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A fractional approach to Hamiltonian-generalized classical fields : The Hamilton-Jacob technique

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Abstract

We explore classical fields by employing fractional differentials via the fractional Hamilton-Jacobi equation. Specifically, the fractional Hamiltonian equations are established for a particular scenario involving classical fields. This formulation gives rise to equations that bear resemblance to those in classical field theory. In the Hamilton-Jacobi system, we handle an inconsistent Lagrangian comprising variables that represent the scalar field and the fermionic field. To represent the formulas of motion, we construct overall differential formulas with multiple parameters. By considering integrability conditions, we express the system through path integration.

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1. Introduction

Fractional calculus, a development of classical mathematics, was introduced by W. Leibniz in 1695, laying the groundwork for this mathematical branch. It defines derivatives and integrals of any noninteger order, encompassing complex cases. Initially, it evolved as a theoretical area explored by eminent mathematicians such as Euler, Lagrange, and Fourier. Various disciplines, including fractal theory, viscoelasticity, electrodynamics, optics, and thermodynamics, have seen renewed interest in fractional calculus [1–4]. Leibniz's early contributions spurred a rapid expansion of knowledge [5–7]. fractional derivatives have gained significance in mathematics, engineering, physics, and applied mathematics [8–11]. Attempts in Lagrangian and Hamiltonian mechanics to incorporate non-conservative forces were made, yielding classical results. In Other study [12], Maxwell's equations were obtained through the fractional Euler-Lagrange formulations and fractional Hamilton's formulations of motion. This particular formulation aligned with classical results. High-level fractional-order derivatives, computed using the fractional derivative [13], revealed conserved variables, including momentum, energy density, and Poynting's. Baleanu et al. [14] proposed an integral identity for partly differentiable functions, extended Raina integral inequalities, and defined Riemann-Liouville and Caputo fractional derivatives with unique properties. On the other hand, the Riemann-Liouville differential operator and Caputo differential operator do not satisfy the Leibniz law and chain law. Khalil, Horani, Yousef, and Sababheh presented conformable derivative, overcoming limitations of other fractional derivatives[15]. This derivative applies the Leibniz law and the chain law. Lobo [16] used set theory to solve a nonlinear Hamilton-Jacobi problem by building characteristic equations, classifying, extracting accurate solutions, and graphically representing results.

Considering these factors, we extend previous analysis on Hamilton's formulations for classical fields to incorporate fractional derivatives, ensuring a more cohesive exploration of mathematical and physical principles.

This essay is presented as organized: Section 2 provides a quick examination of the fractional generalized Euler equation in Lagrangane form. Section 3 presents fractional Hamilton-Jacobi analysis for constrained systems.

Section 4 presents a fractional approach to a complex scalar field using a limited version of Dirac's formulation. Section 5 uses an illustrative example from Hamilton-Jacobi Theory: Major Contributions in Constrained Systems. The conclusions are given in Section 6.

2. Fractional Generalized Euler Equation in Lagrangane Form

We will now examine the concept of conformable calculus [17]. For values of $0 < \alpha \leq 1$, the left conformable derivative (CFD) is described as follows:

$$\mathrm{D}^{\alpha}_{s|t}\mathrm{f}(\mathrm{t}) = \lim_{\epsilon\to\infty} \frac{(t+\epsilon(t-s)^{1-\alpha}) - f(t)}{\epsilon} \tag{2.1}$$

and the right CFD is described as

$$\mathrm{D}^{\alpha}_{t|s'}\mathrm{f}(\mathrm{t}) = -\lim_{\epsilon\to\infty} \frac{\left(\mathrm{t}+\epsilon\left(\mathrm{s}'-\mathrm{t}\right)^{1-\alpha}\right) - \mathrm{f}(\mathrm{t})}{\epsilon} \tag{2.2}$$

In the case where $0 < \alpha \leq 1$, the left conformable integral (CFI) is described as:

$$\mathrm{I}^{\alpha}_{s|t}\mathrm{f}(\mathrm{t}) = \int_{\mathrm{s}}^{\mathrm{t}} (\delta - \mathrm{s})^{1-\alpha}\,\mathrm{f}(\delta)\,\mathrm{d}\delta \tag{2.3}$$

and the right CFI is defined as

$$I^{\alpha}_{t|s'}f(\mathrm{t}) = \int_{\mathrm{t}}^{s'} (s' - \delta)^{1-\alpha}\,\mathrm{f}(\delta)\,\mathrm{d}\delta \tag{2.4}$$

The CFD and CFI obey the following relations:

$$\mathrm{D}^{\alpha}_{s|t} I^{\alpha}_{s|t}\mathrm{f}(\mathrm{t}) = \mathrm{f}(\mathrm{t}) \tag{2.5}$$

$$I^{\alpha}_{t|s'}\,\mathrm{D}^{\alpha}_{t|s'}\mathrm{f}(\mathrm{t}) = \mathrm{f}(\mathrm{t}) - \mathrm{f}(a) \tag{2.6}$$

Moving ahead, our attention will be directed toward the scenario where s = 0, and we will employ the notation $\mathrm{D}^{\alpha}_{s|t}$ to represent $\left(\mathrm{D}^{\alpha}_{s|t} = D^{\alpha}_{t} t\right)$. The conformable derivative (CFD) will then be defined in this context.

$$\mathrm{D}^{\alpha}_{t}\mathrm{f}(\mathrm{t}) = \lim_{\epsilon\to\infty} \frac{(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{2.7}$$

Given that $0 < \alpha \leq 1$ and D^{α}_{x} represents a right CFD, the definition of CFD can be reformulated using l'Hôpital's rule as follows:

$$\mathrm{D}^{\alpha}_{t}\mathrm{f}(\mathrm{t}) = \mathrm{f}'(\mathrm{t})t^{1-t} \tag{2.8}$$

The following properties are satisfied by the CFD:

1. **Linearity**:

$$\mathrm{D}^{\alpha}_{t}\big(af(t) + bg(t)\big) = aD^{\alpha}_{t}f(t) + bg(t) \tag{2.9}$$

2. **Leibniz rule:**

$$\mathrm{D}^{\alpha}_{t}\big(f(t)g(t)\big) = [D^{\alpha}_{t}f(t)]g(t) + f(t)D^{\alpha}_{t}g(t) \tag{2.10}$$

3. **Chain rule:**

$$\mathrm{D}^{\alpha}_{t}f\big(g(t)\big) = \left[D^{\alpha}_{g(t)}f\big(g(t)\big)\right][D^{\alpha}_{t}g(t)]g(t)^{\alpha-1} \tag{2.11}$$

To get the fractional form of the Euler-Lagrange formulations, we shall add a fractional function.

$$S = I^{\alpha}_{0|t_0}\, \mathcal{L}(q_i, D^{\alpha}_t q_i) \tag{2.12}$$

To determine the condition for a local minimum $J^{\alpha}[q_i]$ if we examine a new fractional functional that depends on the parameter ϵ^{α}

$$S = I^{\alpha}_{0|t_0}\, \mathcal{L}\big(q_i(t, \epsilon^{\alpha}), D^{\alpha}_x q_i(t, \epsilon^{\alpha})\big) \tag{2.13}$$

Where

$$q_i(t, \epsilon^{\alpha}) = q(t) + \epsilon^{\alpha} \eta(\mathrm{t}) \tag{2.14}$$

$$D^{\alpha}_t q_i(t, \epsilon^{\alpha}) = D_t\ q(t) + \epsilon^{\alpha} D_t\ \eta(\mathrm{t}) \text{ and } \eta(0) = \eta(t_0) = 0 \tag{2.15}$$

The condition for an extreme value is obtained as $D^{\alpha}_{\epsilon} J^{\alpha}[\epsilon^{\alpha}] = 0$. Where D^{α}_{ϵ} is a brief notation for $D^{\alpha}_{0|\epsilon}$.Then, from the chain rule of the fractional derivative, we get:

$$D^{\alpha}_t S = I^{\alpha}_{0|t_0} \left[D^{\alpha}_{q_i} \mathcal{L}(D^{\alpha}_{\epsilon} q_i {q_i}^{\alpha-1}) + D^{\alpha}_{D^{\alpha}_t q_i} \mathcal{L}(D^{\alpha}_{\epsilon} q_i)(D^{\alpha}_t q_i)(D^{\alpha}_t q_i)^{\alpha-1} \right] \tag{2.16}$$

$$= I^{\alpha}_{0|t_0} \left[D^{\alpha}_{q_i} \mathcal{L}({q_i}^{\alpha-1})\eta + D^{\alpha}_{D^{\alpha}_t q_i} \mathcal{L}(D^{\alpha}_t q_i)^{\alpha-1}(D^{\alpha}_t \eta) \right] = 0 \tag{2.17}$$

We get the following by using Eq. (15) and integration by parts:

$$I^{\alpha}_{0|t_0} \left[D^{\alpha}_{q_i} \mathcal{L}(\ {q_i}^{\alpha-1})\eta - D^{\alpha}_t \left(D^{\alpha}_{D^{\alpha}_t q_i} \mathcal{L}(D^{\alpha}_t q_i)^{\alpha-1} \right)(\eta) \right] \tag{2.18}$$

The fractional Euler-Lagrange equation follows from this.

$$D^{\alpha}_{q_i} \mathcal{L}(q^{\alpha-1}) = D^{\alpha}_t \left(D^{\alpha}_{D^{\alpha}_t q_i} \mathcal{L}(D^{\alpha}_t q)^{\alpha-1} \right) \tag{2.19}$$

From Eq. (2.8), we know

$$D^{\alpha}_q \mathcal{L}(q)^{\alpha-1} = \partial_q \mathcal{L},$$

Taking the conformable derivative on both sides, we obtain:

$$D^{\alpha}_{D^{\alpha}_t q_i} \mathcal{L}(D^{\alpha}_t q_i)^{\alpha-1} = \partial_{D^{\alpha}_t q_i} \mathcal{L} \tag{2.20}$$

Using Eq. (2.20), we can rephrase the fractional Euler-Lagrange equation as follows*:*

$$\frac{\partial \mathcal{L}}{\partial q_i} - D^{\alpha}_t \left(\frac{\partial \mathcal{L}}{\partial D^{\alpha}_t q_i} \right) = 0 \tag{2.21}$$

The equation that results from expanding the expression (f) in terms of (L) and (Y) to the form (ϕ^{+}, ϕ) is as follows.

$$\frac{\partial \mathcal{L}}{\partial \phi^{+}} - \partial_x \left(\frac{\partial \mathcal{L}}{\partial D^{\alpha}_t \phi^{+}} \right) = 0 \tag{2.22}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_x \left(\frac{\partial \mathcal{L}}{\partial D^{\alpha}_t \phi} \right) = 0 \tag{2.23}$$

Upon utilizing Eq.s (2.22 and2.23) from the Euler-Lagrange equation, we obtain the subsequent equation:

$$\frac{\partial L}{\partial (D^{\alpha}_{\tau} \phi)} = D^{\alpha}_{\tau} \phi^{+} \tag{2.24}$$

$$\frac{\partial L}{\partial (D^{\alpha}_{\tau} \phi^{+})} = D^{\alpha}_{\tau} \phi \tag{2.25}$$

If α goes to 1, then Eq.s((2.24) and (2.25)) go to the standard equations. It's significant to note that the results obtained are consistent with those reported in Reference [18].

3. Fractional Hamilton-Jacobi Analysis for Constrained Systems

Within this portion, we analyze restricted models using the Hamilton-Jacobi strategy, which solves the gauge-fixing issue naturally. [19] developed a new way of researching unique systems. He started with the Hess matrix elements Aik of the Lagrangian second derivatives L=$\mathcal{L}(q_i, D_t^{\alpha} q_i, \tau)$ i=1,2,…….,n, which are defined as:

$$A_{ij} = \frac{\partial^2 \mathcal{L}(q_i, D_t^{\alpha} q_i, \tau)}{\partial(D_t^{\alpha} q_i)\partial(D_t^{\alpha} q_j)} \tag{3.1}$$

r < n, with rank (n-r) -dependent moments. The Hamilton-Jacobi Equations (HJPDE) set is derived using an analogous Lagrangian technique. The generalized moments associated with the generalized coordinates Pi are defined as:

$$\begin{cases} p_a = \frac{\partial L}{\partial(D_t^{\alpha} q_a)} \quad a\ increases\ form\ 1\ to\ n-r\ by\ one & (3.2) \\ p_{\mu} = \frac{\partial L}{\partial(D_t^{\alpha} q_{\mu})} \quad \mu\ increases\ form\ n-r+1\ to\ n & (3.3) \end{cases}$$

Due to the system's singularity, we can solve Eq. (3.2) for $D_t^{\alpha} q_a$ as:

$$D_t^{\alpha} q_a = D_t^{\alpha} q_a\left(q_i, D_t^{\alpha} q_{\mu}, p_{a;} \tau\right) = W_b \tag{3.4}$$

We find the constraints by substituting Eq. (3.4) into Eq. (3.3).

$$H'_{\mu} = H_{\mu}(\tau, q_i, p_a) + p_{\mu} \tag{3.5}$$

Where

$$H_{\mu} = -\frac{\partial L}{\partial(D_t^{\alpha} q_{\mu})}\bigg|_{D_t^{\alpha} q_{a=} w_a} \tag{3.6}$$

Within this approach, the established Hamiltonian is specified as

$$\mathrm{H_0 = -L + p_a D_t^{\alpha} w_a - D_t^{\alpha} q_{\mu} H_{\mu}} \tag{3.7}$$

The expression $\mathrm{H_{\mu}}$, like the function $\mathrm{H_0}$, is not clearly defined formula of the velocities $\mathrm{D_t^{\alpha} q_v}$. As a result, for a function extremum, the Hamilton-Jacobi function $\mathrm{S(\tau, q_i)}$ must simultaneously fulfill the following collection of Hamilton-Jacobi expressions (HJPDE):

$$H'_{\alpha}\left(t_{\beta}, q_{\beta}, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial t_0}\right) = 0 \tag{3.8}$$

Where

$\alpha, \beta = 0, n-r+1, \dots \dots, n;$

with a step of one , the sequence a spans from 1 to n-r

$a = 1,2 \dots\dots\dots., n - r$

The sequence a spans from 1 to $n - r$ with an of one for each term

And

$$H'_\alpha = \pi_\alpha + H_\alpha$$

To explain the canonical relations of motion, sum differential mathematical statements in parameters t_B are used.

$$\begin{cases} dq_\alpha = \frac{\partial H'_\alpha}{\partial p_p} dt_\alpha, \begin{cases} Sequence\ p\ from\ 1\ to\ n\ increment\ by\ one; \\ Values\ in\ \alpha\ span\ from\ 0\ to\ n \end{cases} . & (3.9) \\ dp_a = -\frac{\partial H'_\alpha}{\partial q_p} dt_\alpha \quad with\ an\ increment\ of\ one, \mu\ spans\ 1\ to\ n - r & (3.10) \\ dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha \qquad Sequence\ \alpha\ from\ n - r + 1\ to\ n & (3.11) \\ dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_\alpha} dt_\alpha\right), \end{cases}$$

where

$$\mathrm{Z} = \mathrm{S}(t_\alpha, q_a), \tag{3.12}$$

Being the center of attention. As a result, the evaluation of a constrained model includes limited to solving expressions (3.9–3.11) under constraints.

$H'_\alpha = (t_\beta, q_a, p_i) = 0$ In the sequence α, values range from 0 to n

(3.13)

Considering the expressions previously mentioned are sum differential relations, integrability requirements must be verified. These formulas of motion are integrable on the condition that and uniquely if the variations of H'_α dissipate in the same way, specifically

$$dH'_\alpha = 0$$

Should the requirements not be met in the same way, they should be regarded as extra limitations, and the differences should be analyzed again. Until a set of conditions disappears in all its variations, this procedure is repeated. This section will look at constraint systems using Dirac's technique first, then the Hamilton-Jacobi method.

4. Fractional approach to a complex scalar field using a constrained Dirac's formulation

Consider a Lagrangian scalar field and a complex scalar field [19].

$$L = \bar{\psi}(x)\left(i\gamma^\mu D^\alpha_{x_\mu} - m\right)\psi(x) + \frac{1}{2} D^\alpha_{x_\mu}\sigma(x)\, D^\alpha_{x^\mu}\sigma(x) - \frac{1}{2} M^2\, \sigma^2(x) \tag{3.14}$$

We are employ the Minkowski metric $\eta_{\mu\nu} diag((+1, -1, -1, -1))$. The Lagrangian (3.14) is singular, as the order of the Hess matrix (1) can be expressed as one. To rewrite the complex scalar field density in fractional expression, we start with the concept.

$$\begin{cases} D^{\alpha}_{x^{\mu}} = \left(D^{\alpha}_{t}, D^{\alpha}_{x^{i}}\right) = \partial^{\mu} \\ D^{0}_{\mu} = \left(D^{0}_{t}, D^{0}_{x^{i}}\right) = (\partial^{0}, \partial^{i}) \end{cases} \tag{3.15}$$

The statement is indicating that when μ is equal to zero and i takes on the values of 1,2,and 3.

$$L = \bar{\psi}(x)\left(i\gamma^{\mu}D^{\alpha}_{x_{\mu}} - m\right)\psi(x) + \frac{1}{2}\, D^{\alpha}_{x_{\mu}}\sigma(x)\, D^{\alpha}_{x^{\mu}}\sigma(x) - \frac{1}{2}M^{2}\,\sigma^{2}(x) \tag{3.16}$$

The generalized moments (3.3) and (3.4) are expressed as:

$$\begin{cases} p = \frac{\partial L_{\tau}}{\partial(D^{\alpha}_{t}\sigma)} = D^{0}_{t}\sigma & a. \\ p_{\bar{\psi}} = \frac{\partial L_{\tau}}{\partial(\bar{\psi})} = i\bar{\psi}\gamma^{0} = -H_{\psi}\ \ b & b. \\ p_{\psi} = \frac{\partial L_{\tau}}{\partial(D^{\alpha}_{t}\psi)} = i\bar{\psi}\gamma^{0}p_{\psi} = \frac{\partial L_{\tau}}{\partial(D^{\alpha}_{t}\psi)} = i\bar{\psi}\gamma^{0} & c \end{cases} \tag{3.17}$$

We must emphasize the need to use spinor indexes with caution. Given that ψ is a column vector and $\bar{\psi}$ is $(1 \times n)$ vector, $p_{\bar{\psi}}$ will be represented as a column vector

The standard Hamiltonian H_{ψ} is written similar to:

$$H_{0} = \begin{cases} -L + (w)p + (D^{0}_{t}\psi)(p_{\psi})\Big|_{p_{\psi}=H_{\psi}} + (D^{0}_{t}\bar{\psi})(p_{\bar{\psi}})\Big|_{p_{\bar{\psi}}=H_{\bar{\psi}}} \\ or \\ \frac{1}{2}p^{2} - \bar{\psi}(i\gamma^{\mu}D^{0}_{a}\psi - m\psi) + \frac{1}{2}(D^{0}_{a}\sigma D^{0}_{t}\sigma - M^{2}\sigma^{2}) + \\ \lambda_{\psi}\left(p_{\psi} - i\,\bar{\psi}\,\gamma^{0}\right) + \lambda_{\bar{\psi}}p_{\bar{\psi}} \qquad a = 1,2,3. \end{cases} \tag{3.18}$$

Eqs. (3.17.b) and (3.17.c) lead to the primary constraints

$$H_{\psi} = \begin{cases} H'_{\psi} = p_{\psi} + H_{\psi} = p_{\psi} - i\bar{\psi}\gamma^{0} = 0. \\ or \\ H'_{\bar{\psi}} = p_{\bar{\psi}} + H_{\bar{\psi}} = p_{\bar{\psi}} = 0. \end{cases} \tag{3.19}$$

Correspondingly. These constraints result in the overall Hamiltonian

$$H_{T} = \begin{cases} H'_{\psi} = H_{0} + \lambda_{\psi}H'_{\psi} + \lambda_{\bar{\psi}}H'_{\bar{\psi}}\ . \\ or \\ \frac{1}{2}p^{2} - \bar{\psi}(i\gamma^{\mu}D^{0}_{a}\psi - m\psi) + \frac{1}{2}(D^{0}_{a}\sigma D^{0}_{a}\sigma - M^{2}\sigma^{2}) + \lambda_{\bar{\psi}}H'_{\bar{\psi}} \end{cases} \tag{3.20}$$

Based on the coherence requirements, the time derivative of the initial constraints must equal zero, which means

$$D^{\alpha}_{t}H'_{\psi} = \left\{H'_{\psi}, H_{T}\right\} = -(iD^{0}_{a}\bar{\psi}\gamma^{a} + m\bar{\psi}) - i\lambda_{\bar{\psi}}\gamma^{0} \approx 0 \tag{3.21}$$

$$D^{\alpha}_{t}H'_{\bar{\psi}} = \left\{H'_{\bar{\psi}}, H_{T}\right\} = -(iD^{0}_{a}\psi\gamma^{a} + m\psi) - \lambda_{\psi}\gamma^{0} \approx 0 \tag{3.22}$$

The multipliers $\lambda_{\bar{\psi}}$and λ_{ψ} are defined in Eqs. (3.21) and (3.22) as follows:

$$i\lambda_{\bar{\psi}}\gamma^{0} = -(iD^{0}_{a}\bar{\psi}\gamma^{a} + m\bar{\psi}) - i\lambda_{\bar{\psi}}\gamma^{0} \approx 0 \tag{3.23}$$

$$D^{\alpha}_{t}H'_{\bar{\psi}} = i\lambda_{\psi}\gamma^{0} = -(iD^{0}_{a}\psi\gamma^{a} + m\psi) \tag{3.24}$$

By multiplying Eq.s (3.23) on the right and (3.24) on the left by -iγ^{0}, we get the following result:

$$\lambda_{\bar{\psi}} = i\lambda_{\bar{\psi}}\gamma^0(-D_a^0\bar{\psi}\gamma^a\gamma^0 + im\bar{\psi}\gamma^0) \tag{3.25}$$

$$\lambda_{\psi} = -\gamma^0(D_a^0\psi\gamma^a + im\psi) \tag{3.26}$$

It is the case that no other constraints. Using proper linear formulations of constraints, all second-class constraints must be found.

$$\phi_1 = H'_{\psi} = p_{\psi} - i\bar{\psi}\gamma^0 \quad \text{and} \quad \phi_2 = H'_{\bar{\psi}} = p_{\bar{\psi}} \tag{3.27}$$

The Hamiltonian as a whole is gradually vanishing. It is possible to express it entirely in terms of limitations of the second type, as:

$$H_T = \frac{1}{2}p^2 - \bar{\psi}(i\gamma^\mu D_a^0\psi - m\psi) + \frac{1}{2}(D_a^0\sigma D_a^0\sigma - M^2\sigma^2) + \lambda_\psi\phi_1 + \lambda_{\bar{\psi}}\phi_2 \tag{3.28}$$

The laws of motion are read as:

$$D_t^\alpha\sigma = \{\sigma, H_T\} = p \tag{3.29}$$

$$D_t^\alpha\psi = \{\psi, H_T\} = \lambda_\psi \tag{3.30}$$

$$\bar{\psi} = \{\psi\ , H_T\} = \overline{\lambda_\psi} \tag{3.31}$$

$$D_t^\alpha p = \{p, H_T\} = -M^2\sigma \tag{3.32}$$

$$D_t^\alpha p_\psi = \{p_\psi, H_T\} = -\bar{\psi}\left(i\overline{D_a^0}\gamma^a + m\right) \tag{3.33}$$

$$D_t^\alpha \overline{p_\psi} = \{p_{\bar{\psi}}, H_T\} = \psi(iD_a^0\gamma^a - m) + i\psi\lambda_\psi \tag{3.34}$$

Using Eq. (3.25) and differentiating Eq. (3.32) with regard to time, we have:

$$D_t^{2\alpha}\sigma + M^2\sigma = 0 \tag{3.35}$$

Replacing Eqs. (3.25) and (3.24) into Eqs. (3.32) and (3.33) accordingly, we obtain:

$$\psi\left(\ iD_{x_\mu}^\alpha\gamma^\mu - m\right)\psi = 0 \tag{3.36}$$

$$\bar{\psi}\left(i\bar{D}_{x_\mu}^\alpha\gamma^\mu + m\right)\psi = 0 \tag{3.37}$$

Eq. (3.24) can be substituted with Eq. (3.34), producing:

$$D_{x_\mu}^\alpha p_{\bar{\psi}} = 0 \tag{3.38}$$

Now, the definition of the Hamilton-Jacobi partial differential formulas (HJPDE) set (3.8) is:

$$\begin{cases} H'_0 = p_0 + H_0 = p_0 + \frac{1}{2}p^2 - \bar{\psi}(i\gamma^\mu D_a^0\psi - m\psi) + \frac{1}{2}(D_a^0\sigma D_a^0\sigma - M^2\sigma^2) & (3.39) \\ H'_\psi = p_\psi + H_\psi = p_\psi - i\bar{\psi}\gamma^0 = 0 & (3.40) \\ H'_{\bar{\psi}} = p_{\bar{\psi}} + H_{\bar{\psi}} = p_{\bar{\psi}} = 0 & (3.41) \end{cases}$$

Thus, for Eqs. (3.9), (3.10), and (3.11), the overall differential equations are as follows:

$$d\sigma = pd\tau \tag{3.42}$$

$$dp = M^2\sigma d\tau \tag{3.43}$$

$$dp_{\bar{\psi}} = -\bar{\psi}\left(i\overline{D_a^0}\gamma^a + m\right)d\tau \tag{3.44}$$

$$dp_{\bar{\psi}} = (iD_a^0\gamma^a - m)\psi d\tau + i\gamma^0 d\psi \tag{3.45}$$

The prerequisites for integrability $H'_\alpha = 0$ implies that the variation of the restrictions $H'_\psi$and $H'_{\bar{\psi}}$should be zero, that is:

$$dH'_\psi = dp_\psi - id\bar{\psi}\gamma^0 = 0 \tag{3.46}$$

$$dH'_{\bar{\psi}} = dp_{\bar{\psi}} \tag{3.47}$$

We obtain the following equations of motion by inserting Eqs. (3.25) and (3.26) into Eqs. (3.30) and (3.31), respectively.

$$\psi\left(iD^\alpha_{x_\mu}\gamma^\mu - m\right)\psi = 0 \tag{3.48}$$

$$\bar{\psi}\left(i\overline{D^\alpha_{x_\mu}}\gamma^\mu + m\right) = 0 \tag{3.49}$$

$$D_t^\alpha \sigma = p \tag{3.50}$$

We obtain the following findings by Eqs. (3.43) alongside (3.44), respectively:

$$D_t^\alpha p = -M^2\sigma \tag{3.51}$$

$$D_t^\alpha p_\psi = \bar{\psi}\left(i\overline{D_a^0}\gamma^a + m\right) \tag{3.52}$$

$$D_t^\alpha p_{\bar{\psi}} = 0 \tag{3.53}$$

We can derive the subsequent equation of motion with respect to time by applying Eqs. (3.50) and (3.53)

$$D_t^{2\alpha}\sigma + M^2\sigma = 0 \tag{3.54}$$

5. Hamilton-Jacobi Theory in Constrained Systems

This section explores the key contributions of Hamilton-Jacobi's theory, particularly in the study of singular systems. It uses a real-world example to demonstrate its versatility and effectiveness in navigating complex systems. A constrained system example demonstrates the theory's ability to provide meaningful solutions and navigate challenges. It highlights its significance in understanding system behaviors, making it a valuable tool for tackling complexity.

5.1 *Hamilton-Jacobi Theory Contributions for Singular Conformable Lagrangian*

In recent times, there has been significant research focus on Hamilton-Jacobi theory, owing to its versatility in addressing singular systems. To propel advancements in this domain, numerous major contributions can be explored.

Comprehensive Analysis: The Hamilton-Jacobi formalism provides a comprehensive framework for studying singular systems, offering a holistic view of their dynamics.

Comparative Perspective: By utilizing both Hamilton-Jacobi and Dirac’s generalized Hamiltonian formalism, the study allows for a comparative analysis. This comparison enhances our understanding by highlighting differences and similarities in the insights gained from different formalisms.

Application to Physical Systems: The Hamilton-Jacobi formalism has been successfully applied to a range of physical systems [20], showcasing its adaptability and relevance in diverse scenarios.

This approach is widely applied in the physical sciences, and its contrast with Dirac's method advances our knowledge of theoretical physics' single-system dynamics.

5.2 *Illustrating Hamilton and Jacobi's Theory in Restricted Systems Through Ball Movement.*

The Hamilton-Jacobi theory simplifies trajectory determination in complex fields mathematically. It focuses on motion as a constrained system with intricate configurations, aiding boundary definition, speed monitoring, and assessing influencing forces. This approach assists researchers in studying motion aspects such as acceleration, velocity, and geometric constraints. Contributing to understanding dynamics in complex systems, the theory may yield more accurate trajectory determination, a clearer grasp of motion boundaries, and diverse force effects. It also enables a deeper examination of acceleration and velocity, offering comprehensive insights into influencing geometric constraints. This could enhance the understanding of complex system dynamics, potentially useful in designing devices or systems with intricate configurations. Applying the theory to ball motion on a complex surface could result in more precise path determination, a deeper understanding of motion boundaries, and the effects of interacting forces, ultimately improving the interpretation of acceleration and velocity over time.

6. Conclusion

Within this research, this have studied the restricted model of Lagrangians, including a fermionic and a scalar field, employing Dirac's Hamiltonian formalism and Hamilton-Jacobi formalism.

The total Hamiltonian, which is added to the canonical Hamiltonian, is obtained by multiplying the limitations by Lagrange multipliers using the Dirac technique. With the aim of to obtain the equations of motion, one must reframe such undefined unknown factors arbitrarily.

However, in the Hamilton-Jacobi formalism, Lagrange multipliers are not included in the canonical Hamiltonian. Both the consistency conditions and the integrability conditions provide constraints. The equations of motion in the Hamilton-Jacobi formulation are directly derived utilizing HJPDES as complete differential systems. Additionally, it is not essential to differentiate between the limitations of the first and second**.** The conformable derivative approach has attracted a lot of attention due to its accuracy in solving fractional differential equations and systems. where it is an adaptable tool that can solve singular fractional differential equations and generalize well-known transforms like Laplace and Sumudu. In addition to providing novel applications and comparisons, this work shows its effectiveness in fractional physical system solutions using conformable derivatives.

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