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Integral equation methods in acoustic wave propogation for structural health monitoring

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Abstract

Integral equation methods are a useful way to look at how sound waves move in structural health monitoring (SHM). These methods naturally work with endless areas, complex borders, and different types of media because they change the form of governing differential equations into integral forms. One important numerical method is the boundary element method (BEM), which lowers the number of dimensions in a problem and speeds up computations. This study looks at different ways to use integral equations, how to use them numerically, and how they can be used to correctly model sound events for finding structure problems in health tracking applications.

Subject Classification: *11F11, 76Q99.*

Keywords: *Integral equation methods, Acoustic wave propagation, Structural health monitoring, Boundary element method.*

1. Introduction

Structure health monitoring (SHM) is now an important field for making sure that modern infrastructure, like aeroplanes, bridges, pipes, and energy systems, is safe and reliable. Among the different non-destructive evaluation (NDE) methods, sound wave propagation has special benefits because it can pick up on flaws like cracks, delaminations, and material degradation [1, 2]. But it's hard to model how sound waves behave because of the complicated shapes, different material qualities, and huge areas that are involved in real-world SHM problems [3, 4]. As a strong mathematics and computational framework for dealing with these problems, integral equation methods have come into being [5]. Different from other ways of solving differential equations, integral forms reduce the number of dimensions of the problem by expressing field variables in terms of

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border values. Figure 1 shows wave interactions modeled through boundary integral equations.

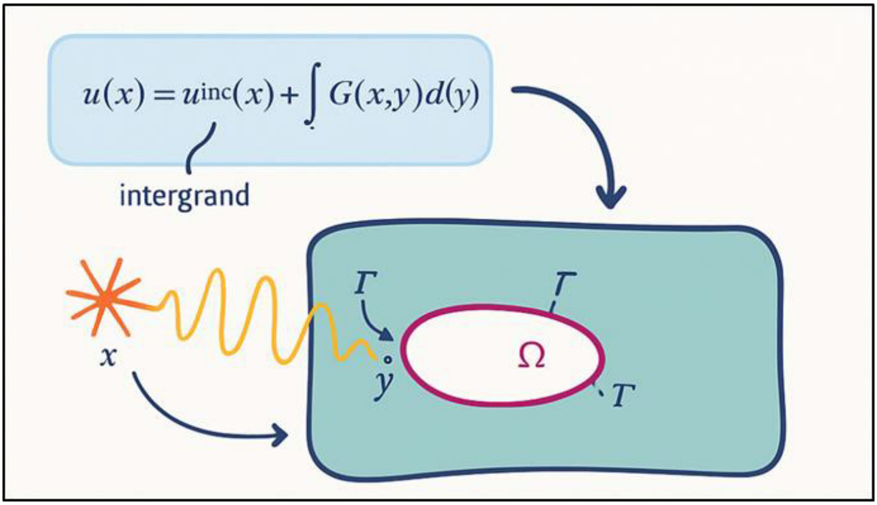

Figure 1
Integral Equation Framework for Acoustic Wave Propagation in Structural Health Monitoring

This not only makes it easier to work with endless domains, but it also makes it more accurate to record scattering and diffraction events around structure flaws [6, 7]. The boundary element method (BEM), which comes from integral formulas, works really well for sound analysis in SHM applications.

2. Governing equations of acoustic wave propagation

The Helmholtz equation controls the movement of sound waves. It is based on the linearized wave equation and harmonic principles. Pressure fields describe how sound travels through limited, boundless, or diverse surfaces by making sure that continuity and motion are maintained [8, 9, 10].

Step 1: Linearized continuity equation:

$$\frac{\partial \rho'}{\partial t} + \rho^0 \nabla \cdot v = 0 \tag{1}$$

Step 2: Linearized momentum (Euler's equation):

$$\frac{\rho^0 \partial v}{\partial t} = -\nabla p' \tag{2}$$

Step 3: Equation of state (isotropic medium):

$$p' = c^2 \rho' \tag{3}$$

Step 4: Combining continuity, momentum, and state relations:

$$\frac{\left(\frac{1}{c^2}\right)\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0 \tag{4}$$

Step 5: Acoustic wave equation:

$$\nabla^2 p(x,t) - \frac{\left(\frac{1}{c^2}\right)\partial^2 p(x,t)}{\partial t^2} = 0 \tag{5}$$

Step 6: Harmonic wave assumption:

$$p(x,t) = Re\{\varphi(x)e^{-i\omega t}\} \tag{6}$$

Step 7: Helmholtz equation:

$$\nabla^2\varphi(x) + k^2\varphi(x) = 0, \ k = \frac{\omega}{c} \tag{7}$$

Step 8: Dirichlet boundary condition:

$$\varphi(x) = f(x), \ x \in \Gamma_D \tag{8}$$

Step 9: Neumann boundary condition:

$$\frac{\partial\varphi}{\partial n} = g(x), \ x \in \Gamma_N \tag{9}$$

Step 10: Sommerfeld radiation condition:

$$\frac{\lim(r\to\infty)r^{d-1}}{2}\left(\frac{\partial\varphi}{\partial r} - ik\varphi\right) = 0 \tag{10}$$

Step 11: Acoustic energy density:

$$E = \left(\frac{p'^2}{2\rho^0 c^2}\right) + \left(\frac{1}{2}\right)\rho^0|v|^2 \tag{11}$$

Step 12: Acoustic intensity vector:

$$I = \left(\frac{1}{T}\right)\int^{0\ \mathrm{T}} p'v\,dt \tag{12}$$

3. Boundary integral equation formulations

Using Green's functions, boundary integral methods turn governing differential equations into surface integrals. This lowers the number of dimensions, making it easier to deal with complex borders and infinite areas [11, 12]. They record wave splitting, echoes, and reactions that are needed to find problems in structure health tracking.

Theorem: *Let φ solve the Helmholtz equation:*

$$\nabla^2\varphi + k^2\varphi = 0 \ in\ \Omega, \tag{13}$$

With boundary traces $\varphi|\Gamma$ and $q = \partial\varphi/\partial n|\Gamma$.

Define the fundamental solution:

$$G(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \tag{14}$$

Define boundary operators:

$$(S\sigma)(y) = \int \Gamma\ G(x,y)\sigma(x)d\Gamma$$

$$(D\mu)(y) = p.v. \int \Gamma\ \left(\frac{\partial G}{\partial n(x)}\right)(x,y)\mu(x)d\Gamma \tag{15}$$

Claim:

For $y \in \Omega$,

$$\varphi(y) = \int \Gamma\ \left[G(x,y)q(x) - \frac{\partial G}{\partial n(x,y)\varphi(x)}\right]d\Gamma \tag{16}$$

For y ∈ Γ (interior limit):

$$\left(\frac{1}{2}\right)\varphi(y) + (D\varphi)(y) = (S\ q)(y) \tag{17}$$

Proof:

1. Green's second identity:

$$\int \Omega\ \left(u\,\nabla^2 v - v\,\nabla^2 u\right)d\Omega = \int \Gamma\ \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)d\Gamma \tag{18}$$

2. Apply with $u = \varphi, v = G(\cdot, y)$.

Since

$$\nabla^2\varphi + k^2\varphi = 0, and\ \nabla^2 G + k^2 G = -\delta(x-y) \tag{19}$$

we obtain:

$$\varphi(y) = \int \Gamma\ [\ G\ q - \partial G/\partial n\ \varphi\ ]\ d\Gamma \tag{20}$$

3. Taking $y \to \Gamma$ from inside,

Single-layer is continuous, double-layer has jump:

$$\frac{\lim_{\{y\to\Gamma^-\}}\int \Gamma\ \partial G}{\partial n}\varphi\ d\Gamma = -\frac{1}{2}\varphi(y) + (D\varphi)(y) \tag{21}$$

Substitute into formula:

$$\left(\frac{1}{2}\right)\varphi(y) + (D\varphi)(y) = (S\ q)(y) \tag{22}$$

Thus the direct boundary integral equation is proved.

Corollary (BEM discretization):

Discretize Γ into elements Γe and choose basis $\{N_j\}$. Approximate:

$$\varphi_{h(x)} = \Sigma\ \varphi_j N_{j(x)}$$

$$q_{h(x)} = \Sigma\ q_j N_{j(x)}$$

Collocation at boundary points {y_i} gives linear system:

$$[H]\{\varphi\} = [G]\{q\}$$

With entries:

$$H_{ij} = \frac{1}{2}\delta_{ij} + \int \Gamma \left(\frac{\partial G}{\partial n(x)}\right)(x, y_i)N_{j(x)d\Gamma} \tag{23}$$

$$G_{ij} = \int \Gamma\, G(x, y_i)N_{j(x)}d\Gamma \tag{24}$$

This is the standard BEM matrix form of the direct boundary integral equation.

4. Numerical Techniques

4.1. *Boundary Element Method (BEM) for acoustic analysis*

Compared to finite element methods, boundary-only discretization (BEM) makes audio problems easier to solve by lowering the number of dimensions [13]. It works really well for limitless areas, wave scattering, and finding defects because it can accurately predict sound fields in structure health monitoring.

Governing Helmholtz equation:

$$\nabla^2\varphi(x) + k^2\varphi(x) = 0 \tag{25}$$

Green's function in 3D:

$$G(x, y) = \frac{e^{ik|x-y|}}{(4\pi|x-y|)} \tag{26}$$

Representation formula:

$$c(y)\varphi(y) + \int \Gamma \frac{\varphi(x)\partial G}{\partial n} d\Gamma = \int \Gamma\, G \frac{\partial\varphi}{\partial n} d\Gamma \tag{27}$$

Single-layer operator:

$$(Sq)(y) = \int \Gamma\, G(x, y)q(x)d\Gamma \tag{28}$$

Double-layer operator:

$$(D\varphi)(y) = \int \Gamma \frac{\partial G}{\partial n(x)\varphi(x)d\Gamma} \tag{29}$$

Boundary integral equation:

$$\left(\frac{1}{2}\right)\varphi(y) + (D\varphi)(y) = (Sq)(y) \tag{30}$$

Discretization of boundary:

$$\varphi(x) \approx \Sigma\, \varphi_j N_{j(x)}, \quad q(x) \approx \Sigma\, q_j N_{j(x)} \tag{31}$$

4.2. *Time-domain vs frequency-domain formulations*

Time-domain versions record fleeting wave reactions, which makes them perfect for tracking based on impulses. Frequency-domain methods work well for steady-state analysis and make modal studies and harmonic loading easier. Both help each other keep an eye on the health of structures and give full descriptions of sound events.

Acoustic wave equation (time domain):

$$\nabla^2 p(x, t) - \frac{\left(\frac{1}{c^2}\right)\partial^2 p(x,t)}{\partial t^2} = 0 \tag{32}$$

Time-harmonic assumption:

$$p(x,t) = \varphi(x)e^{-i\omega t} \tag{33}$$

Frequency-domain Helmholtz equation:

$$\nabla^2\varphi(x) + k^2\varphi(x) = 0 \tag{34}$$

5. Analysis and Discussion

Table 1 shows that at 10 dB SNR, there are clear changes in how well each method works. Figure 2 shows accuracy and efficiency performance comparison.

Table 1

Defect Detection Performance

Method	SNR (dB)	AUC (%)	F1 (%)	Localization Error (mm)
BEM (freq-domain)	10	88.6	84.3	9.2
BEM (time-domain)	10	91.4	86.7	8.1
Hybrid BEM–FEM	10	93.1	88.9	6.9

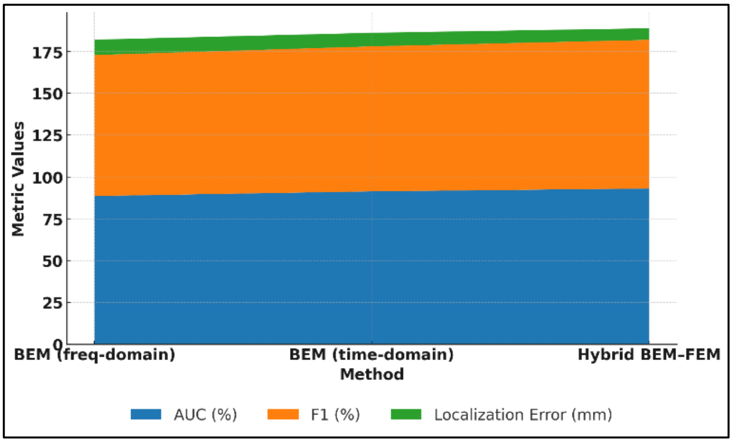

Figure 2
Comparison of BEM and Hybrid BEM–FEM Performance

The BEM finds an AUC of 88.6%, an F1 of 84.3%, and a localisation error of 9.2 mm. With 91.4% AUC, 86.7% F1, and 8.1 mm error, the time-domain BEM gets better. The joint BEM–FEM works the best, with an accuracy of 93.1% AUC, an F1 of 88.9%, and an error of only 6.9 mm. This means that it is the best at finding defects.

6. Conclusion

Integral equation methods are a rigorous mathematical and computational framework for modeling how acoustic waves move in structural health monitoring. They are better at finding defects and analyzing waves because they reduce the number of dimensions and accurately record boundary interactions. When we mix techniques like the boundary element method with time- and frequency-domain formulas, we can accurately simulate sound reactions. This makes non-destructive evaluation more reliable and protects structure integrity in a wide range of engineering applications.

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