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Exact and numerical solutions for two nonlinear regime switching models

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Abstract

This paper investigates two nonlinear Black-Scholes models, which are fundamental in options pricing under conditions where market volatility is non-constant. To address the inherent complexity in these models, a regime-switching approach is employed, decomposing each model into a system of differential equations, with each equation representing a distinct market regime. Leveraging Lie group algebra, we derive exact analytical solutions for the

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equations within this system, providing critical insights into the model dynamics under various regimes. Numerical solutions are obtained using the Chebyshev spectral method, a high-precision approach known for its efficiency in solving differential equations to complement the analytical results.

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1. Introduction

In general, the calculation of option's prices turned into one of the foundational and cornerstone branches of financial mathematics that offers crucial instruments necessary for managing and hedging various risks linked with fluctuating markets. As for these instruments, contingent claims, especially options, can be considered as effective tools for taking out the risk cover off the investments and making them more accessible to those investors who would otherwise be unable to take the risk. The most basic known Black-Scholes model gotten in the 1970s tops every one of these as they provided a systematic approach in an organized approach that is still a popular model widely used for pricing European options in various financial settings like France and Australia. However, the generic Black-Scholes model is flawed due to numerous market assumptions that include free and fast trading and stead state price volatility.

Market conditions often deviate significantly from the assumptions underlying classical models, leading to their failure in scenarios involving stochastic volatility or transaction costs [6]. To address these challenges, numerous extensions of the Black-Scholes model have been proposed. For example, the double Heston model and its variant incorporating jumps have been utilized to study power option pricing under stochastic volatility [5]. Other modifications introduce nonlinearities to better reflect market realities; for instance, [7] derives analytical solutions and Greek parameters for nonlinear Black-Scholes systems, while [8] presents particular solutions for reduced nonlinear models.

A major limitation of the traditional Black-Scholes model is its assumption of a frictionless market—namely, the absence of transaction costs and the constancy of market dynamics. However, real markets with transaction costs exhibit time-varying drift and volatility parameters, which align more closely with the observed variability in option price volatilities. To address these discrepancies, this study replaces the classical Black-Scholes model with a nonlinear framework that accounts for transaction costs. By incorporating advanced methods from stochastic analysis and Lie group theory, we aim to rigorously analyze these models and derive solutions within the framework of nonlinear Black-Scholes equations for European options.

Lie group analysis has seen growing application in financial modeling, offering powerful techniques for simplifying and solving complex differential

equations that arise in options pricing models. This approach has been applied in numerous financial studies, as documented in [2, 3, 4, 12, 11, 13]. For instance, in [4], the fast Fourier transform is applied to model power option pricing in the global gold market under a fractional Heston framework, with profit and loss calculations for discrete intervals over the study period.

In this paper, we focus on the Black-Scholes model which is a model that includes two assets; risk-free asset and risky asset which respectively have the following dynamics:

$$dB_t = rB_t dt, B(0) = 1, \tag{1}$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2}$$

where σ is volatility of risky underlying assets S, r interest rate , W_t a standard Brownian motion and μ is drift. The transaction price at time t,

$$S_t(\alpha) = e^{\rho\alpha} S_t^0, \tag{3}$$

when the liquidity parameter $0 \leq \rho \leq 1$, shares valued at α are exchanged [7].

A parameterized model for the liquidity impacts of asset trading was created by Bakstein and Howison. They explored potential model extensions, such as stochastic liquidity, and conducted an empirical test of their model setup using high-frequency data of German blue chips [1].

$$\frac{\partial U_1}{\partial t} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 U_1}{\partial S^2}\left(1 + 2\rho S \frac{\partial^2 U_1}{\partial S^2}\right) + rS\frac{\partial U_1}{\partial S} - rU_1 = q_{12}(U_1 - U_2), \tag{4}$$

$$\frac{\partial U_2}{\partial t} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 U_2}{\partial S^2}\left(1 + 2\rho S \frac{\partial^2 U_2}{\partial S^2}\right) + rS\frac{\partial U_2}{\partial S} - rU_2 = q_{21}(U_2 - U_1),$$

σ represents volatility, $\rho \geq 0$, represents the liquidity parameter, and for $r = 0$,

$$\frac{\partial U_1}{\partial t} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 U_1}{\partial S^2}\left(1 + 2\rho S \frac{\partial^2 U_1}{\partial S^2}\right) = q_{12}(U_1 - U_2), \tag{5}$$

$$\frac{\partial U_2}{\partial t} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 U_2}{\partial S^2}\left(1 + 2\rho S \frac{\partial^2 U_2}{\partial S^2}\right) = q_{21}(U_2 - U_1).$$

The above formula is known as the Cetin et al. model.

This study assumes two nonlinear Black-Scholes models with two assets: a bond, which is assumed to be a risk-free and liquid asset, and an illiquid stock. Analytical solutions for the models under consideration are offered in this inquiry using instrumentals from stochastic analysis and Lie group analysis. This study examines two nonlinear Black-Scholes models involving two assets: a bond, assumed to be a risk-free, liquid asset, and an illiquid stock. Analytical solutions for these models are provided using tools from stochastic analysis and Lie group analysis.

In this study, the numerical solutions of equations (4) and (5) are obtained using the Chebyshev Spectral Collocation (CSC) method. The method employs Chebyshev polynomials to approximate the equations, and the convergence of these approximations to the exact solutions is rigorously analyzed and demonstrated.

This study consists of four sections. Section 2 presents the fundamental concepts of Lie symmetries. The reduction method is applied to yield symmetry

operators and to construct exact solutions. Section 3 uses the CSC technique to obtain numerical solutions for both equations, (4) and (5).

2. Lie symmetry analysis

Lie symmetry, also known as Lie group analysis, is a powerful method for determining exact solutions to systems of differential equations. A symmetry group of a differential equation is characterized as a set of transformations that preserve the solution space of the equation by mapping one solution to another [9]. Consider a system of differential equations where the dependent variables are (u_1, u_2) and the independent variables are (S, t). For the considered equations, the infinitesimal Lie transformations are

$$\begin{aligned} x &\mapsto S + \varepsilon\xi_1(S,t,u_1,u_2) + \mathcal{O}(\varepsilon^2),\\ t &\mapsto t + \varepsilon\xi_2(S,t,u_1,u_2) + \mathcal{O}(\varepsilon^2),\\ u_1 &\mapsto u_1 + \varepsilon\phi_1(S,t,u_1,u_2) + \mathcal{O}(\varepsilon^2),\\ u_2 &\mapsto u_2 + \varepsilon\phi_2(S,t,u_1,u_2) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

where $\varepsilon \ll 1$. The vector field associated with the aforementioned transformation group can be represented as

$$X = \xi_1(S,t,u_1,u_2)\frac{\partial}{\partial S} + \xi_2(S,t,u_1,u_2)\frac{\partial}{\partial t} + \phi_1(S,t,u_1,u_2)\frac{\partial}{\partial u_1} + \phi_2(S,t,u_1,u_2)\frac{\partial}{\partial u_2}.$$

For the order of equations (4) and (5) we apply the second prolongation of operator

$$X^{(2)} = X + \phi^S\frac{\partial}{\partial u_S} + \phi^t\frac{\partial}{\partial u_t} + \phi^{SS}\frac{\partial}{\partial u_{SS}} + \phi^{St}\frac{\partial}{\partial u_{St}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}, \tag{6}$$

coefficients of prolongation expressed by

$$\begin{aligned} \phi^S &= D_S(\phi - \xi_1 u_S - \xi_2 u_t) + \xi_1 uSS + \xi_2 u_{St},\\ \phi^t &= D_t(\phi - \xi_1 u_x - \xi_2 u_t) + \xi_1 uSt + \xi_2 u_{tt},\\ \phi^{SS} &= D_S(D_S(\phi - \xi_1 u_S - \xi_2 u_t)) + \xi_1 uSSS + \xi_2 u_{SSt},\\ \phi^{St} &= D_S(D_t(\phi - \xi_1 u_S - \xi_2 u_t)) + \xi_1 uSSt + \xi_2 u_{Stt},\\ \phi^{tt} &= D_t(D_t(\phi - \xi_1 u_S - \xi_2 u_t)) + \xi_1 uStt + \xi_2 u_{ttt}. \end{aligned}$$

The operators D_S and D_t denote the total derivatives with respect to S and t, respectively [9]. Applying the standard Lie symmetry technique, it is established that equation (4) admits a six-dimensional Lie group, with its symmetries defined by the following infinitesimal generators

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, X_2 = S\frac{\partial}{\partial u_1} + S\frac{\partial}{\partial u_2},\\ &X_3 = S\frac{\partial}{\partial S} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}, X_4 = e^{rt}\frac{\partial}{\partial u_1} + e^{rt}\frac{\partial}{\partial u_2},\\ &X_5 = \exp(t(q_{12}+q_{21}))S\frac{\partial}{\partial u_1} - \frac{Sq_{21}\exp(t(q_{12}+q_{21}))}{q_{12}}\frac{\partial}{\partial u_2},\\ &X_6 = \exp(t(q_{12}+q_{21}+r))\frac{\partial}{\partial u_1} - \frac{q_{21}\exp(t(q_{12}+q_{21}+r))}{q_{12}}\frac{\partial}{\partial u_2}. \end{aligned}$$

Similarly, for equations (5)

$$X_1^* = \frac{\partial}{\partial t}, X_2^* = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}, X_3^* = S\frac{\partial}{\partial u_1} + S\frac{\partial}{\partial u_2}, X_4^* = S\frac{\partial}{\partial S} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2},$$
$$X_5^* = \exp(t(q_{12}+q_{21}))S\frac{\partial}{\partial u_1} - \frac{Sq_{21}\exp(t(q_{12}+q_{21}))}{q_{12}}\frac{\partial}{\partial u_2},$$
$$X_6^* = \exp(t(q_{12}+q_{21}))\frac{\partial}{\partial u_1} - \frac{q_{21}\exp(t(q_{12}+q_{21}))}{q_{12}}\frac{\partial}{\partial u_2},$$

is the foundation for generators of symmetry.

2.1 *Reduced equations for eq.* (4)

This section provides reduced form and exact solutions for the eq. (4) using the symmetries' retrieved invariants.

1. Integrating the group of trajectories, the classical similarity solution of (4) for generator X_1 is achieved.

$$\frac{dS}{d\varepsilon} = 0, \frac{dt}{d\varepsilon} = 1, \frac{du_1}{d\varepsilon} = 0, \frac{du_2}{d\varepsilon} = 0,$$

for each trajectory, ε serves as a parameter. The invariant transformations obtained from the above integrations, $S = y$, $u_1 = f(y)$ and $u_2 = g(y)$ are substituted into (4).

$$\frac{1}{2}\sigma_1^2 y^2 f''(1+2\rho y f'') + ryf' - rf_1 = q_{12}(f-g), \tag{7}$$
$$\frac{1}{2}\sigma_2^2 y^2 g''(1+2\rho y g'') + ryg' - rg_2 = q_{21}(g-f).$$

2. Consider the generator X_3. Integrating the group of trajectories, the traditional similarity solution of (4) is obtained.

$$\frac{dS}{d\varepsilon} = S, \frac{dt}{d\varepsilon} = 0, \frac{du_1}{d\varepsilon} = u_1, \frac{du_2}{d\varepsilon} = u_2,$$

ε is a parameter along the paths. The invariant transformations of this system are $u_1 = f(y)e^z, u_2 = g(y)e^z$, $S = e^z$, and $t = y$. After incorporating the modifications into (4),

$$f'e^z - q_{12}(fe^z - ge^z) = 0,$$
$$g'e^z - q_{21}(ge^z - fe^z) = 0.$$

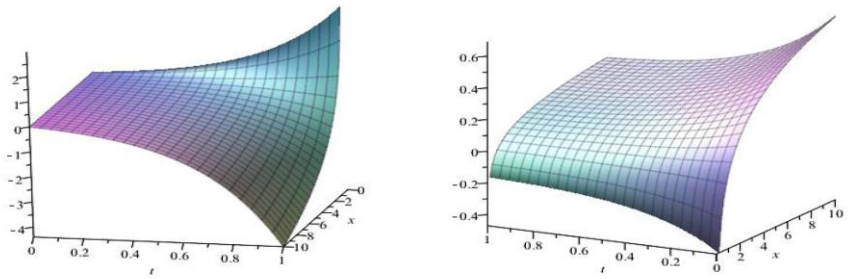
So

$$f(y) = C_1 - \frac{C_2 q_{12}\exp(y(q_{21}+q_{12}))}{q_{21}},$$
$$g(y) = C_1 + C_2\exp(y(q_{21}+q_{12})),$$

changing variables yields

$$u_1(S,t) = \ln(S)\left(C_1 - \frac{C_2 q_{12}\exp(t(q_{21}+q_{12}))}{q_{21}}\right), \tag{8}$$
$$u_2(S,t) = \ln(S)(C_1 + C_2\exp(t(q_{21}+q_{12}))). \tag{9}$$

Figure 1 show the process of changing the above answers in an example.



(a) $u_1(S,t)$ (b) $u_2(S,t)$

Figure 1
Plots for (8) and (9) with $q_{12} = 1, q_{21} = 2, C_1 = 0.1,$ and $C_2 = 0.2$.

The one-parameter group of the generators can yield additional solutions derived from equation (8).

The flow $\exp(\lambda X_2)(S,t,u_1,u_2) = (\bar{S},\bar{t},\bar{u}_1,\bar{u}_2)$ is

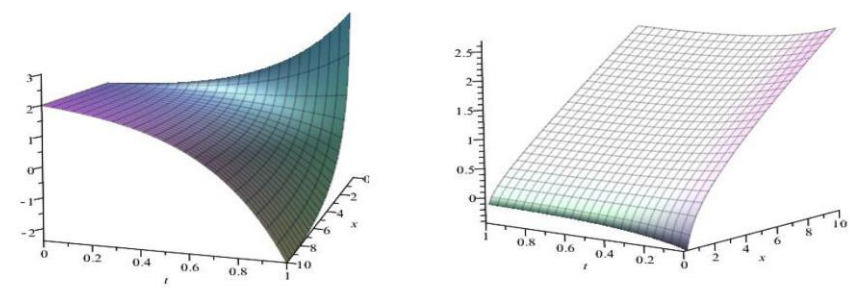
$$\begin{aligned} \bar{S} &= S, \bar{t} = t, \\ \bar{u}_1 &= \lambda S + u_1, \\ \bar{u}_2 &= \lambda S + u_2. \end{aligned} \tag{10}$$

Upon substituting (8) into this flow, the non-trivial precise solution that follows is obtained.

$$u_1(S,t) = \lambda S + \ln(S)\left(C_1 - \frac{C_2 q_{12}\exp(t(q_{21}+q_{12}))}{q_{21}}\right), \tag{11}$$

$$u_2(S,t) = \lambda S + \ln(S)(C_1 + C_2\exp(t(q_{21}+q_{12}))). \tag{12}$$

Figure 2 show the process of changing the above answers in an example.



(a) $u_1(S,t)$ (b) $u_2(S,t)$

Figure 2
Plots for (11) and (12) with $q_{12} = 1, q_{21} = 2, C_1 = 0.1, C_2 = 0.2,$ and $\lambda = 0.2$.

The flow $\exp(\lambda X_6)(S,t,u_1,u_2) = (\bar{S},\bar{t},\bar{u}_1,\bar{u}_2)$ is

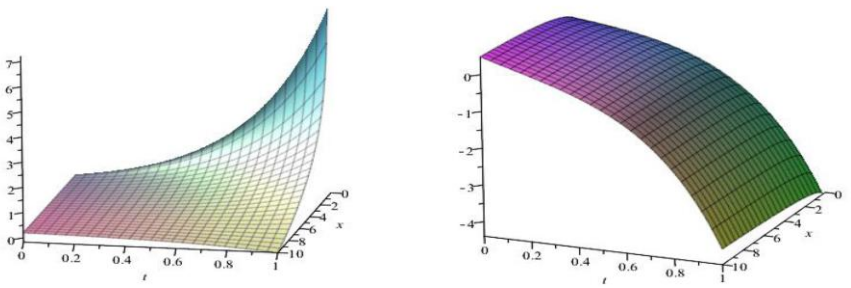
$$\begin{aligned} &\bar{S} = S, \bar{t} = t, \\ &\bar{u}_1 = \lambda \exp(t(q_{12} + q_{21} + r)) + u_1, \\ &\bar{u}_2 = -\frac{\lambda q_{21} \exp(t(q_{12}+q_{21}+r))}{q_{12}} + u_2. \end{aligned} \tag{13}$$

Adding (8) to the above flow yields

$$u_1(S,t) = \lambda \exp(t(q_{12} + q_{21} + r)) + \ln(S)\left(C_1 - \frac{C_2 q_{12} \exp(t(q_{21}+q_{12}))}{q_{21}}\right), \tag{14}$$

$$u_2(S,t) = -\frac{\lambda q_{21} \exp(t(q_{12}+q_{21}+r))}{q_{12}} + \ln(S)(C_1 + C_2 \exp(t(q_{21} + q_{12}))). \tag{15}$$

Figure 3 show the process of changing the above answers in an example.



(a) $u_1(S,t)$ (b) $u_2(S,t)$

Figure 3
Plots for (14) and (15) with $q_{12} = 1, q_{21} = 2, C_1 = 0.1, C_2 = 0.2, \lambda = 0.2,$ and $r = 0.05$.

2.2 *Reduced equations for* (5)

To determine an accurate solution for eq. (5), the same procedure is used.

1. Integrating the group of trajectories, the classical similarity solution of (5) for generator X_1^* is achieved.

$$\frac{dS}{d\varepsilon} = 0, \frac{dt}{d\varepsilon} = 1, \frac{du_1}{d\varepsilon} = 0, \frac{du_2}{d\varepsilon} = 0,$$

for each trajectory, ε serves as a parameter. The invariant transformations derived from the previous integrations are given by $u_1 = f(y)$, $u_2 = g(y)$, and $S = y$. Substituting $u_1 = f(y)$ and $u_2 = g(y)$ into equation (5) provides solutions within this invariant framework.

$$\begin{aligned} &\frac{1}{2}\sigma_1^2 y^2 f''(1 + 2\rho f'') = q_{12}(f - g), \\ &\frac{1}{2}\sigma_2^2 y^2 g''(1 + 2\rho g'') = q_{21}(g - f). \end{aligned} \tag{16}$$

2. A different approach would be to look at the generator X_4^*. Thus, integrating the group of trajectories, the conventional similarity solution of (5) may be achieved.

$$\frac{dS}{d\varepsilon} = S, \frac{dt}{d\varepsilon} = 0, \frac{du_1}{d\varepsilon} = u_1, \frac{du_2}{d\varepsilon} = u_2,$$

ε is a parameter along the paths. The invariant transformations from the foregoing integrations are $u_1 = f(y)e^z, u_2 = g(y)e^z$, $S = y$, and $t = z$. Adding $t = y$ and $S = e^z$ to (5)

$$f'e^z - q12(fe^z - ge^z) = 0,$$
$$g'e^z - q21(ge^z - fe^z) = 0.$$

So

$$f(y) = C_1 - \frac{C_2 q_{12}\exp(y(q_{21}+q_{12}))}{q_{21}},$$
$$g(y) = C_1 + C_2\exp(y(q_{21} + q_{12})),$$

changing variables

$$u_1(S,t) = \ln(S)\left(C_1 - \frac{C_2 q_{12}\exp(t(q_{21}+q_{12}))}{q_{21}}\right), \tag{17}$$

$$u_2(S,t) = \ln(S)(C_1 + C_2\exp(t(q_{21} + q_{12}))). \tag{18}$$

Using the one-parameter group for generators obtains another solutions.

Consider the flow of generator X_3^* as follows.

$$\begin{aligned} \bar{S} &= S, \bar{t} = t, \\ \bar{u}_1 &= \lambda S + u_1, \\ \bar{u}_2 &= \lambda S + u_2. \end{aligned} \tag{19}$$

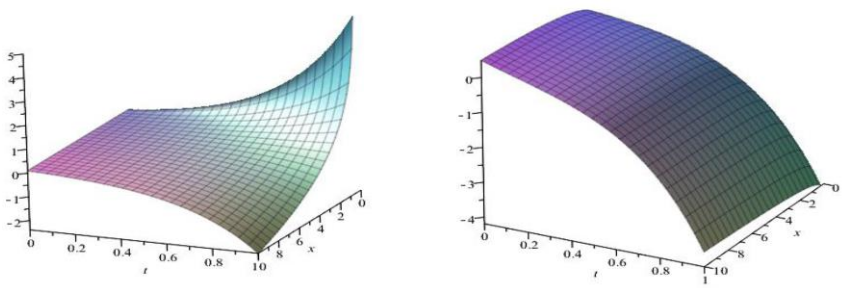
The following non-trivial solution is provided by this case.

$$u_1(S,t) = \lambda S + \ln(S)\left(C_1 - \frac{C_2 q_{12}\exp(t(q_{21}+q_{12}))}{q_{21}}\right), \tag{20}$$

$$u_2(S,t) = \lambda S + \ln(S)(C_1 + C_2\exp(t(q_{21} + q_{12}))). \tag{21}$$

Consider the following flow of generator X_5^*.

$$\begin{aligned} \bar{S} &= S, \bar{t} = t, \\ \bar{u}_1 &= \lambda S\exp(t(q_{12} + q_{21})) + u_1, \\ \bar{u}_2 &= -\frac{\lambda S q_{21}\exp(-t(q_{12}+q_{21}))}{q_{12}} + u_2. \end{aligned} \tag{22}$$



(a) $u_1(S,t)$ (b) $u_2(S,t)$

Figure 4

Plots for (23) and (24) with $q_{12} = 1, q_{21} = 2, C_1 = 0.1, C_2 = 0.2,$ and $\lambda = 0.2$.

This situation yields non-trivial solution

$$u_1(S,t) = \lambda S\exp(t(q_{12}+q_{21})) + \ln(S)\left(C_1 - \frac{C_2 q_{12}\exp(t(q_{21}+q_{12}))}{q_{21}}\right), \tag{23}$$

$$u_2(S,t) = -\frac{\lambda S q_{21}\exp(-t(q_{12}+q_{21}))}{q_{12}} + \ln(S)(C_1 + C_2\exp(t(q_{21}+q_{12}))). \tag{24}$$

Figure 4 show the process of changing the above answers in an example.

3. Numerical simulation with the CSC method

One of the most effective and widely used techniques for solving differential equations is the Chebyshev Spectral Collocation (CSC) method, known for its higher accuracy compared to alternative approaches. This method employs Chebyshev polynomials to approximate the solutions of differential equations. For discretizing equations (4) and (5), the CSC method utilizes collocation points within the interval $[-1,1]$ [10]. The variables S and t in system (4) (or (5)) are transformed to correspond to this interval, enabling the application of the CSC method.

$$\begin{cases} t = \frac{T_1-T_0}{2}\bar{t} + \frac{T_1+T_0}{2}, \\ S = \frac{b-a}{2}\bar{S} + \frac{b+a}{2}, \end{cases} \tag{25}$$

where $t \in [T_0, T_1], S \in [a,b]$ and $\bar{S}, \bar{t} \in [-1,1]$. Take

$$\begin{aligned} U^1(S,0) &= f_1(S), \\ U^2(S,0) &= f_2(S), \\ U^1(a,t) &= g_1(S), \\ U^2(a,t) &= g_2(S), \\ U^1(b,t) &= h_1(S), \\ U^2(b,t) &= h_2(S), \end{aligned} \tag{26}$$

where $a \le S \le b$ and $t \in [T_0, T_1]$. So

$$\begin{aligned} U^1_{\bar{t}} &= \psi_1(\bar{t}, U^1, U^1_{\bar{S}}, U^1_{\bar{S}\bar{S}}, U^2), \\ U^2_{\bar{t}} &= \psi_2(\bar{t}, U^2, U^2_{\bar{S}}, U^2_{\bar{S}\bar{S}}, U^1), \\ U^1(\bar{S},-1) &= F_1(\bar{S}), \\ U^2(\bar{S},-1) &= F_2(\bar{S}), \\ U^1(-1,\bar{t}) &= G_1(\bar{S}), \\ U^2(-1,\bar{t}) &= G_2(\bar{S}), \\ U^1(1,\bar{t}) &= H_1(\bar{S}), \\ U^2(1,\bar{t}) &= H_2(\bar{S}), \end{aligned} \tag{27}$$

where

$$\psi_1(\bar{t}, U^1, U^1_{\bar{S}}, U^1_{\bar{S}\bar{S}}) = \frac{T_1-T_0}{2}\Lambda\left(\frac{T_1-T_0}{2}\bar{t} + \frac{T_1+T_0}{2}, U^1, U^1_{\bar{S}}, U^1_{\bar{S}\bar{S}}, U^2\right),$$

$$\psi_2(\bar{t}, U^2, U^2_{\bar{S}}, U^2_{\bar{S}\bar{S}}) = \frac{T_1-T_0}{2}\Lambda\left(\frac{T_1-T_0}{2}\bar{t} + \frac{T_1+T_0}{2}, U^2, U^2_{\bar{S}}, U^2_{\bar{S}\bar{S}}, U^1\right),$$

$$
\begin{aligned}
&U^1(\bar{S},\bar{t}) = U^1\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2},\tfrac{b-a}{2}\bar{S}+\tfrac{b+a}{2}\right),\\
&U^1_{\bar{S}}(\bar{S},\bar{t}) = \tfrac{2}{b-a}U^1_{\bar{S}}\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2},\tfrac{b-a}{2}\bar{S}+\tfrac{b+a}{2}\right),\\
&U^1_{\bar{S}\bar{S}}(\bar{S},\bar{t}) = \left(\tfrac{2}{b-a}\right)^2 U^1_{\bar{S}\bar{S}}\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2},\tfrac{b-a}{2}\bar{S}+\tfrac{b+a}{2}\right),\\
&U^2(\bar{S},\bar{t}) = U^2\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2},\tfrac{b-a}{2}\bar{S}+\tfrac{b+a}{2}\right),\\
&U^2_{\bar{S}}(\bar{S},\bar{t}) = \tfrac{2}{b-a}U^2_{\bar{S}}\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2},\tfrac{b-a}{2}\bar{S}+\tfrac{b+a}{2}\right),\\
&U^2_{\bar{S}\bar{S}}(\bar{S},\bar{t}) = \left(\tfrac{2}{b-a}\right)^2 U^2_{\bar{S}\bar{S}}\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2},\tfrac{b-a}{2}\bar{S}+\tfrac{b+a}{2}\right),\\
&F_1(\bar{S}) = f_1\left(\tfrac{b-a}{2}\bar{S},\tfrac{b+a}{2}\right),\\
&F_2(\bar{S}) = f_2\left(\tfrac{b-a}{2}\bar{S},\tfrac{b+a}{2}\right),\\
&G_1(\bar{S}) = g_1\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2}\right),\\
&G_2(\bar{S}) = g_2\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2}\right),\\
&H_1(\bar{S}) = h_1\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2}\right),\\
&H_2(\bar{S}) = h_2\left(\tfrac{T_1-T_0}{2}\bar{t}+\tfrac{T_1+T_0}{2}\right).
\end{aligned}
\tag{28}
$$

The Chebyshev-Gauss-Lobatto (CGL) points are defined within the interval $[-1,1]$.

$$\bar{t}_k = \bar{S}_k = \cos\left(\frac{\pi(N-k)}{n}\right), \tag{29}$$

for $k = 0,1,\dots,N$. $\bar{t}_k$ (or $\bar{S}_k$) are solutions of $(1+\bar{t}^2)\frac{dT_N}{d\bar{t}}$ where $T_N(.)$ is the Chebyshev polynomial of order N [14].

$$T_j(\bar{t}) = \cos(j\cos^{-1}(\bar{t})), \tag{30}$$

where $j = 0,1,\dots,N$, $\bar{t} \in [-1,1]$. Substituting CGL points into (30),

$$L_k(\bar{t}) = \prod_{\substack{j=0\\ j\neq k}}^{N} \frac{\bar{t}-\bar{t}_j}{\bar{t}_k-\bar{t}_j} = \frac{2}{N\mu_k}\sum_{j=0}^{N}\frac{1}{\mu_j}T_j(\bar{t}_k)T_j(\bar{t}), \tag{31}$$

where $k = 0,1,\dots,N, \bar{t} \in [-1,1]$ and

$$\mu_j = \begin{cases} 2, & j = 0, N,\\ 1, & 1 \le j \le N-1, \end{cases} \qquad L_j(\bar{t}_k) = \begin{cases} 1, & j = k,\\ 0, & j \neq k. \end{cases} \tag{32}$$

Take an approximation of the (27).

$$U^1(\bar{S},\bar{t}) \approx U^{1,N}(\bar{S},\bar{t}) = \sum_{i=0}^{N}\sum_{j=0}^{N}\bar{a}^N_{ijk}L_i(\bar{t})L_j(\bar{S}), \tag{33}$$

$$U^2(\bar{S},\bar{t}) \approx U^{2,N}(\bar{S},\bar{t}) = \sum_{i=0}^{N}\sum_{j=0}^{N}\bar{b}^N_{ijk}L_i(\bar{t})L_j(\bar{S}). \tag{34}$$

From (32),

$$\begin{cases} U^{1,N}(\bar{S}_j,\bar{t}_i) = \bar{a}^N_{ij},\\ U^{2,N}(\bar{S}_j,\bar{t}_i) = \bar{b}^N_{ij}. \end{cases} \tag{35}$$

Take notations $U_{\bar{t}}^{1,N}(.,.)$, $U_{\bar{S}}^{1,N}(.,.)$, $U_{\bar{S}\bar{S}}^{1,N}(.,.)$ for the derivatives of $U^{1,N}(.,.)$ w.r.t. t, S and SS, respectively and $U_{\bar{t}}^{2,N}(.,.)$, $U_{\bar{S}}^{2,N}(.,.)$, $U_{\bar{S}\bar{S}}^{2,N}(.,.)$ the derivatives of $U^{2,N}(.,.)$ w.r.t. t, S, and SS. So for $k, j = 0,1, \dots, N$

$$\begin{cases} U_{\bar{t}}^{1,N}(\bar{S}_k, \bar{t}_p) = \sum_{i=1}^{N} \bar{a}_{ik}^{N} D_{pi}, \\ U_{\bar{S}}^{1,N}(\bar{S}_k, \bar{t}_p) = \sum_{j=1}^{N} \bar{a}_{pj}^{N} D_{kj}, \\ U_{\bar{S}\bar{S}}^{1,N}(\bar{S}_k, \bar{t}_p) = \sum_{j=1}^{N} \bar{a}_{pj}^{N} \widehat{D}_{kj}, \\ U_{\bar{t}}^{2,N}(\bar{S}_k, \bar{t}_p) = \sum_{i=1}^{N} \bar{b}_{ik}^{N} D_{pi}, \\ U_{\bar{S}}^{2,N}(\bar{S}_k, \bar{t}_p) = \sum_{j=1}^{N} \bar{b}_{pj}^{N} D_{kj}, \\ U_{\bar{S}\bar{S}}^{2,N}(\bar{S}_k, \bar{t}_p) = \sum_{j=1}^{N} \bar{b}_{pj}^{N} \widehat{D}_{kj}, \end{cases} \tag{36}$$

$$D_{kj} = L'_j(\bar{t}_k) = \begin{cases} \frac{\mu_k}{\mu_j}(-1)^{k+j} \frac{1}{\bar{t}_k - \bar{t}_j}, & if\ j \neq k, \\ -\frac{\bar{t}_k}{2-2\bar{t}_k^2}, & if\ 0 \leq j = k \leq N-1, \\ -\frac{2N^2+1}{6}, & if\ j = k = 0, \\ \frac{2N^2+1}{6}, & if\ j = k = N. \end{cases} \tag{37}$$

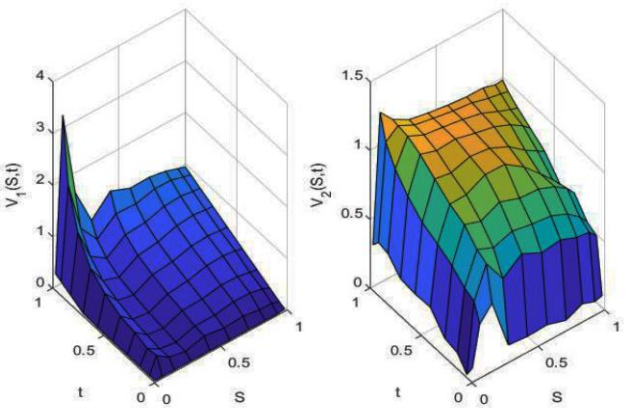


Figure 5
Plots of CSC method for Bakstein and Howison model with $q_{12} = 1, q_{21} = 2, \sigma_1 = 0.4, \sigma_2 = 0.7, r = 0.01$ and $\rho = 0.6$.

$\widehat{D} = D.D = (\widehat{D}_{kj}), \widehat{D}_{kj} = \sum_{l=0}^{N} D_{kl} D_{lj}$. From (35) and (36), eq. (27) can be rewritten as

$$\sum_{i=1}^{N} \bar{a}_{ik}^{N} D_{pi} - \psi_1\left(\bar{t}_p, \bar{a}_{pk}^{N}, \sum_{j=1}^{N} \bar{a}_{pj}^{N} D_{kj}, \sum_{j=1}^{N} \bar{a}_{pj}^{N} \widehat{D}_{kj}, \bar{b}_{pk}^{N}\right) = 0,$$
$$\sum_{i=1}^{N} \bar{b}_{ik}^{N} D_{pi} - \psi_2(\bar{t}_p, \bar{b}_{pk}^{N}, \sum_{j=1}^{N} \bar{b}_{pj}^{N} D_{kj}, \sum_{j=1}^{N} \bar{b}_{pj}^{N} \widehat{D}_{kj}, \bar{a}_{pk}^{N}) = 0,$$

$$\begin{aligned} \bar{a}_{0k}^{N} &= F_1(\bar{S}_k), \\ \bar{b}_{0k}^{N} &= F_2(\bar{S}_k) \\ \bar{a}_{p0}^{N} &= G_1(\bar{t}_p), \\ \bar{b}_{p0}^{N} &= G_2(\bar{t}_p), \\ \bar{a}_{pN}^{N} &= H_1(\bar{t}_p), \\ \bar{b}_{pN}^{N} &= H_2(\bar{t}_p), \end{aligned} \tag{38}$$

for $k = 0,1,\dots,N$ and $p = 1,\dots,N$. The coefficients $\bar{a}_{pk}^{N}$ and $\bar{b}_{pk}^{N}$ are obtained substituting the aforementioned system with (4) (or (5)). Ultimately, the approximation polynomials $U^{1,N}(.,.)$ and $U^{2,N}(.,.)$ are given

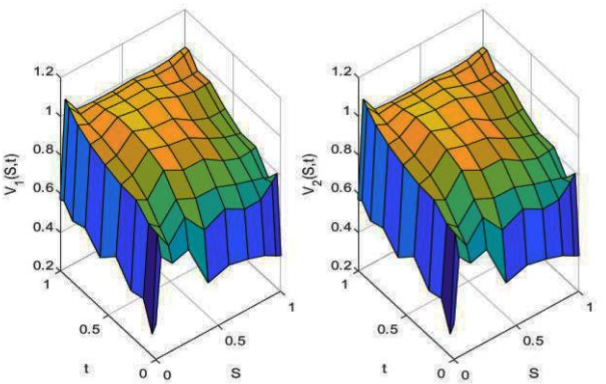


Figure 6
Plots of CSC method for Cetin et'al model with $q_{12} = 1, q_{21} = 2, \sigma_1 = 0.4, \sigma_2 = 0.7$ and $\rho = 0.6$.

Figures 5 and 6 are numerical examples of the CSC method for Bakstein and Howison and Cetin et'al models, respectively. The convergence of the CSC method has been studied in numerous articles, so its repetition has been avoided.

4. Conclusion

It is evident that financial markets therefore tend to be unpredictable by nature and trends may be determined by changing economic, political and social events. The structural choice model can capture these volatilities through the regime-switching models, and transaction costs also improve the options pricing. This work concerned a system of nonlinear partial differential equations obtained from a regime-switching model with transaction costs. In particular, we considered a nonlinear regime-switching model based on Black-Scholes model. Reducing the equations order and using Lie symmetry methods, we have arrived at analytical solutions consequently. Furthermore, computational experiments involving the application of the Chebyshev pseudo-spectral method for numerical integration of the Keller–Miksis equations were conducted, and the efficiency of

the idea under consideration was studied in detail. Thus, these results serve to enhance the knowledge regarding price determination under the transformation from transaction cost and shifts in regimes in financial markets.

References

[1] D. Bakstein and S. Howison, "A non-arbitrage liquidity model with observable parameters for derivatives," Working Paper (2003).

[2] E. Dastranj and S. R. Hejazi, "Exact solutions for Fokker-Planck equation of geometric Brownian motion with Lie point symmetries," *Computational Methods for Differential Equations,* vol. 6, no. 3, pp. 372–379 (2018).

[3] E. Dastranj and S. R. Hejazi, "New solutions for Fokker-Planck equation of special stochastic process via Lie point symmetries," *Computational Methods for Differential Equations,* vol. 5, no. 1, pp. 30–42 (2017).

[4] E. Dastranj, M. R. Lotfi, R. Latifi, and A. R. Javan, "Power option pricing under the unstable conditions (evidence of power option pricing under fractional Heston model in the Iran gold market)," *Physica A: Statistical Mechanics and its Applications,* vol. 537, p. 122690 (2020).

[5] E. Dastranj and R. Latifi, "A comparison of option pricing models," *International Journal of Financial Engineering,* vol. 4, no. 02n03, p. 1750024 (2017).

[6] E. Dastranj and H. Sahebi Fard, "Exact solutions and numerical simulation for Bakstein–Howison model," *Computational Methods for Differential Equations,* vol. 10, no. 2, pp. 461–474 (2022).

[7] J. E. Esekon, "Analytic solution of a nonlinear Black–Scholes equation," *International Journal of Pure and Applied Mathematics,* vol. 82, no. 4, pp. 547–555 (2013).

[8] J. E. Esekon, "A particular solution of a nonlinear Black–Scholes partial differential equation," *International Journal of Pure and Applied Mathematics,* vol. 81, no. 5, pp. 715–721 (2012).

[9] S. R. Hejazi, "Lie group analysis, Hamiltonian equations and conservation laws of Born–Infeld equation," *Asian-European Journal of Mathematics*, vol. 7, no. 3, p. 1450040 (2014).

[10] Y. Huang, X. Gao, C. Xu, and H. Liang, "Space-time spectral collocation method for solving Burgers equations with the convergence analysis," *Symmetry*, vol. 11, no. 12, p. 1439 (2019).

[11] E. Lashkarian, S. R. Hejazi, N. Habibi, and A. Motamednezhad, "Symmetry properties, conservation laws, reduction and numerical approximations of time-fractional cylindrical-Burgers equation," *Computational Methods for Differential Equations,* vol. 10, no. 4, pp. 655–669 (2022).

[12] H. Sahebi Fard, E. Dastranj, and A. Abdolbaghi Ataabadi, "Analytical and numerical solutions for the pricing of a combination of two financial derivatives in a market under Hull–White model," *Advances in Mathematical Finance and Applications,* vol. 7, no. 4, pp. 1013–1023 (2022).

[13] H. Sahebi Fard, E. Dastranj, R. Hejazi, and A. Jajarmi, "Analytical and numerical solutions for a special nonlinear equation," *International Journal of Financial Engineering*, vol. 11, no. 1, p. 2350057 (2024).

[14] J. Shen, T. Tang, and L. L. Wang, Spectral Methods: Algorithms, Analysis and Applications, vol. 41. Berlin, Germany: Springer Science & Business Media (2011).

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