

Co-intersection graph of subsemimodules of a semimodule

Zahraa A. Nema *

Ahmed H. Alwan †

Department of Mathematics

College of Education for Pure Sciences

University of Thi-Qar

Thi-Qar

Iraq

Abstract

Assume that U is a left R -semimodule, R is an abelian semiring with nonzero identity. The co-intersection graphs of U , indicated by $\Gamma(U)$, an undirected graph whose nodes set is the collection of all non-trivial subsemimodules of U with two distinct nodes X, Y are adjacent if and only if $X + Y \neq U$. These graphs were studied to relate the combinatorial properties of $\Gamma(U)$ to the algebraic properties of the R -semimodule U . Connectedness and planarity were for $\Gamma(U)$. Also, studies are done on the girth, diameter, clique and chromatic number of this graphs.

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1. Introduction

Bosak in 1964 [7] defined the graph of semigroups. In [13], the authors introduced the open subset inclusion graph of a topological space. In [2], [4] Alwan introduced the maximal ideal graph, maximal submodule graph. In [11], the present authors introduced the inclusion subsemimodule graph. Mahdavi and Talebi [10] introduced the co-intersection graph of a module. In this work, the co-intersection graph is defined of subsemimodules of a semimodule.

A semiring R is an algebraic structure $(R, +, \cdot)$ where $(R, +)$, $(R, \cdot)$ are commutative semigroups, joined by $a(t + s) = at + as$ for all $a, t, s \in R$, there exist $0 \neq 1 \in R$ with $d + 0 = d$, $d0 = 0d = 0$ and $d1 = 1d = d$ for each $d \in R$. In this work, it is presumed that U is a left R -semimodule and that all semirings are commutative semirings with identities. We imply that a non-trivial

* E-mail: zahalshabib@gmail.com (Corresponding Author)

† E-mail: ahmedha_math@utq.edu.iq

submodule of U is a suitable non-zero subsemimodule. A semimodule U is minimal if and only if it does not have any non-trivial subsemimodule. A subsemimodule of U is minimal if and only if it does not have any subsemimodule of U other than 0 and itself. By $\min(U)$, is the set of minimal subsemimodules of U . The length of U is the length of the composition series of U , symbolized as $l_R(U)$. A subsemimodule N of U named small in U (we write $N \ll U$), if for every subsemimodule $F \subseteq U$, the equality $N + F = U$ implies $F = U$, a subsemimodule N is named small in U , if $N + K \neq U$ for all proper subsemimodule K of U . The radical of U written $\text{Rad}(U)$, is the sum of all small subsemimodules of U . A semimodule U is named hollow, if every one proper subsemimodule of U is small in U , [3]. $M \subset U$ is named maximal in U if $M \not\subseteq Q$ for all $Q \subset U$. A semimodule U is named local if it has a single maximal subsemimodule. By $\max(U)$, we mean the set of all maximal subsemimodules of U . A subtractive subsemimodule N is a subsemimodule of U where if $x, x + y \in N$, thus $y \in N$. We say an R -semimodule is subtractive if each of its R -subsemimodules is subtractive [1]. If each proper subsemimodule of U is contained in a maximal subsemimodule, we say that U is coatomic [5]. For graph theory, the reference is [6].

In a graph Γ , $E(\Gamma)$ and $V(\Gamma)$ are the set of all edges and vertices, respectively. For two distinct vertices, Q and P symbolized by $Q - P$ means that Q and P are adjacent. A complete graph with n vertices is a graph which all two vertices are adjacent, symbolized by (K_n) . A bipartite graph is a graph whose vertices can be separated into two sets W and V such that every edge joins a vertex in W to one in V , and W and V are each independent is named bipartite. The vertices u and v of a graph, Γ are named joined in Γ if a path exists between them. If a path exists between any two distinct vertices of Γ , we say that a graph Γ is connected (or joined). Or else, Γ is named disjointed. Let Γ be a joined graph. The distance between two distinct vertices u and v , is symbolized by $d(u, v)$, is the length of the shortest path connecting them, if such a path exists [12]. By $\text{diam}(\Gamma) = \text{Max}\{d(u, v) : u, v \in V(\Gamma)\}$ A diameter of the graph Γ is The length of the shortest cycle in Γ , in case Γ contains a cycle is the girth of a graph Γ , symbolized $\text{gr}(\Gamma)$; else, $\text{gr}(\Gamma) = \infty$. A graph with no edges is a null graph.

In Section 2, it is ascertained that if U is a subtractive semimodule, $\Gamma(U)$ is not connected if and only if $U = T^1 \oplus T^2$, where T^1, T^2 are simples. We characterize all semimodules for which $\Gamma(U)$ is not joined. Besides, the girth and the diameter of $\Gamma(U)$ are fixed. We revise some semimodules whose co-intersection of graphs are complete. In Section 3, investigate the clique number of $\Gamma(U)$. We fix the structure of U for which the graph $\Gamma(U)$ is planar.

2. Connectedness of $\Gamma(U)$

Definition 2.1: Let R is a semiring with identity and U is a unitary left R -semimodule. The co-intersection graph U , symbolized $\Gamma(U)$, is an undirected simple graphs with all nontrivial subsemimodules of U as nodes and two distinct nodes N, L are adjacent if and only if $N + L \neq U$.

Remark 2.2: For a semiring R , $\Gamma(R)$ is the co-intersection graph of ideals of R , where R is regarded as a left R -semimodule.

Theorem 2.3: Let U be a subtractive R -semimodule. So the graph $\Gamma(U)$ is not connected if and only if $U = T^1 \oplus T^2$, where T^1, T^2 are two simple R -semimodules.

Proof: Suppose that $\Gamma(U)$ is non-connected. Assume Γ_1 and Γ_2 are two components of $\Gamma(U)$. Let T^1, T^2 are two subsemimodules of U such that $T^1 \in \Gamma_1$ and $T^2 \in \Gamma_2$. As there is no T^1, T^2 -path, at that point $U = T^1 + T^2$. Currently, if $T^1 \cap T^2 \neq (0)$, so $T^1 \cap T^2 + T^1 = T^1 \neq U$ besides $T^1 \cap T^2 + T^2 = T^2 \neq U$ indicates that there is a path $T^1 - T^1 \cap T^2 - T^1$, a contradiction. Hence, $T^1 \cap T^2 = (0)$. Now, we exhibition that T^1 and T^2 belong to $\min(U)$. To realize this, let T be a subsemimodule of U such that $(0) \neq T \subseteq T^1$ then $T + T^1 = T^1 \neq U$. Hence T and T^1 are adjacent vertices, that indicates that $T \in \Gamma_1$. So there is no T, T^2 -path and by arguing as above, we have $U = T + T^2$, since T and T^2 are not adjacent vertices. But since $T^1 = T^1 \cap U = T^1 \cap (T + T^2) = T + (T^1 \cap T^2) = U$ by semimodular law, since T is a k -subsemimodule [5]. As a result, $T^1 \in \min(U)$. A like argument shows that $T^2 \in \min(U)$ and in factual each $(0) \neq N \subset U$, then $N \in \min(U)$, which yields that each $(0) \neq M \subset U$, then $M \in \max(U)$. But T^1 and T^1 belong to $\min(U)$ so that, they are simple semimodules, such that the semimodule U is a sum $U = T^1 \oplus T^2$, by [9].

On the other hand, suppose that $\Gamma(U)$ is connected. Let $U = T^1 \oplus T^2$, where T^1 and T^2 are simple R -semimodules. Let $U_1 = T^1 \times \{0\}$ and $U_2 = \{0\} \times T^2$. Then $U_1, U_2 \in \min(U)$. In addition, U_1 and U_2 are simple R -semimodules. But, $U = U_1 \oplus U_2$ and $U_1 \cong \frac{U}{U_2}$ and $U_2 \cong \frac{U}{U_1}$. So, U_1 and $U_2 \in \max(U)$. Thus, $U_1, U_2 \in \max(U)$ and $U_1, U_2 \in \min(U)$. to prove U_1 is an isolated node in $\Gamma(U)$, take Θ_ℓ be a vertex in $\Gamma(U)$, with $\Theta_\ell + U_1 \neq U$. Then, the maximality of U_1 implies that $\Theta_\ell + U_1 = U_1$, and hence $\Theta_\ell \subseteq U_1$. Then, the minimality of U_1 implies that $U_1 = \Theta_\ell$. So, U_1 is an isolated vertex in $\Gamma(U)$. So, $\Gamma(U)$ is not connected, a contradiction.

Remark 2.4: Let $\mathbb{Z}_n$ be a $\mathbb{Z}$ -semimodule. Then $\Gamma(\mathbb{Z}_n)$ is disconnected iff $n = q_1 q_2$, wherever q_1 and q_2 are distinct primes.

Example 2.5: Consider $\mathbb{Z}_{sqr}$ as a $\mathbb{Z}$ -semimodule, such that s, q , and r three distinct primes. We know $\langle s \rangle = s\mathbb{Z}_{sqr}$, $\langle q \rangle = q\mathbb{Z}_{sqr}$ and $\langle r \rangle = r\mathbb{Z}_{sqr}$ are the only maximal subsemimodules of $\mathbb{Z}_{sqr}$. Also, $\langle sq \rangle = sq\mathbb{Z}_{sqr}$, $\langle sr \rangle = sr\mathbb{Z}_{sqr}$ and $\langle qr \rangle = qr\mathbb{Z}_{sqr}$ are the other subsemimodules and $\mathbb{Z}_{sqr} = \langle sq \rangle \oplus \langle qr \rangle \oplus \langle sr \rangle$. So, $\Gamma(\mathbb{Z}_{sqr})$ is a joined graph see Figure 1.

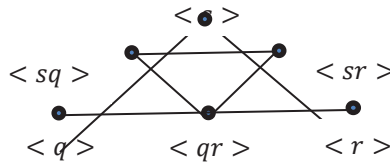


Figure 1
 $\Gamma(\mathbb{Z}_{sqr})$

Corollary 2.6: Let U be a subtractive R -semimodule. If $\Gamma(U)$ is not joined and $|\Gamma(U)| \geq 2$, so

- (a) $\Gamma(U)$ is a null graph;
- (b) $l_R(U) = 2$.

Corollary 2.7: Assume that U be a subtractive R -semimodule. If $\Gamma(U)$ is connected, so

- (1) For any $N, L \in \max(U)$, then $N \cap L \neq (0)$, and there exists a path between them.
- (2) For any $U_1, U_2 \in \min(U)$, then $U_1 + U_2 \neq U$.

Proof:

- (1) It can be seen by the alike proof of [10, Corollary 2.3 (1)].
- (2) Let U_1 and $U_2 \in \min(U)$ with $U = U_1 + U_2$. If $U_1 \cap U_2 = (0)$, then $U = U_1 \oplus U_2$, according to [9], such that U_1 and U_2 are simple semimodules. By Th. 2.3, $\Gamma(U)$ is non-connect., a contradiction. Also if $U_1 \cap U_2 \neq (0)$, since $(0) \subsetneq U_1 \cap U_2 \subsetneq U_i \subsetneq U$, for $i = 1, 2$, by minimality of U_1 and U_2 indicates that $U_1 \cap U_2 = U_1 = U_2$, an illogicality. So $U \neq U_1 + U_2$.

Remark 2.8: If U is a subtractive coatomic semimodule, then any non-maximal subsemimodule is adjacent to M for some $M \in \max(U)$.

Corollary 2.9: Let U be a subtractive semimodule with $(0) \neq \text{Rad}(U) \ll U$. If $\Gamma(U)$ is joined, then $\text{Rad}(U)$ is adjacent toward every other vertex in $\Gamma(U)$.

Remark 2.10: Let U be a semimodule. Then $\Gamma(U)$ isomorphic to K_2 if and only if U has precisely two non-trivial subsemimodules which are one maximal besides another one minimal.

Remark 2.11: $\Gamma(U)$ is complete if subsemimodules of U form a chain

Theorem 2.12: Let U be an R -semimodule. Then $\Gamma(U)$ is a complete graph if and only if U is a hollow.

Proof: Take $N_1, N_2 \in V(\Gamma(U))$. Since U is hollow, then $N_1 + N_2 \ll U$. So, $N_1 + N_2 \neq U$ and $\Gamma(U)$ is complete. In opposition, let $\Gamma(U)$ be a complete. Let $N \in \Gamma(U)$. Then $N + X \neq U$ for any subsemimodule X , so $N \ll U$, as required.

Proposition 2.13: Let U be a subtractive R -semimodule and $\Gamma(U)$ is a connected graph. If U Contains three minimums at least minimal subsemimodules, then $\Gamma(U)$ is not a bipartite graph.

Proof: Let M_1, M_2 and $M_3 \in \min(U)$ and $\Gamma(U)$ is connected. By Corollary 2.7 (2), (M_1, M_2, M_3) is a 3-cycle of $\Gamma(U)$. knows that a simple graph is bipartite if it has no odd cycle.

Theorem 2.14: Let U be a subtractive semimodule. If $\Gamma(U)$ is connected, then $\text{diam}(\Gamma(U)) \leq 3$.

Proof: Let N, L be two non-zero subsemimodule of U . If $N + L \neq U$ then there is an edge between them, so $d(N, L) = 1$. Presume that $N + L = U$. If $N \cap L \neq (0)$, then there exists a path $N - N \cap L - L$ is length 2, so $d(N, L) = 2$. if $N \cap L = (0)$, since $\Gamma(U)$ is connected, by Corollary 2.7(1), indicates at least one of N and L must be non-maximal. Let $L \notin \max(U)$. Henceforth there is a subsemimodule $Y^u \subset U$ such that $L \subsetneq Y^u \subsetneq U$, and $L + Y^u = Y^u \neq U$. Now, if $N + Y^u \neq U$, hence there is a path $N - Y^u - L$ of length 2, then $d(N, L) = 2$. But, if $N + Y^u = U$, then by semimodular law [5], $Y^u = Y^u \cap (N + L) = (Y^u \cap N) + L$. Now, if $Y^u \cap N = (0)$, then $Y^u = L$, a contradiction with the existence Y^u . Also, if $Y^u \cap N \neq (0)$, then there is a path $N - Y^u \cap N - Y^u - L$, so $d(N, L) \leq 3$. As a result, $\text{diam}(\Gamma(U)) \leq 3$.

Proposition 2.15: Let U be a semimodule, and $\Gamma(U)$ contains a cycle. Then $\text{gr}(\Gamma(U)) = 3$.

Proof: Assume on the conflicting that, let $\text{gr}(\Gamma(U)) \geq 4$. This implies that every $(0) \neq \mathcal{F}_i, \mathcal{F}_j \subset U$ with $\mathcal{F}_i \cap \mathcal{F}_j \neq (0)$ must be comparable, else, $\Gamma(U)$ will have a circuit of distance 3, a conflict. Now, since $\text{gr}(\Gamma(U)) \geq 4$, $\Gamma(U)$ has a path of distance 3, say $\mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3 - \mathcal{F}_4$. Since any series of non-trivial subsemimodules of length 2 generates a circuit of length 3 in $\Gamma(U)$ also any two $\mathcal{F}_i, \mathcal{F}_j$ in this trail are comparable, the just two likely cases are $\mathcal{F}_1 \subseteq \mathcal{F}_2$, $\mathcal{F}_3 \subseteq \mathcal{F}_2$, $\mathcal{F}_3 \subseteq \mathcal{F}_4$ or $\mathcal{F}_2 \subseteq \mathcal{F}_1$, $\mathcal{F}_2 \subseteq \mathcal{F}_3$. The first case yields $\mathcal{F}_3 \subseteq \mathcal{F}_2 \cap \mathcal{F}_4$ and hence $\mathcal{F}_2 \cap \mathcal{F}_4 \neq (0)$. As a result $(\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ is a circuit in $\Gamma(U)$, a conflict. In the another case, we get $\mathcal{F}_2 \subseteq \mathcal{F}_1 \cap \mathcal{F}_3$. So $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is a circuit in $\Gamma(U)$, a conflict. Thus $\text{gr}(\Gamma(U)) = 3$.

3. Clique Number and Planarity of $\Gamma(U)$

A clique in Γ is a complete sub-graph. The number of vertices in a largest clique of Γ , symbolized $\omega(\Gamma)$, named the clique number of Γ . The smallest

amount of colors that may be used to color each pair of neighboring vertices in such a way that dissimilar colors is named the chromatic number of Γ , symbolized by $\chi(\Gamma)$. A graph Γ in which $\omega(\Gamma) = \chi(\Gamma)$ is named weakly perfect. Γ is if anything can be drawn in a plane, it is planar. consequently that its edges intersect only on their ends.

Proposition 3.1: Let U be a subtractive R -semimodule and $\omega(\Gamma(U)) < \infty$. Then the next hold.

- (1) $l_R(U)$ is finite,
- (2) $\omega(\Gamma(U)) = 1$ if and only if either $|\Gamma(U)| > 2$ or $|\Gamma(U)| = 1$ and $U = T^1 \oplus T^2$, where T^1, T^2 are simple subsemimodules of U .

Proof:

- (1) Let $U_0 \subset U_1 \subset \dots \subset U_q \subset U_{q+1} \subset \dots$, is an infinite strictly increasing chain of subsemimodules of U . For all $q < j$, $U_q + U_j = U_j \neq U$, so likewise for the infinite strictly decreasing chain of subsemimodules of U . So, any severely increasing or decreasing infinite chain of subsemimodules of U generates a clique $\Gamma(U)$ that opposes the finiteness of $\omega(\Gamma)$. This implies that technically speaking (increasing and decreasing) infinite chain of subsemimodules of U , $U_n = U_{n+i}$ for $i = 1, 2, \dots$. So U is Noetherian & Artinian. So $l_R(U)$ is finite.
- (2) Suppose $\omega(\Gamma) = 1$ and $|\Gamma(U)| > 2$. This indicates that $\Gamma(U)$ is not joined. So, by Theorem 2.3, $U = T^1 \oplus T^2$, where T^1, T^2 are simple subsemimodules of U . Conversely, clearly by Theorem 2.3.

Remark 3.2: Assume $l_R(U)$ be the length of U and $\Delta(\Gamma) = \max\{\deg(q_i) \mid q_i \in V(\Gamma)\}$, then, $l_R(U) \leq \omega(\Gamma(U)) + 1$. If $\Gamma(U)$ is joined, then $\omega(\Gamma(U)) \leq \chi(\Gamma(U)) \leq \Delta(\Gamma) + 1$ by Theorem 10.3 of [8]. So, $\Delta(\Gamma) < \infty$, infers that $\chi(\Gamma(U)) < \infty$, $\omega(\Gamma(U)) < \infty$ & $l_R(U) < \infty$.

Theorem 3.3: If a semimodule U is subtractive and $|\Gamma(U)| > 2$. Then the next is equal:

- (a) $\Gamma(U)$ be a star graph;
- (b) $\Gamma(U)$ be a tree;
- (c) $\chi(\Gamma(U)) = 2$;
- (d) $l_R(U) = 3$, U has a unique minimal subsemimodule Z such that each proper subsemimodule containing Z belongs to $\max(U)$.

Proof: The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (d), Let $\chi(\Gamma(U)) = 2$. Thus $\Gamma(U)$ is not null. So, by Corollary 2.6, $\Gamma(U)$ is joined. By Remark 3.2, $\omega(\Gamma(U)) \leq \chi(\Gamma(U))$, as a result,

$\omega(\Gamma(U)) < \infty$ and by Proposition 3.1(1), $l_R(U) < \infty$. Then U is Artinian. As a result, U contains a simple subsemimodule Z . We show that Z is single. Let L_h and L_g two minimal subsemimodules of U . Then by part (2) of Corollary 2.7, $L_h + L_g \neq U$. Since $(L_h + L_g) + L_h = L_h + L_g \neq U$ and $(L_h + L_g) + L_g = L_h + L_g \neq U$, so $(L_h, L_h + L_g, L_g)$ is a circuit has length 3, this contradicts $\chi(\Gamma(U)) = 2$. So, Z is a single simple subsemimodule of U . Let's say $Z \subseteq M$ for all $M \subset U$. If $(0) \neq L_k \subset U$ with $Z \subsetneq L_k$, we exhibition that $L_k \in \max(U)$. Let $Z \subsetneq L_k \subsetneq X \subsetneq U$, since $Z + L_k = L_k$, $L_k + X = X$ and $Z + X = X$, (Z, L_k, X) is a cycle of length 3, an illogicality. So, $L_k \in \max(U)$, $Z \subset L_k$, and $(0) \subsetneq Z \subsetneq L_k \subsetneq U$, is a composition series of U with a length 3. So, $l_R(U) = 3$.

(d) $\Rightarrow$ (a) Let $l_R(U) = 3$ and U has a unique simple subsemimodule Z , such that every unique non-trivial subsemimodule L_i , ($i \in I$) of U contains Z , is belong to $\max(U)$. Then $(0) \subsetneq Z \subsetneq L_i \subsetneq U$, for all $i \in I$, are composition series of U with length 3, such that $L_i + Z = L_j \neq U$ and $L_i + L_j = U$ for $i \neq j$, So, $\Gamma(U)$ is a star graph.

Proposition 3.4: Assume that U be a subtractive semimodule. If $|\Gamma(U)| \geq 2$ and $\Gamma(U)$ is not connected, so the graph $\Gamma(U)$ named weakly perfect.

Proof: Assume $|\Gamma(U)| \geq 2$. $\Gamma(U)$ is a null graph, since $\Gamma(U)$ is not connected. Thus, $\Gamma(U)$ has no cycle, $\omega(\Gamma(U)) = \chi(\Gamma(U)) = 1$.

Remark 3.5: The opposite off Proposition 3.4 in general is not true ,to see this consider the $\mathbb{Z}$ -semimodule $\mathbb{Z}_{pqr}$, where p , q , and r be distinct primes. $\omega(\Gamma(\mathbb{Z}_{pqr})) = \chi(\Gamma(\mathbb{Z}_{pqr})) = 3$.

Theorem 3.6: [6, Theorem 10.30] *A graph is planar if and only if the graph contains no subdivision of either K_5 or $K_{3,3}$.*

Example 3.7: For every one prime number p , $\mathbb{Z}_{p^\infty}$ is hollow $\mathbb{Z}$ -semimodule, then $\Gamma(\mathbb{Z}_{p^\infty})$ is not a planar graph and complete .

Theorem 3.8: *Let U be a subtractive R -semimodule. Then*

- (1) *If $\Gamma(U)$ is connected and planar graph, then $|\min(U)| \leq 3$.*
- (2) *If $U = \bigoplus_{i=1}^n U_i$, where U_i are non-isomorphic simple R -semimodules, $1 \leq i \leq n$, then for all $n \leq 3$, $\Gamma(U)$ is planar and for all $n \geq 4$, $\Gamma(U)$ is not planar.*

Proof:

- (1) assume that $|\min(U)| \geq 4$. If $|\min(U)| \geq 5$, then $\Gamma(U)$ have a subgraph of the complete graph K_5 , if $|\min(U)| = 4$, we can see easily that

$\Gamma(U)$ contains $K_{3,3}$ as a subgraph. By Theorem 3.6, $\Gamma(U)$ is not planar, a contradiction. Thus $|\min(U)| \leq 3$.

(2) For all $n \leq 3$ it is trivial and for all $n \geq 4$ it follows by (1).

Theorem 3.9: *Let U be a subtractive semimodule. If $\Gamma(U)$ is planar, so every chain of proper subsemimodules of U has a length at most five.*

Proof: Assume that $N_1 \subset N_2 \subset \cdots \subset N_5$ be a chain of nontrivial subsemimodules of U . Since $N_i + N_j = N_j \neq U$ for $i < j$, $1 \leq i, j \leq 5$, so the set $\{N_i : 1 \leq i \leq 5\}$ induces K_5 as a subgraph in $\Gamma(U)$. By Th. 3.6, $\Gamma(U)$ is not planar, a cont.

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