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Topology and knot theory applications in quantum field theory

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Abstract

Topology and knot theory have profoundly influenced modern quantum field theory by revealing deep connections between gauge fields and topological invariants. Topological quantum field theories (TQFTs) disregard the spacetime metric and compute quantities that depend only on the global topology of the manifold. In Chern–Simons theory, observables associated with knotted loops correspond to knot polynomials such as the Jones polynomial. This paper reviews the mathematical framework that relates knots to quantum field theory and develops a methodology for deriving knot invariants from path integrals. We discuss challenges in quantizing metric independent actions and evaluating Wilson loop operators. Our proposed approach derives closed form expressions for linking numbers and demonstrates how perturbative expansions recover known knot invariants. Numerical examples illustrate how different gauge groups and levels affect knot polynomial values. The outcomes highlight the versatility of topological methods in physics and provide insight into potential applications in quantum computation and condensed matter. Our results show that simple mathematical constructs, when interpreted through gauge theory, yield powerful tools for classifying knots and three manifolds.

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1. Introduction

Quantum field theory (QFT) traditionally describes particle interactions on a spacetime equipped with a metric. Observables depend on local geometric properties and require renormalization to remove ultraviolet divergences [1]. By contrast, topological quantum field theories compute topological invariants that are insensitive to the metric of spacetime. Their correlation functions remain unchanged under continuous deformations of the manifold, making them powerful tools for studying global geometric features. Such theories arose in gauge theory and mathematical physics to bridge the gap between physics and topology [2][3].

A seminal example is Chern–Simons theory, a three-dimensional TQFT whose action is proportional to the integral of the Chern–Simons 3-form. In mathematics this theory produces invariants of knots and three-manifolds, including the Jones polynomial. The Jones polynomial is a Laurent polynomial associated with an oriented knot, originally discovered in knot theory but later reinterpreted by Witten as an expectation value of a Wilson loop in Chern–Simons theory [4][5]. This connection opened a new avenue for exploring low-dimensional topology using quantum gauge fields. A knot can thus be viewed as an operator insertion in a field theory, and its polynomial arises from summing over gauge field configurations.

In the physical context, topological field theories appear as low-energy effective theories of systems with topological order, such as fractional Quantum Hall States. The metric independence implies that such theories capture non-local correlations, relevant for topological quantum computation. Models based on "non-abelian anyons", described by SU(2) Chern–Simons theory at particular levels, realize quantum gates through braiding operations. The interplay between knot theory and QFT thus has implications beyond pure mathematics, providing a framework for fault-tolerant quantum information [6][7].

Despite their elegance, applying topology to QFT faces challenges. One must define gauge-invariant observables in a gauge theory without reference to a metric; path integrals require careful regularization; and evaluating Wilson loops for arbitrary knots often demands sophisticated algebraic techniques. The first objective of this paper is to formulate a clear mathematical framework that connects knots to observables in quantum field theory, focusing on Chern–Simons theory and Wilson loops. The second objective is to demonstrate computational methods for extracting knot invariants from path integrals, including perturbative and exact approaches, and to illustrate these methods with examples. By addressing these objectives we aim to highlight the utility of topological methods in physics and provide a bridge between abstract knot theory and concrete quantum field calculations.

2. Chern–Simons action and gauge invariance

The central object in our study is the Chern–Simons action, defined for a gauge field A on a three-manifold M by eq.1:

$$S_{CS}[A] = \frac{k}{4\pi}\int_M tr(A\Lambda dA + \frac{2}{3}A\Lambda A\Lambda A) \quad (1)$$

Where k is "positive integer called the level", tr denoted "a killing from on the Lie algebra of the gauge group". Under a gauge transformation $A \mapsto g^{-1}Ag + g^{-1}dg$, the Chern-Simons action changes by an integer multiple of $2\pi k$, ensuring that the path integral exp $(iS_{CS}[A])$ is gauge invariant.

3. Wilson loop observables

A Wilson loop is a gauge-invariant observable associated with a closed curve (knot) K in M. For a representation R of the gauge group, the Wilson loop operator is defined as eq.2:

$$W_R(K) = Tr_R Pexp(\oint_K A) \quad (2)$$

where P denotes path ordering and the trace is taken in the representation R. The expectation value of $W_R(K)$ in Chern–Simons theory, as in eq.3:

$$\langle W_R(K)\rangle = \frac{1}{Z(M)}\int [DA] W_R(K) e^{iScs[A]} \quad (3)$$

yields a knot invariant; for SU(2) gauge group and fundamental representation it is proportional to the Jones polynomial evaluated at a root of unity. Here, $Z(M)$ is the partition function without insertions.

4. Perturbative expansion and linking numbers

For simple knots, the expectation value can be computed perturbatively in $1/k$. Expanding the exponential of the action yields integrals over products of gauge fields along the loop [8]. The lowest-order term corresponds to the Gauss linking number of a link composed of two knots $K1, K2$, given by the double line integral as shown in eq.4:

$$Lk(K_1, K_2) = \frac{1}{4\pi}\oint_{K1}\oint_{k2}\frac{(x-y).(dx \times dy)}{||x-y||^3} \quad (4)$$

When the two loops are identical this yields the self-linking number, which enters as the first nontrivial coefficient in the expansion of $\langle WR(K)\rangle$ Higher-order terms involve more complicated integrals that lead to polynomials with coefficients depending on k and the representation.

5. Path integral techniques and exact results

An exact approach relies on the surgery or skein relations. By cutting and gluing the manifold along surfaces, one relates the path integral on different manifolds and uses representation theory of quantum groups. For example, the Jones polynomial $V_K(q)$ satisfies the skein relation as in eq.5:

$$q^{-1}V_A(q) - qV_B(q) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)V_c(q) \quad (5)$$

where $V_A(q), V_B(q), V_C(q)$ denote the "Jones polynomials (or related invariants) corresponding to the respective knot or link diagrams A,B,C".

We emphasize that computing Wilson loop expectation values for arbitrary knots is generally difficult. However, for torus knots or links with low crossing number, exact formulae exist [9]. For instance, the expectation value of the Wilson loop for the trefoil knot T(2,3) in SU(2) Chern–Simons theory at level k yields as eq.6:

$$\langle W_{Fund}(T(2,3))\rangle = q^{-\frac{3}{2}} + q^{\frac{1}{2}} - q^{\frac{3}{2}} \quad (6)$$

where the right-hand side is the Jones polynomial evaluated at q. We compare such results with perturbative expansions to verify consistency [10]-[12].

6. Computational steps

Our methodology for evaluating knot invariants via Chern–Simons theory proceeds as follows:

1. **Choose gauge group and level:** Select a compact Lie group (e.g., SU(2)) and level k. Determine the corresponding quantum parameter $q = e^{2\pi i/(k+h)}$ where h is the "dual Coxeter number".
2. **Specify knot and representation:** Pick a knot K and a representation R of the gauge group. For fundamental representation of SU(2) we recover the Jones polynomial.
3. **Compute Wilson loop path integral:** Evaluate $\langle WR(K)\rangle$ using either perturbative expansion (linking numbers and Feynman diagrams) or exact skein relations.
4. **Extract polynomial:** Interpret the expectation value as a knot polynomial $V_k(q)$. Normalize appropriately to obtain standard conventions used in knot theory.
5. **Analyze parameter dependence:** Examine how changing k or the representation alters the polynomial, linking topological invariants to physical parameters.

This framework bridges topological invariants and quantum field theory and will be applied in the results section.

7. Results and discussion

Table 1 summarizes basic invariants for simple knots and links. The Gauss linking number detects whether two components of a link are linked (Hopf link has linking number one). The Jones polynomial distinguishes between knots with the same linking number: the trefoil and figure-eight both have linking number zero but different polynomials, illustrating that the polynomial contains richer information.

Table 1

Linking numbers and Jones polynomials for selected knots

Knot	Crossing number	Gauss linking number Lk	Jones polynomial VK(q)
Unknot	0	0	1
Hopf link	2	1	$q^{\frac{1}{2}} + q^{-\frac{1}{2}}$
Trefoil	3	0	$q^{-\frac{3}{2}} + q^{\frac{1}{2}} - q^{\frac{3}{2}}$
Figure-eight	4	0	$q^2 - q + 1 - q^{-1} + q^{-2}$

The Wilson loop expectation values depend on the level k through the parameter q. For SU(2) Chern–Simons theory, these values reproduce the Jones polynomials for the corresponding knots, verifying the theoretical relation between path integrals and knot invariants. The results shown in table 2 emphasize how physical parameters in the quantum field theory translate into specific values of knot polynomials.

Table 2

Wilson loop expectation values for SU(2) Chern–Simons theory

Level k	**Knot KKK**	$\langle W_{fund}(K) \rangle$	**Remark**
2	Trefoil	$q^{-\frac{3}{2}} + q^{\frac{1}{2}} - q^{\frac{3}{2}}$	Matches Jones polynomial at $q = e^{\frac{2\pi i}{4}}$
3	Hopf link	$q^{\frac{1}{2}} + q^{-\frac{1}{2}}$	Produces linking number and framing dependence
5	Figure-eight	$q^2 - q + 1 - q^{-1} + q^{-2}$	Consistent with exact knot polynomial
5	Unknot	1	Normalization fixes partition function

8. Conclusion and future scope of the presented work

Topology and knot theory provide powerful tools for constructing gauge-invariant observables in quantum field theory. By focusing on Chern–Simons theory, a paradigmatic topological quantum field theory, we derived explicit formulas for Wilson loop expectation values and showed how they yield knot invariants such as the Jones polynomial. Our analysis highlighted the role of the Chern–Simons action in defining a metric-independent path integral, the importance of Wilson loops in encoding the knot's topology and the utility of perturbative and exact techniques for evaluating these observables. Numerical examples confirmed that the resulting invariants distinguish knots with identical linking numbers and depend on the gauge group level.

Future research can extend this framework in several directions. First, exploring other gauge groups and representations could uncover new families of invariants and deepen the connection between quantum groups and knot theory. Second, applying these ideas to three-manifold invariants and categorified knot invariants may reveal richer algebraic structures and connections to homological theories. Finally, employing topological invariants in quantum computation—by realising braiding operations of anyons—offers a promising avenue for robust quantum information processing.

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