\right\|\right) > \frac{1}{2}\right\}$ has natural density zero. Hence$(T_{j_i})_{i\in\mathbb{N}}$ is FS-convergent in norm.

Theorem 8: *On a BFRS* $\mathfrak{S}$ *if every*Fsc_n^1*-Cauchy sequence is*Fsc_n^1*-Convergent to an element* $T_0 \in \mathfrak{S}$.

Proof: Suppose a sequence $(T_j)_{j\in\mathbb{N}}$ be Fsc_n^1-Cauchy. Now to show that $(T_j)_{j\in\mathbb{N}}$ be Fsc_n^1-Convergent. Assume that$(T_j)_{j\in\mathbb{N}}$ is not convergent to an element $x_0 \in \mathfrak{S}$. Then there exist a positive integer $\mathfrak{P}$ STfor every$j, \mathcal{K} \geq \mathfrak{P}$, the set $\left\{j, \mathcal{K} \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_{\mathcal{K}}\right\|\right) > \frac{1}{2}\right\}$ for every $\in > 0$, has natural density zero. Also

we can write for every $\in > 0$ the set $\left\{j, \mathcal{K} \in \mathbb{N}: \dot{Q}\left(\left\|T_j - T_{\mathcal{K}}\right\|, \epsilon\right) > \frac{1}{2}\right\}$ has natural density 1. In particular, we can write $\dot{Q}(\left\|T_j - T_{\mathcal{K}}\right\|, 2\left\|T_{\mathrm{j}} - T_0\right\|) > \frac{1}{2}$ and

If $\dot{Q}(\left\|T_j - T_0\right\|, \frac{\epsilon}{2}) > \frac{1}{2}$. But $(T_j)_{j\in\mathbb{N}}$ is not convergent, means that the set $\left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{\mathrm{j}} - T_0\right\|\right) > \frac{1}{2}\right\}$ for every $\in > 0$, has natural density 1, i.e., the set $\left\{j \in \mathbb{N}: \dot{Q}\left(\left\|T_{\mathrm{j}} - T_0\right\|, \epsilon\right) > \frac{1}{2}\right\}$ for every$\in > 0$, has natural density zero. By using (6), we see that the set $\left\{j, \mathcal{K} \in \mathbb{N}: \dot{Q}\left(\left\|T_j - T_{\mathcal{K}}\right\|, \epsilon\right) > \frac{1}{2}\right\}$ has natural density zero, i.e., the set $\left\{j, \mathcal{K} \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_{\mathrm{j}} - T_{\mathcal{K}}\right\|\right) > \frac{1}{2}\right\}$ has natural density 1. This contradicts our assumption and therefore $(T)_{j\in\mathbb{N}}$ isFsc_n^1-convergent. Thus, we have that $\mathfrak{S}$ is BFRS.

Definition 9: A F-norm $\|\cdot\|$ in $\mathfrak{S}$ is called F-order continuous if for any$(T_{\mathrm{j}})_{\mathrm{j}\in\mathbb{N}}$ of element in $\mathfrak{S}$ with $x_j \downarrow_{Fst} 0$ implies that$\| x_j \| \xrightarrow{Fsc_n^1} 0$.

Theorem 9: *A real sequence* $(T_j)_{j\in\mathbb{N}}$ *is FS-convergent in norm to an element*$T_0 \in \mathfrak{S}$ *if and only if there exists a subset* $F = \{\mathrm{j}_1 < \mathrm{j}_2 < \cdots\} \subset \mathbb{N}$ $ST\delta(F) = 1$ *and* $\lim_{\substack{j\to\infty \\ j\in F}} T_j = T_0$.

Proof: Let $(T_j)_{j\in\mathbb{N}}$ be a FS-convergent in norm to T_0. Put

$F_l = \left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\frac{1}{l}, \left\|T_j - T_0\right\|\right) > \frac{1}{2}\right\}$ and $H_l = \left\{\mathrm{j} \in \mathbb{N}: \dot{Q}\left(\frac{1}{l}, \left\|T_j - T_0\right\|\right) > \frac{1}{2}\right\}$, $(l = 1, 2, 3, \ldots.)$. Then $\delta(F_l) = 0$,

$$H_1 \supset H_2 \supset \cdots \supset H_j \supset H_{j+1} \supset \cdots \tag{7}$$

and
$$\delta(H_l) = 1, l = 1, 2, 3, \ldots. \tag{8}$$

Now we have to show that for $j \in H_l$ that $(T_j)_{j\in\mathbb{N}}$ is convergent toT_0. Suppose that $(T_j)_{j\in\mathbb{N}}$ is not convergent to T_0. Therefore there is $\in > 0$ such that$\dot{Q}\left(\epsilon, \left\|T_j - T_0\right\|\right) > \frac{1}{2}$ for infinitely many terms. Let $H_\epsilon = \left\{j \in \mathbb{N}: \dot{Q}\left(\left\|T_j - T_0\right\|, \epsilon\right) > \frac{1}{2}\right\}$ and $\dot{Q}\left(\frac{1}{l}, \epsilon\right) > \frac{1}{2}, (l = 1, 2, 3, \ldots.)$. Then $\delta(H_\epsilon) = 0$ and by (7), $H_l \supset H_\epsilon$. Hence $\delta(H_l) = 0$ which contradicts (8). Therefore $(T_j)_{j\in\mathbb{N}}$ is convergent to T_0.

Conversely, suppose that exists a subset $F = \{j_1 < j_2 < \cdots\} \subset \mathbb{N}$ ST$\delta(F) = 0$ and $\lim_{\substack{j\to\infty \\ j\in F}} T_j = T_0$, i.e., there is $\mathfrak{P} \in \mathbb{N}$ ST $\forall \in > 0$, $\dot{Q}\left(\left\|T_j - T_0\right\|, \epsilon\right) > \frac{1}{2}, \forall j \geq \mathfrak{P}$. Now $F_\epsilon = \left\{j \in \mathbb{N}: \dot{Q}\left(\epsilon, \left\|T_j - T_0\right\|\right) > \frac{1}{2}\right\} \subseteq \mathbb{N} - \left\{j_{\mathfrak{P}+1} < j_{\mathfrak{P}+2} < \cdots\right\}$ therefore $\dot{Q}(\delta(F_\epsilon), 0) > \frac{1}{2}$. Hence $(T_j)_{j\in\mathbb{N}}$ is FS- convergent in norm to T_0.

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Received November, 2024