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A novel approach to the Routh-Hurwitz criterion

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Abstract

The Routh-Hurwitz Criterion is closely related to the stability of a continuous-time system of differential equations. Its main concern is the determination of the number of zeros of the characteristic polynomial of the system with positive or negative real parts and the number of imaginary zeros. We propose a new algorithm to achieve this goal. The new technique is extremely simple to apply since it only requires elementary row operations of an associated determinant. The asymptotic stability of the system falls out as a special case of the newly proposed algorithm. Several randomly chosen examples are given to illustrate the simplicity and the effectiveness of the novel approach.

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1. Introduction

The Routh-Hurwitz criterion is an important tool to test the stability of continuous-time systems of differential equations. The stability of such systems is required in a variety of mathematical and engineering applications such as control theory, numerical computations, image processing, and systems theory. For some recent references see [1] and [2]. Stability is granted when all zeros of the characteristic polynomial of the system have negative real parts. For some references in this respect see [3], and [4].

The Routh-Hurwitz criterion has a more general objective, that is to determine the number of zeros of the characteristic polynomial with positive or negative real parts and the number of imaginary zeros. Several authors addressed this issue in great details [5], [6], and [7]

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In the current work we address this issue, and we propose a new algorithm to achieve this objective. The proposed technique is characterized by its simplicity since it only requires elementary row operations of a determinant associated with the system. We support our claim by a list of randomly chosen examples.

In Section 2, we lay out the new algorithm. In Section 3 we provide a list of examples of polynomials of different degrees. We end up with some concluding remarks in Section 4.

2. The New Algorithm

Given a linear continuous-time system with characteristic polynomial of degree n

$$f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-2} z^2 + a_{n-1} z + a_n, \quad a_0 \neq 0.$$

The proposed new algorithm consists of four steps.

Step 1: From the coefficients of $f(z)$, construct the $n\times n$ determinant

$$D = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & \cdots \\ a_0 & a_2 & a_4 & \cdots & \cdots & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots & \cdots \\ 0 & a_0 & a_2 & a_4 & \cdots & \cdots \\ 0 & 0 & a_1 & a_3 & a_5 & \cdots \\ 0 & 0 & a_0 & a_2 & a_4 & \cdots \\ & & \cdots\cdots\cdots & & & \end{vmatrix}$$

Step 2: Apply elementary row operations to bring D to an upper triangular determinant.

Step 3: Define the row

$$d = [d_{11}, d_{22}, d_{33}, \ldots, d_{nn}]$$

consisting of the entries on the main diagonal of the determinant arising in Step 2.

Step 4: The number of sign changes in row d equals the number of roots of $f(z)$ with positive real parts.

Therefore, if all entries in d are positive, then $f(z)$ has no zeros with positive real parts.

If in Step 2 a zero row arises, the technique proposed in [8] is applied. This technique consists of replacing the zero row by the derivative of the auxiliary polynomial constructed from the coefficients of the row which precedes the zero row. Concrete highlights of this case will be given in Examples 3, 4, and 5 below.

3. Illustrative Examples

In this section we illustrate the simplicity of the new algorithm with 5 examples of arbitrary polynomials of various degrees.

Example 1: Consider the polynomial

$$f(z) = z^5 + 7z^4 + 23z^3 + 47z^2 + 52z + 30.$$

Then,

D =

7	47	30	0	0
1	23	52	0	0
0	7	47	30	0
0	1	23	52	0
0	0	7	47	30

By elementary row operations, we get:

D =

7	47	30	0	0
1	114/7	334/7	0	0
0	0	1510/57	30	0
0	0	0	4420/151	0
0	0	0	0	30

In the sequence 7, 114/7, 1510/57, 4429/151, 30 there is no sign change. Therefore $f(z)$ has no zeros with positive real part.

In fact, the zeros of $f(z)$ are:

$$-2, -1 \pm 2i, -1 \pm 3i.$$

Example 2: Consider the polynomial

$$f(z) = z^5 + 5z^4 + 8z^3 + 16z^2 + 55z + 75.$$

Then,

D =

5	16	75	0	0
1	8	55	0	0
0	5	16	75	0
0	1	8	55	0
0	0	5	16	75

By elementary row operations, we get:

D =

5	16	75	0	0
0	24/5	40	0	0
0	0	−77/3	75	0
0	0	0	4160/77	0
0	0	0	0	75

In the sequence 5, 24/5, −77/3, 4160/77, 75 there are two sign changes. According to our claim, $f(z)$ should have two zeros with positive real parts. In fact, this is true since the zeros of $f(z)$ are

$$-3, -2 \pm i, 1 \pm 2i.$$

Example 3: Consider the polynomial

$$f(z) = z^6 + 6z^5 + 17z^4 + 38z^3 + 58z^2 + 56z + 24.$$

Then,

D =

6	38	56	0	0	0
1	17	58	24	0	0
0	6	38	56	0	0
0	1	17	58	24	0
0	0	6	38	56	0
0	0	1	17	58	24

By elementary row operations, we get:

D =

6	38	56	0	0	0
0	32/3	146/3	24	0	0
0	0	85/8	85/2	0	0
0	0	0	6	24	0
0	0	0	0	0	0
0	0	0	0	6	24

A zero row appears in the 5th row.

According to [8], the zero row will be replaced by the coefficients of the derivative of the auxiliary polynomial corresponding to the previous 4th row which is

$$\begin{vmatrix} 0 & 0 & 0 & 6 & 24 & 0 \end{vmatrix}$$

So, the auxiliary polynomial and its derivative are

$$g(z) = 6z^2 + 24$$

and

$$g'(z) = 12z$$

Note that the zeros of $g(z)$ are $\pm 2i$ which are indeed roots of the original function $f(z)$.

Now, the fifth zero row will be replaced by the row

0	0	0	0	12	0

and the modified determinant is

6	38	56	0	0	0
0	32/3	146/3	24	0	0
0	0	85/8	85/2	0	0
0	0	0	6	24	0
0	0	0	0	12	0
0	0	0	0	6	24

In the sequence 6, 32/3, 85/8, 6, 12, 24 there is no sign change. According to our claim,

$f(z)$ should have no zeros with positive real parts. In fact, this is true since the zeros of

$f(z)$ are

$$-1, -3, \pm 2i, -1 \pm i.$$

Example 4: Consider the polynomial

$$f(z) = z^5 + 5z^4 + 7z^3 - 5z^2 - 44z - 60.$$

Since the coefficients have different signs, $f(z)$ must have at least one zero with positive real part.

D =

5	–5	–60	0	0
1	7	–44	0	0
0	5	–5	–60	0
0	1	7	–44	0
0	0	5	–5	–60

By elementary row operations, we get:

D =

5	–5	–60	0	0
0	8	–32	0	0
0	0	15	–60	0
0	0	0	0	0
0	0	0	15	–60

A zero row appears in the 4th row.

As in the previous example, the zero row will be replaced by the coefficients of the derivative of the auxiliary polynomial corresponding to the previous 3rd row which is

0	0	15	–60	0

So, the auxiliary polynomial and its derivative are

$$g(z) = 15z^2 - 60$$

and

$$g'(z) = 30z$$

Note that the zeros of $g(z)$ are ± 2 which are indeed roots of the original function $f(z)$.

Now, the fourth zero row will be replaced by the row

0	0	0	30	0

and the modified determinant is

5	–5	–60	0	0
0	8	–32	0	0
0	0	15	–60	0
0	0	0	30	0
0	0	0	15	–60

In the sequence 5, 8, 15, 30, −60 there is only one sign change. According to our claim,

$f(z)$ should have only one zero with positive real part. In fact, this is true since the zeros of $f(z)$ are

$$\pm 2, -3, -1 \pm 2i\,.$$

Example 5: Consider the polynomial

$$f(z) = z^7 + 5z^6 + 15z^5 + 19z^4 + 31z^3 + 171z^2 - 47z - 195.$$

Since the coefficients have different signs, $f(z)$ must have at least one zero with positive real part.

D =

5	19	171	–195	0	0	0
1	15	31	–47	0	0	0
0	5	19	171	–195	0	0
0	1	15	31	–47	0	0
0	0	5	19	171	–195	0
0	0	1	15	31	–47	0
0	0	0	5	19	171	–195

By elementary row operations, we get:

D =

5	19	171	–195	0	0	0
0	56/5	–16/5	–8	0	0	0
0	0	143/7	1222/7	–195	0	0
0	0	0	–1088/11	1088/11	0	0
0	0	0	0	195	–195	0
0	0	0	0	0	0	0
0	0	0	0	0	195	–195

A zero row appears in the 6^{th} row.

The zero row will be replaced by the coefficients of the derivative of the auxiliary polynomial corresponding to the previous 5^{th} row which is

0	0	0	0	195	–195	0

So, the auxiliary polynomial and its derivative are

$$g(z) = 195z^2 - 195$$

and

$$g'(z) = 390z$$

Note that the zeros of $g(z)$ are ± 1 which are indeed roots of the original function $f(z)$.

Now, the sixth zero row will be replaced by the row

0	0	0	0	0	390	0

and the modified determinant is

D =

5	19	171	−195	0	0	0
0	56/5	−16/5	−8	0	0	0
0	0	143/7	1222/7	−195	0	0
0	0	0	−1088/11	1088/11	0	0
0	0	0	0	195	−195	0
0	0	0	0	0	390	0
0	0	0	0	0	195	−195

In the sequence 5, 56/5, 143/7, −1088/11, 195, 390, −195 there are three sign changes. According to our claim, $f(z)$ should have three zeros with positive real part. In fact, this is true since the zeros of $f(z)$ are

$$\pm 1, -3, -2 \pm 3i, 1 \pm 2i$$

4. Concluding Remarks

A novel algorithm is proposed to determine the number of zeros of a real polynomial with positive or negative real parts and the number of imaginary zeros. This novel approach has three main advantages:

1. Its elegant simplicity illustrated in the given examples where only elementary row operations of a certain determinant are required.
2. Its ability to determine the number of zeros of the characteristic polynomial of the system with positive or negative real parts and the number of imaginary zeros.
3. The concept of stability of continuous-time systems of differential falls out as a corollary of the new algorithm. This is Step 4 of the algorithm which states that if all entries in row d (entries on the main diagonal) are positive, then $f(z)$ has no zeros with positive real parts.

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