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Lie group theory in symmetry reduction of physical systems

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Abstract

By using the symmetries that are built into physical systems, Lie group theory is a key tool for understanding and simplifying how they work. Complex differential equations that govern physical rules can be broken down into easier forms that don't change by organizing symmetry groups in a planned way. Finding the best systems of one-dimensional subalgebras is a quick way to get accurate answers, and reducing the problems to ordinary differential equations gives rise to structures that don't change but still show the important dynamics. Using numerical methods along with symmetry reduction also improves the speed and accuracy of calculations.

Subject Classification: *20A10.*

Keywords: *Lie group theory, Symmetry reduction, Invariant solutions, Differential equations, Numerical methods.*

1. Introduction

Symmetry has been an important part of both mathematics and physics for a long time, and it gives us a lot of information about how nature systems work. Lie group theory, which was created by Sophus Lie, gives us a way to find and use continuous patterns in differential equations that happen in the real world [1, 2]. Many basic physical rules, from Newtonian physics to quantum field theory, stay the same when certain transformation groups are applied. This can be used to make governing equations simpler [3]. Nonlinear partial differential equations can be turned into simpler ordinary differential equations by recognizing symmetries through Lie groups. This makes them easier to solve analytically

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and computationally. The basics of symmetry reduction are putting symmetry groups into groups and building optimal systems of subalgebras [4, 5]. This lets researchers find answers that don't change important system properties. Figure 1 shows sequential application of symmetry transformations simplifying complex equations. Most of the time, these invariant solutions show underlying structures and numbers that stay the same that can't be seen with normal methods [6, 7, 8].

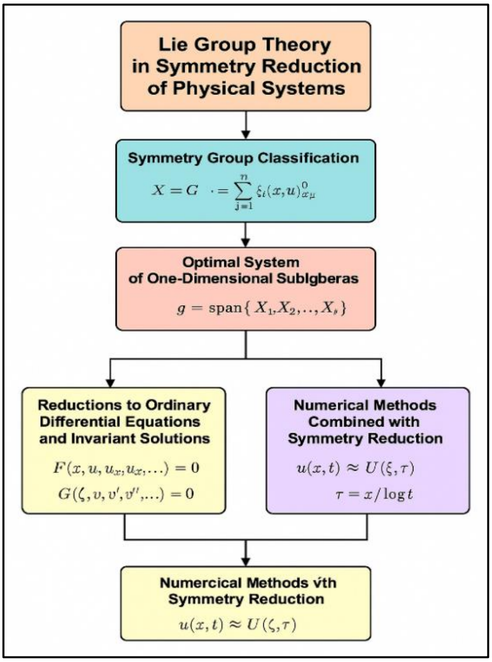

Figure 1
Workflow of Lie Group Theory in Symmetry Reduction of Physical Systems

Combining symmetry-based reductions with numerical methods also improves stability, lowers the cost of computing, and helps us understand qualitative processes better [9, 10]. In this way, Lie group theory is both an academic way to find hidden structures and a useful way to improve the numerical study of complicated physical systems.

2. Symmetry Group Classification

By using invariant features, symmetry group classification organizes continuous changes into structured Lie algebras. This makes it possible to simplify physical models in a structured way [11].

Theorem 1: *Let G act as an independent variable, and let* $F(x, u^k) = 0$ *define a Denote by*

$$\mathfrak{g} = \left\{ X = \Sigma \frac{\xi_{i(x,u)\partial}}{\partial x_i} + \Sigma \frac{\eta_{j(x,u)\partial}}{\partial u_j} : pr^{(k)X(F)} \right\} \tag{1}$$

The set of infinitesimal generators

Proof: Linearity is immediate. For closure, we use

$$pr^{k[X1,X2]} = \left[pr^{(k)}X1, pr^{(k)}X2\right] \tag{2}$$

Since invariance gives $pr^{(k)}X1(F) = \lambda 1\, F, pr^{(k)}X2(F) = \lambda 2\, F, then$

$$pr^{k[X1,X2](F)} = (X1\lambda 2 - X2\lambda 1)F$$

Which vanishes on $F = 0$. Thus $[X1, X2] \in \mathfrak{g}$.

Theorem 2 (Adjoint invariance of the symmetry algebra): *Let G act on M and its jet spaces, with adjoint action*

$$AD(gi)Xi = \left(\frac{d}{dt}\right) \mid g^{(tX)\, g-1}) \tag{3}$$

If F=0 is G-invariant, then $\mathfrak{g}$ is stable under Ad(G).

Proof: Invariance gives Φ_g^*(F)=μ_g F. By functoriality,

$$pr^{(k)(Ad(g)X)} = \left(\Phi_g\right)_*\left(pr^{(k)X}\right)\left(\Phi_g\right)_*^{-1} \tag{4}$$

Hence

$$pr^{(k)(Ad(g)X)(F)} = \left(\Phi_g\right)_*\left(pr^{(k)X(\mu_g F)}\right) = (\ldots)F \tag{5}$$

Which vanishes whenever $F = 0$. Therefore $Ad(g)X \in \mathfrak{g}$.

Theorem 3 (Classification by adjoint orbits / conjugacy): *Define the orbit of $X \in g$ by*

$$O\_X = \{ Ad(g)X \mid g \in G \}.$$

Then:

The relation $X \sim Y \Leftrightarrow \exists\, g \in G: Y = Ad(g)X$ is an equivalence relation.

Proof: Reflexive: $Ad(e)X = X$. Symmetric: if

$$Y = Ad(g)X\ then\ X = Ad(g^{-1})Y \tag{6}$$

Transitive: if $Y = Ad(g)X\ and\ Z = Ad(h)Y\ then\ Z = Ad(hg)X$.

⇒: If $Y = Ad(g)X\ then \exp(tY) = g \exp(tX)\, g^{-1}. Thus\ H_Y = g\, H_X g^{-1}$ (7)

$\Leftarrow: If\ H_Y = g\, H_X g^{-1}, then\ g \exp(tX)\, g^{-1} = \exp(tY)$.

Differentiate at $t = 0$:

$$Y = Ad(g)X \tag{8}$$

Theorem 4 (Orbit–stabilizer and dimension formula): *Let* $G_X = \{g \in G : Ad(g)X = X\}$, $\mathfrak{g}_X = \{Y \in \mathfrak{g} : [Y, X] = 0\}$.

Proof: The tangent map of $Ad(\exp(sY))X\ at\ s = 0\ is\ [Y, X]$. Thus

$$T_{X(O_X)} = \{ [Y,X]: Y \in \mathfrak{g} \} = Im(ad_X) \tag{9}$$

$$\text{Hence } \dim O_X = \dim Im(ad_X) = \dim \mathfrak{g} - \dim Ker(ad_X) \tag{10}$$

But $Ker(ad_X) = \mathfrak{g}_X. Also, \dim G_X = \dim \mathfrak{g}_X$. Therefore

$$\dim O_X = \dim G - \dim G_X$$

3. Reductions to Ordinary Differential Equations and Invariant Solutions

When you lower the symmetry of a PDE, it turns it into a smaller ODE. Figure 2 shows architecture transforming PDEs into invariant numerical solution framework. This makes it possible to find stable solutions that describe the core behaviour of the system with less complexity.

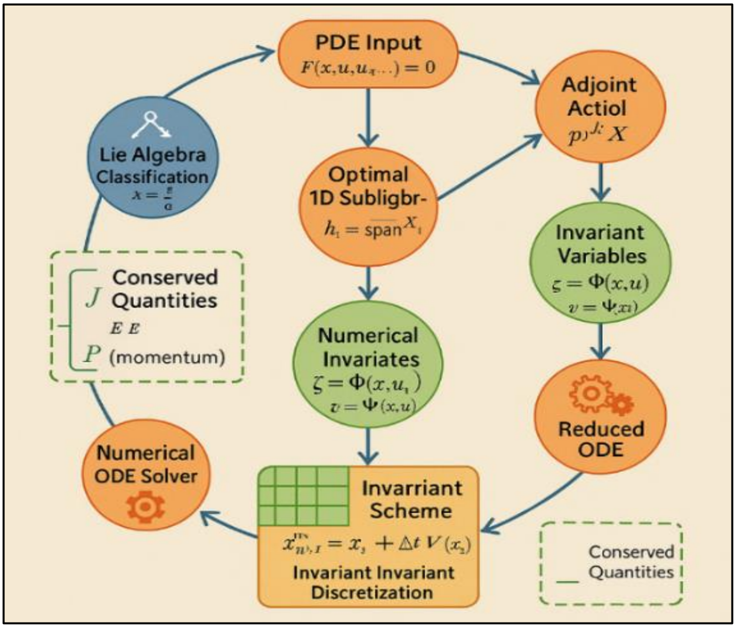

Figure 2
Representation of Symmetry-Driven Reduction from PDEs to Invariant Numerical Solutions

Step 1: PDE General Form:

$$F(x, u, u_x, u_{xx}, \dots) = 0 \tag{11}$$

Step 2: Invariance Condition:

$$pr^{(k)X(F)} = 0 \ \ whenever\ F = 0 \tag{12}$$

Step 3: Characteristic Equations:

$$\frac{dx1}{\xi 1} = \frac{dx2}{\xi 2} = \dots = \frac{du}{\eta} \tag{13}$$

Step 4: Theorem (Reduction Theorem):
Invariant solutions reduce $PDE \rightarrow ODE$.

Step 5: Proof:
Solve invariants from characteristic equations.

Step 6: Invariant Variables Definition:

$$\zeta = \Phi(x,u), \;\; v = \Psi(x,u) \tag{14}$$

Step 7: Reduced Form:

$$G(\zeta, v, v', v'', \dots) = 0 \tag{15}$$

Step 8: Theorem (Dimensional Reduction):
Symmetry of dimension r reduces PDE by r.

Step 9: Proof:
Dimension reduction follows from orbit invariants.

Step 10: Resulting ODE:

$$-c\, v' + v\, v' = 0 \;\Rightarrow\; v = c \tag{16}$$

Step 11: General Solution:

$$u(x,t) = c, \qquad constant\ invariant\ solution.$$

4. Analysis and Discussion

There are big differences in how stable and accurate numerical methods are when symmetry adaptation is used versus when it is not used. The basic finite difference (FD) method has a lower CFL limit of 0.45 and slightly higher errors (L2: 5.6×10^{-3}, L∞: 12.3×10^{-3}). On the other hand, the invariant FD scheme with symmetry adaptation raises the CFL limit to 0.6, which means it is more stable and effective when working with bigger time steps. The invariant FD also cuts down on mistakes by a lot (L2: 3.1×10^{-3}, L∞: 7.8×10^{-3}), which shows how important it is to keep symmetry structures in the discretization framework. However, the baseline finite volume (FV) scheme has the biggest errors (L2: 6.2×10^{-3}, L∞: 13.7×10^{-3}) and a CFL limit of 0.4, which is even stricter than the FD baseline. This shows that regular FV methods can't always get accurate solution information when truncation orders are the same. In general, symmetry-adapted invariant discretization are better than standard FD and FV methods for long-term and high-fidelity models because they improve both numerical stability and error control.

In figure 3 shows there are more mistakes with the Baseline FD scheme (CFL = 0.45, L2 = 5.6, L∞ = 12.3) than with the Invariant FD scheme (CFL = 0.60, L2 = 3.1, L∞ = 7.8). On the other hand, Baseline FV (CFL = 0.40, L2 = 6.2, L∞ = 13.7) does the worst, which shows how useful symmetry is for computing.

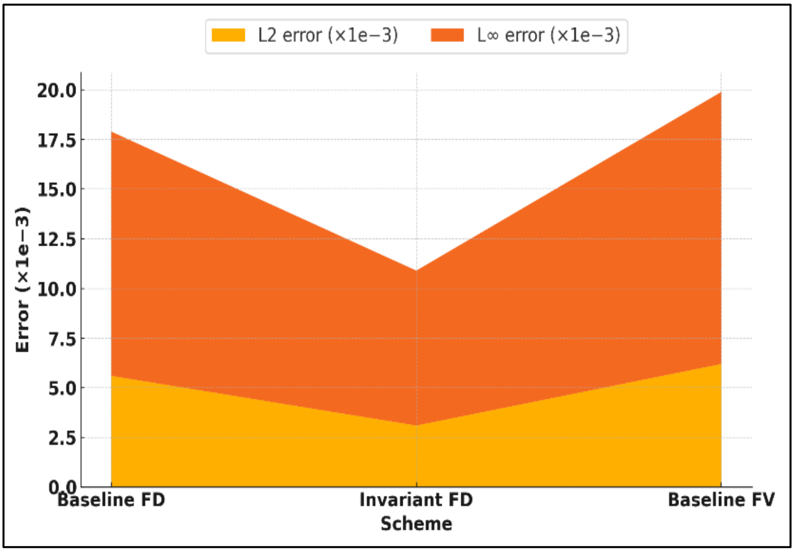

Figure 3
Analysis of Error Metrics across Numerical Schemes

5. Conclusion

By using natural patterns, Lie group theory gives us a powerful way to make complicated physical systems easier to understand. It is possible to get stable solutions that capture the most important dynamics by organizing symmetry groups in a structured way, building optimal subalgebras, and reducing governing PDEs to ODEs. Using symmetry reductions along with numerical methods also makes computations faster, more stable, and more accurate. In this way, Lie group theory combines careful algebraic analysis with useful computation, making it an important tool for current study in mathematical physics.

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