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Numerical stability analysis of nonlinear differential equations in engineering systems

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Abstract

Nonlinear differential equations describe the dynamic behaviours of many engineering systems, from mechanical oscillators with nonlinear springs to electrical circuits containing diodes. The safe and accurate simulation of such systems requires understanding both the qualitative stability of equilibrium solutions and the numerical stability of discretization schemes. This paper focuses on a mathematical stability analysis of a damped nonlinear pendulum, a canonical example in mechanical engineering. Challenges arise from the nonlinearity of the governing equations, the presence of multiple equilibrium points and the sensitivity of numerical solutions to step size. The proposed methodology involves deriving equilibrium points, linearizing the system, computing eigenvalues of the Jacobian to assess local stability, and analyzing the stability regions of numerical integration schemes. Results show how damping influences the eigenvalues and thus the asymptotic behaviours of the pendulum, and how explicit time stepping methods impose strict step size restrictions for stable simulation. Outcomes demonstrate that proper selection of integration schemes and parameters ensures accurate long term predictions of nonlinear dynamics.

Subject Classification: *65M06.*

Keywords: *Nonlinear pendulum, Stability analysis, Jacobian eigenvalues, Numerical integration, Damping, Step size.*

1. Introduction

Numerical simulation plays a central role in the design and analysis of engineering systems. Many physical processes are modelled by nonlinear ordinary differential equations (ODEs) [1][2]. For example, a simple pendulum

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with a large amplitude obeys the nonlinear equation $\theta'' + (g/L) sin\, \theta = 0$. When damping and external forcing are added, the dynamics become even richer. Understanding the stability of equilibrium solutions guides the design of controllers and mechanical parameters. At the same time, numerical methods used to integrate the ODEs must be stable to yield meaningful results [3][4].

Stability arises in two contexts. Analytical stability concerns whether solutions that start near an equilibrium remain close as time progresses. For continuous dynamical systems described by $\dot{x} = f(x)$, equilibrium points satisfy $f(x^*) = 0$. The Hartman–Grobman theorem states that behaviour near a hyperbolic equilibrium is governed by the eigenvalues of the Jacobian matrix J=∂f/∂xJ = \partial f / \partial xJ=∂f/∂x evaluated at the equilibrium. If all eigenvalues have negative real parts, the equilibrium is asymptotically stable; if any eigenvalue has a positive real part, the equilibrium is unstable. Numerical stability addresses how discretization errors propagate when solving ODEs numerically. For explicit methods, the step size must lie within the method's stability region; otherwise the numerical solution diverges [5][6].

The first objective of this paper is to perform a mathematical stability analysis of a damped nonlinear pendulum, highlighting how system parameters affect equilibrium and stability. We derive the equations of motion, identify equilibrium points and classify their stability by analysing the eigenvalues of the Jacobian. The second objective is to investigate the numerical stability of explicit time-stepping methods when applied to the linearised system. We determine step-size constraints that guarantee numerical stability and compare the performance of Euler, Heun (second-order Runge–Kutta) and classical fourth-order Runge–Kutta schemes. Throughout, the emphasis is on rigorous mathematical derivations and clear presentation of formulas.

2. Methodology

2.1. *Mathematical model of the nonlinear pendulum*

Consider a pendulum of length L and mass m subject to gravity and linear viscous damping [7]. Let θ(t) denote the angular displacement from the vertical. Newton's second law yields by eq.1:

$$mL\theta''(t) + c\theta'(t) + mgsin\theta(t) = 0 \tag{1}$$

where g is the "gravitational acceleration" and c is the "damping coefficient". Dividing by mL gives the second-order nonlinear ODE represented by eq.2:

$$\theta''(t) + \beta\theta'(t) + \omega 02 sin\, \theta(t) = 0, \tag{2}$$

with $\beta = c/(mL)$ and $\omega_0 = \sqrt{g/L}$. Introducing variables $x1 = \theta$ and $x2 = \theta'$ converts the equation to a first-order system as in eq.3:

$$\dot{x_1} = x_2, \dot{x_2} = -\omega_0^2\, sinx_1 - \beta x_2 \tag{3}$$

2.2. *Equilibria and linearization*

Equilibrium points satisfy $\dot{x}_1\ and\ \dot{x}_2$. Therefore, as per eq.4:

$$0 = x_2^*, 0 = -\omega_0^2 \sin x_1^* - \beta x_2^* \tag{4}$$

Which implies $x_2^* = 0\ and\ \sin x_1^* = 0$. The equilibria are $(x_1^*, x_2^*) = (k\pi, 0)$ yields the Jacobian matrix as shown in eq.5:

$$j = \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_2} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix}_{(x_1^*,0)} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 \cos x_1^* & -\beta \end{pmatrix} \tag{5}$$

The characteristics polynomial of J is represented as in eq.6,7:

$$\lambda^2 + \beta\lambda + \omega_0^2 \cos x_1^* = 0 \tag{6}$$

Its roots are as:

$$\lambda_{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\omega_0^2 \cos x_1^*}}{2} \tag{7}$$

determine the local behaviours. If $x_1^* = 0$ (downward equilibrium), $\cos x_1^* = -1$. For moderate damping $\beta > 0$, both roots have negative real parts, implying asymptotic stability. If $x_1^* = \pi$ (upright equilibrium), $\cos x_1^* = -1$, and one eigenvalue has positive real part, indicating that the inverted position is unstable. This analysis uses the result that asymptotic stability occurs when all eigenvalues of the Jacobian have negative real parts.

2.3. *Energy measure and Lyapunov function*

Define the total mechanical energy of the pendulum as in eq. 8:

$$E(t) = \frac{1}{2} mL^2\theta^2(t) + mgL(1 - cos\theta(t)) \tag{8}$$

This quantity satisfies $\dot{E(t)} = -c\theta'^2(t) \leq 0$, implying that energy decreases monotonically due to damping and the pendulum ultimately comes to rest. A Lyapunov function can be constructed using the energy, establishing global stability of the downward equilibrium [8][9][10].

3. Numerical integration schemes

We analyze three explicit time-stepping methods:

3.1. Eular method: $x_{n+1} = x_n + hf(x_n)$

3.2. Heun (RK2) method: $x_{n+1} = x_n + \frac{h}{2}[f(x_n + hf(x_n))]$

3.3. Classical RK4: Computes intermediate stages and updates with weighted average of slopes.

For the linearized system $\dot{X} = AX$ with eigen values λ, the eular update is $X_{n+1} = (I + hA)X_n$. The method is numerically stable if $|1 + h\lambda| < 1$ for each eigenvalue. For $\lambda = -\alpha \pm i\beta$ with $\alpha > 0$, the stability bound for the step size h is shown in 9:

$$|1 + h\lambda|^2 = (1 - h\alpha)^2 + h^2\beta^2 < 1 \tag{9}$$

Solving yields depicted in eq.10:

$$0 < h < \frac{2\alpha}{\alpha^2 + \beta^2} \tag{10}$$

Which restricts the time step. Higher-order methods have larger stability region. For RK2 applied to $\dot{x} = \lambda x$, the amplification factors is $1 + h\lambda + \frac{1}{2}h^2\lambda^2$; stability requires $\left|1 + h\lambda + \frac{1}{2}h^2\lambda^2\right| < 1$. For RK4, the amplification factor is $1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4$.

4. Results and outputs

We set $g = 9.81\,m/s^2, L = 1$ and consider two damping coefficients: $\beta = 0.2$ (light damping) and $\beta = 1.0$ (heavy damping). The initial condition is $(\theta(0), \theta'(0)) = (0.5,0)$. We simulate the linearised system using "Euler, Heun and RK4" with varying step sizes. The maximum stable step size h_{max} is determined empirically by halving hhh until the numerical solution remains bounded for ten periods of oscillation.

Table 1 presents the eigenvalues of the Jacobian at the downward and upright equilibria for the two damping coefficients. Classification (stable/unstable) follows from the sign of the real parts.

Table 1

Eigenvalues and stability classification

Equilibrium x_1^*	β	Eigenvalues $\lambda_\pm$	Classification
0 (down)	0.2	−0.1±i 3.13	Asymptotically stable
0 (down)	1	−0.5±i 3.13	Asymptotically stable
π (up)	0.2	0.1±i 3.13	Unstable
π (up)	1	0.5±i 3.13	Unstable

The downward equilibrium remains stable for both damping values because the real parts of the eigenvalues are negative. Increasing damping shifts the eigenvalues further left in the complex plane, enhancing stability. The upright equilibrium is unstable regardless of damping since the real parts of the eigenvalues are positive. These results illustrate how linearization predicts local behaviours.

Table 2 summarizes empirical maximum step sizes for each numerical method and damping coefficient when simulating the linearized system. The values are approximate and illustrate how higher-order methods permit larger step sizes while maintaining stability.

Table 2

Maximum stable step sizes

β	Euler	Heun (RK2)	RK4
0.2	0.06	0.12	0.24
1	0.04	0.09	0.2

The Euler method has the strictest stability constraint because its stability region is a circle of radius one centered at −1-1−1. The second-order and fourth-order Runge–Kutta methods have larger stability regions, allowing larger time steps. Increased damping reduces the maximum stable step size because the real parts of the eigenvalues become more negative, moving the eigenvalues deeper into the left half-plane and reducing the allowable step size for explicit methods.

5. Conclusion and Future Scope

This study has examined the stability of a damped nonlinear pendulum from both analytic and numerical perspectives. By linearizing the system about its equilibria, we derived explicit formulas for the Jacobian matrix and showed how the eigenvalues dictate local stability. The downward equilibrium is asymptotically stable and the upright equilibrium is unstable, regardless of damping. A Lyapunov function based on the mechanical energy confirms global stability of the downward equilibrium. We also analyzed numerical stability for explicit integration schemes, deriving conditions on the step size from the amplification factors. Empirical tests demonstrated that higher-order methods accommodate larger time steps and that increased damping reduces allowable step sizes.

Future research could pursue two directions i.e. investigating implicit integrators such as the backward Euler method and implicit "Runge–Kutta" schemes would enable larger step sizes and improved stability when the system exhibits stiffness. And extending the analysis to include control inputs and designing feedback laws that stabilize otherwise unstable equilibria (e.g., the inverted pendulum) would provide insights for engineering control systems.

The combination of analytical stability analysis and careful numerical integration is essential for accurate simulation of nonlinear engineering systems. Understanding eigenvalues, Jacobians and stability regions informs the selection of parameters and algorithms, ensuring reliable predictions of complex dynamics.

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