

Fourier series analysis and computation based on function characteristics

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Abstract

This paper aims to study the Fourier series of a function defined on the interval $[-l/2, 0]$ under various conditions. Furthermore, when a function is given on $[a, b]$, its Fourier series is obtained by applying a suitable transformation to an associated function defined on either $[-l, l]$ or $[0, l]$.

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1. Preliminary knowledge

Now we recall a definition which is useful in this paper. The following concept has been introduced in [2]-[3].

Definition 1.1: Let f be an integrable function on $[-l, l]$. The Fourier series of f is the trigonometric series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \quad \text{for } n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad \text{for } n = 1, 2, \dots$$

and are called the Fourier coefficients of f .

Definition 1.2: Let f be an integrable function on $[0, l]$. The Fourier cosine series of f on $[0, l]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \quad n = 0, 1, 2, \dots$

The Fourier sine series of f on $[0, l]$ is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \quad n = 1, 2, \dots$

Lemma 1.3 [1]: Let f be an integrable function on $[-l, l]$ with period of $2l$, the Fourier coefficients are calculated according to period of $2kl$ ($l \in \mathbb{N}$). The calculation indicates there are same results between Fourier series with period of $2l$ and $2kl$.

Zhang [4] has thoroughly proven the following points through extensive research and analysis, demonstrating significant findings in the field of study:

Theorem 1.4: Let f be an integrable function on $[-l, l]$, which satisfies the condition $f(x+l) = -f(x)$. Then, we have

$$f(x) \sim \sum_{k=1}^{\infty} \left[a_{2k-1} \cos\left(\frac{(2k-1)\pi x}{l}\right) + b_{2k-1} \sin\left(\frac{(2k-1)\pi x}{l}\right) \right]$$

where

$$a_{2k-1} = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{(2k-1)\pi x}{l}\right) dx \text{ and } b_{2k-1} = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{(2k-1)\pi x}{l}\right) dx$$

for $k = 1, 2, \dots$

Theorem 1.5: Let f be an integrable function on $[-l, l]$, which satisfies the condition $f(x+l) = f(x)$. Then, we have

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_{2k} \cos\left(\frac{2k\pi x}{l}\right) + b_{2k} \sin\left(\frac{2k\pi x}{l}\right) \right]$$

where

$$a_{2k} = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{2k\pi x}{l}\right) dx, \quad k = 0, 1, 2, \dots,$$

$$b_{2k} = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{2k\pi x}{l}\right) dx, \quad k = 1, 2, \dots$$

Theorem 1.6: Let f be an integrable function on $[0, l]$. When $\frac{l}{2} < x \leq l$, it satisfies the condition $f(x) = -f(l-x)$. Then, we have

(1) While $f(x)$ is an even function in $(-l, l)$, then we get

$$f(x) \sim \sum_{k=1}^{\infty} a_{2k-1} \cos\left(\frac{(2k-1)\pi x}{l}\right)$$

$$\text{where } a_{2k-1} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{(2k-1)\pi x}{l}\right) dx, \quad k = 1, 2, \dots$$

(2) While $f(x)$ is an odd function in $(-l, l)$, then we get

$$f(x) \sim \sum_{k=1}^{\infty} b_{2k} \sin\left(\frac{2k\pi x}{l}\right)$$

$$\text{where } b_{2k} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{2k\pi x}{l}\right) dx, \quad k = 1, 2, \dots$$

Theorem 1.7: Let f be an integrable function on $[0, l]$. When $\frac{l}{2} < x \leq l$, it satisfies the condition $f(x) = f(l-x)$. Then, we have

(1) While $f(x)$ is an even function in $(-l, l)$, then we get

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k} \cos\left(\frac{2k\pi x}{l}\right)$$

$$\text{where } a_{2k} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{2k\pi x}{l}\right) dx, \quad k = 0, 1, 2, \dots$$

(2) While $f(x)$ is an odd function in $(-l, l)$, then we get

$$f(x) \sim \sum_{k=1}^{\infty} b_{2k-1} \sin\left(\frac{(2k-1)\pi x}{l}\right)$$

$$\text{where } b_{2k-1} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{(2k-1)\pi x}{l}\right) dx, \quad k = 1, 2, \dots$$

2. Fourier series for functions defined on $\left[-\frac{l}{2}, 0\right]$

The results that we present in this work are primarily inspired by the findings and methodologies discussed in [4], which served as a key reference and foundation for our analysis and conclusions.

Consider a function f defined on $\left[-\frac{l}{2}, 0\right]$. To analyze its Fourier series, we first extend f from $\left[-\frac{l}{2}, 0\right]$ to the interval $[-l, 0]$ in two ways. We can then state the following results, which hold for both cases, as presented in the subsequent theorem.

Theorem 2.1: Let f be an integrable function on $[-l, 0]$, which satisfies the condition $f(x) = -f(l+x)$ where $-l \leq x < -\frac{l}{2}$, then

(1) the Fourier cosine series of f is

$$f(x) \sim \sum_{k=1}^{\infty} a_{2k-1} \cos\left(\frac{(2k-1)\pi x}{l}\right)$$

$$\text{where } a_{2k-1} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{(2k-1)\pi x}{l}\right) dx, \quad k = 1, 2, \dots$$

(2) the Fourier sine series of f is

$$f(x) \sim \sum_{k=1}^{\infty} b_{2k-1} \sin\left(\frac{(2k-1)\pi x}{l}\right)$$

$$\text{where } b_{2k-1} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{(2k-1)\pi x}{l}\right) dx, \quad k = 1, 2, \dots$$

Proof: We use the method of periodic extension to $f(x)$ with a period of $2l$ by extending f to even and odd functions. If we extend $f(x)$ to an even function on $(-l, l)$, then $b_n = 0$ where $n = 1, 2, \dots$ and

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) dx + \int_{\frac{l}{2}}^l f(x) dx \right].$$

Sine $f(x) = f(-x) = -f(l-x)$ where $\frac{l}{2} \leq x \leq l$ and

$$\int_{\frac{l}{2}}^l f(l-x) dx = \int_{\frac{l}{2}}^0 f(t)(-dt) = \int_0^{\frac{l}{2}} f(t) dt = \int_0^{\frac{l}{2}} f(x) dx,$$

then we have

$$a_0 = \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) dx - \int_{\frac{l}{2}}^l f(l-x) dx \right] = \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) dx - \int_0^{\frac{l}{2}} f(x) dx \right] = 0.$$

For $n = 1, 2, \dots$, we have

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) \cos\left(\frac{n\pi x}{l}\right) dx - \int_{\frac{l}{2}}^l f(l-x) \cos\left(\frac{n\pi x}{l}\right) dx \right].$$

Since

$$\begin{aligned} \int_{\frac{l}{2}}^l f(l-x) \cos\left(\frac{n\pi x}{l}\right) dx &= \int_{\frac{l}{2}}^0 f(t) \cos\left(\frac{n\pi(l-t)}{l}\right) (-dt) \\ &= \int_0^{\frac{l}{2}} f(t) \left[\cos(n\pi) \cos\left(\frac{n\pi t}{l}\right) + \sin(n\pi) \sin\left(\frac{n\pi t}{l}\right) \right] dt \\ &= \int_0^{\frac{l}{2}} f(t) \cos(n\pi) \cos\left(\frac{n\pi t}{l}\right) dt = (-1)^n \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \end{aligned}$$

then we obtain

$$\begin{aligned} a_n &= \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) \cos\left(\frac{n\pi x}{l}\right) dx - \int_{\frac{l}{2}}^l f(l-x) \cos\left(\frac{n\pi x}{l}\right) dx \right] \\ &= \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) \cos\left(\frac{n\pi x}{l}\right) dx + (-1)^{n+1} \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \right] \\ &= \frac{2}{l} [1 + (-1)^{n+1}] \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\ &= \begin{cases} 0 & , n = 2k \\ \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{(2k-1)\pi x}{l}\right) dx & , n = 2k-1 \end{cases} \end{aligned}$$

for $k = 1, 2, \dots$. Therefore, the Fourier cosine series of f is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) = \sum_{k=1}^{\infty} a_{2k-1} \cos\left(\frac{(2k-1)\pi x}{l}\right)$$

$$\text{where } a_{2k-1} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{(2k-1)\pi x}{l}\right) dx, \quad k = 1, 2, \dots$$

If we extend $f(x)$ to an odd function on $(-l, l)$, then $a_n = 0$ where $n = 0, 1, 2, \dots$ and

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) \sin\left(\frac{n\pi x}{l}\right) dx + \int_{\frac{l}{2}}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \right].$$

Sine $f(x) = -f(-x) = f(l-x)$ where $\frac{l}{2} \leq x \leq l$ and

$$\begin{aligned} \int_{\frac{l}{2}}^l f(l-x) \sin\left(\frac{n\pi x}{l}\right) dx &= \int_{\frac{l}{2}}^0 f(t) \sin\left(\frac{n\pi(l-t)}{l}\right) (-dt) \\ &= (-1)^{n+1} \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \end{aligned}$$

we then have

$$\begin{aligned} b_n &= \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) \sin\left(\frac{n\pi x}{l}\right) dx + \int_{\frac{l}{2}}^l f(l-x) \sin\left(\frac{n\pi x}{l}\right) dx \right] \\ &= \frac{2}{l} \left[\int_0^{\frac{l}{2}} f(x) \sin\left(\frac{n\pi x}{l}\right) dx + (-1)^{n+1} \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{n\pi x}{l}\right) dx \right] \\ &= \frac{2}{l} [1 + (-1)^{n+1}] \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{n\pi x}{l}\right) dx \\ &= \begin{cases} 0 & , n = 2k \\ \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{(2k-1)\pi x}{l}\right) dx & , n = 2k-1 \end{cases} \end{aligned}$$

for $k = 1, 2, \dots$. Therefore, the Fourier sine series of f is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \sum_{k=1}^{\infty} b_{2k-1} \sin\left(\frac{(2k-1)\pi x}{l}\right)$$

$$\text{where } b_{2k-1} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{(2k-1)\pi x}{l}\right) dx \text{ for } k = 1, 2, \dots$$

Theorem 2.2: Let f be an integrable function on $[-l, 0]$, which satisfies the condition $f(x) = f(l+x)$ where $-l \leq x < -\frac{l}{2}$, then

(1) the Fourier cosine series of f is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k} \cos\left(\frac{2k\pi x}{l}\right)$$

$$\text{where } a_{2k} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \cos\left(\frac{2k\pi x}{l}\right) dx, \quad k = 0, 1, 2, \dots$$

(2) the Fourier sine series of f is

$$f(x) \sim \sum_{k=1}^{\infty} b_{2k} \sin\left(\frac{2k\pi x}{l}\right)$$

$$\text{where } b_{2k} = \frac{4}{l} \int_0^{\frac{l}{2}} f(x) \sin\left(\frac{2k\pi x}{l}\right) dx, \quad k = 1, 2, \dots$$

Proof: We can prove by the same way as in Theorem 2.1.

3. Fourier series for functions defined on $[a, b]$

The focus of this paper is to investigate a generalized form of the results started in Theorem 1.4 and Theorem 1.5.

Definition 3.1: Let $f(t)$ be an integrable function on $[a, b]$ and assume that $g(x) = f(x(t))$ is an integrable function on $[-l, l]$. We can say that the Fourier series of g on $[-l, l]$ is the Fourier series of f on $[a, b]$.

Theorem 3.2: Let f be an integrable function on $[a, b]$, which satisfies the condition $f(t) = -f\left(t + \frac{b-a}{2}\right)$. Then, we have

$$f(t) \sim \sum_{k=1}^{\infty} \left[A_{2k-1} \cos\left(\frac{2(2k-1)\pi t}{b-a}\right) + B_{2k-1} \sin\left(\frac{2(2k-1)\pi t}{b-a}\right) \right]$$

$$\text{where } A_{2k-1} = \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \cos\left(\frac{2(2k-1)\pi t}{b-a}\right) dt,$$

$$B_{2k-1} = \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \sin\left(\frac{2(2k-1)\pi t}{b-a}\right) dt \quad \text{for } k = 1, 2, \dots$$

Proof: Let $t = \frac{x(b-a) + b+a}{2}$ or $x = \frac{2t-b-a}{b-a}$, then $x \in [-1, 1]$ where

$t \in [a, b]$. Let $g(x) = f\left(\frac{x(b-a) + b+a}{2}\right)$. Since

$$\begin{aligned} g(x) = f(t) &= -f\left(t + \frac{b-a}{2}\right) = -f\left(\frac{x(b-a)+b+a}{2} + \frac{b-a}{2}\right) \\ &= -f\left(\frac{(x+1)(b-a)+b+a}{2}\right) = -g(x+1), \end{aligned}$$

it then follows that g satisfies the condition of Theorem 1.4 with $l = 1$, and its Fourier series on $[-1, 1]$ can be expressed as

$$g(x) \sim \sum_{k=1}^{\infty} [a_{2k-1} \cos((2k-1)\pi x) + b_{2k-1} \sin((2k-1)\pi x)] \quad (3.1)$$

where

$$a_{2k-1} = 2 \int_0^1 g(x) \cos((2k-1)\pi x) dx = 2 \int_0^1 f\left(\frac{x(b-a)+b+a}{2}\right) \cos((2k-1)\pi x) dx,$$

$$b_{2k-1} = 2 \int_0^1 g(x) \sin((2k-1)\pi x) dx = 2 \int_0^1 f\left(\frac{x(b-a)+b+a}{2}\right) \sin((2k-1)\pi x) dx$$

for $k = 1, 2, \dots$. Substituting $x = \frac{2t-b-a}{b-a}$ into (3.1) we obtain the Fourier series of $f(t)$ on $[a, b]$, i.e.,

$$f(t) \sim \sum_{k=1}^{\infty} \left[a_{2k-1} \cos\left((2k-1)\pi \left(\frac{2t-b-a}{b-a}\right)\right) + b_{2k-1} \sin\left((2k-1)\pi \left(\frac{2t-b-a}{b-a}\right)\right) \right]$$

where
$$a_{2k-1} = \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \cos\left((2k-1)\pi \left(\frac{2t-b-a}{b-a}\right)\right) dt,$$

$$b_{2k-1} = \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \sin\left((2k-1)\pi \left(\frac{2t-b-a}{b-a}\right)\right) dt$$

for $k = 1, 2, \dots$. We apply the addition identifies for trigonometric functions and rearrange the terms:

$$\begin{aligned} f(t) &\sim \sum_{k=1}^{\infty} \left[a_{2k-1} \cos\left(\frac{(2k-1)\pi(2t-b-a)}{b-a}\right) + b_{2k-1} \sin\left(\frac{(2k-1)\pi(2t-b-a)}{b-a}\right) \right] \\ &= \sum_{k=1}^{\infty} \left[a_{2k-1} \left(\cos\left(\frac{2(2k-1)\pi t}{b-a}\right) \cos\left(\frac{(2k-1)\pi(b+a)}{b-a}\right) \right. \right. \\ &\quad \left. \left. + \sin\left(\frac{2(2k-1)\pi t}{b-a}\right) \sin\left(\frac{(2k-1)\pi(b+a)}{b-a}\right) \right) \right] \end{aligned}$$

$$\begin{aligned}
& + b_{2k-1} \left(\sin \left(\frac{2(2k-1)\pi t}{b-a} \right) \cos \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) \right. \\
& \left. - \cos \left(\frac{2(2k-1)\pi t}{b-a} \right) \sin \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) \right) \Bigg] \\
& = \sum_{k=1}^{\infty} \left[\cos \left(\frac{2(2k-1)\pi t}{b-a} \right) \left(a_{2k-1} \cos \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) - b_{2k-1} \sin \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) \right) \right. \\
& \left. + \sin \left(\frac{2(2k-1)\pi t}{b-a} \right) \left(a_{2k-1} \sin \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) + b_{2k-1} \cos \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) \right) \right].
\end{aligned}$$

$$\begin{aligned}
\text{Let } A_{2k-1} &= a_{2k-1} \cos \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) - b_{2k-1} \sin \left(\frac{(2k-1)\pi(b+a)}{b-a} \right), \\
B_{2k-1} &= a_{2k-1} \sin \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) + b_{2k-1} \cos \left(\frac{(2k-1)\pi(b+a)}{b-a} \right).
\end{aligned}$$

Then

$$f(x) \sim \sum_{k=1}^{\infty} \left[A_{2k-1} \cos \left(\frac{2(2k-1)\pi t}{b-a} \right) + B_{2k-1} \sin \left(\frac{2(2k-1)\pi t}{b-a} \right) \right].$$

Since

$$\begin{aligned}
A_{2k-1} &= \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \left[\cos \left(\frac{(2k-1)\pi(2t-b-a)}{b-a} \right) \cos \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) \right. \\
& \quad \left. - \sin \left(\frac{(2k-1)\pi(2t-b-a)}{b-a} \right) \sin \left(\frac{(2k-1)\pi(b+a)}{b-a} \right) \right] dt \\
&= \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \cos \left(\frac{(2k-1)\pi(2t-b-a)}{b-a} + \frac{(2k-1)\pi(b+a)}{b-a} \right) dt \\
&= \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \cos \left(\frac{2(2k-1)\pi t}{b-a} \right) dt.
\end{aligned}$$

In a similar manner, B_{2k-1} is also derived in the same way as A_{2k-1} , we get

$$B_{2k-1} = \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \sin \left(\frac{2(2k-1)\pi t}{b-a} \right) dt.$$

This completes the proof of the theorem.

Theorem 3.3: Let f be an integrable function on $[a, b]$, which satisfies the condition $f(t) = f\left(t + \frac{b-a}{2}\right)$. Then, we have

$$f(t) \sim \frac{A_0}{2} + \sum_{k=1}^{\infty} \left[A_{2k} \cos\left(\frac{4k\pi t}{b-a}\right) + B_{2k} \sin\left(\frac{4k\pi t}{b-a}\right) \right]$$

$$\text{where } A_{2k} = \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \cos\left(\frac{4k\pi t}{b-a}\right) dt, \quad k = 0, 1, 2, \dots,$$

$$B_{2k} = \frac{4}{b-a} \int_{\frac{a+b}{2}}^b f(t) \sin\left(\frac{4k\pi t}{b-a}\right) dt, \quad k = 1, 2, \dots$$

Proof: We can prove by the same way as in Theorem 3.2.

Next, we will study the Fourier series of a function f defined on $\left[a, \frac{a+b}{2}\right]$ by extending the function $f(t)$ to the interval $[a, b]$ in two ways. We can then state the following results, which hold for both cases, as presented in the subsequent theorem.

Theorem 3.4: Let f be an integrable function on $[a, b]$, which satisfies the condition $f(t) = -f(b+a-t)$ where $\frac{a+b}{2} < t \leq b$, then

(1) the Fourier cosine series of f is

$$f(t) \sim \sum_{k=1}^{\infty} a_{2k-1} \cos\left((2k-1)\pi\left(\frac{t-a}{b-a}\right)\right)$$

$$\text{where } a_{2k-1} = \frac{4}{b-a} \int_a^{\frac{a+b}{2}} f(t) \cos\left((2k-1)\pi\left(\frac{t-a}{b-a}\right)\right) dt, \quad k = 1, 2, \dots$$

(2) the Fourier sine series of f is

$$f(t) \sim \sum_{k=1}^{\infty} b_{2k} \sin\left(2k\pi\left(\frac{t-a}{b-a}\right)\right)$$

$$\text{where } b_{2k} = \frac{4}{b-a} \int_a^{\frac{a+b}{2}} f(t) \sin\left(2k\pi\left(\frac{t-a}{b-a}\right)\right) dt, \quad k = 1, 2, \dots$$

Proof: Let $t = x(b-a) + a$ or $x = \frac{t-a}{b-a}$, then $x \in [0, 1]$ where $t \in [a, b]$. Let $g(x) = f(x(b-a) + a)$. Since

$$\begin{aligned}
 g(x) &= f(t) = -f(b+a-t) \\
 &= -f(b+a-x(b-a)-a) \\
 &= -f((1-x)(b-a)+a) = -g(1-x),
 \end{aligned}$$

it then follows that g satisfies the condition of Theorem 1.6 with $l=1$, and its Fourier series on $[-1, 1]$ can be expressed as

$$g(x) \sim \sum_{k=1}^{\infty} a_{2k-1} \cos((2k-1)\pi x) \quad (3.2)$$

where

$$a_{2k-1} = 4 \int_0^{\frac{1}{2}} g(x) \cos((2k-1)\pi x) dx = 4 \int_0^{\frac{1}{2}} f(x(b-a)+a) \cos((2k-1)\pi x) dx$$

for $k=1, 2, \dots$. Substituting $x = \frac{t-a}{b-a}$ into (3.2) we obtain the Fourier cosine series of $f(t)$ on $[a, b]$, i.e.,

$$f(t) \sim \sum_{k=1}^{\infty} a_{2k-1} \cos\left((2k-1)\pi \left(\frac{t-a}{b-a}\right)\right)$$

where $a_{2k-1} = \frac{4}{b-a} \int_a^{\frac{a+b}{2}} f(t) \cos\left((2k-1)\pi \left(\frac{t-a}{b-a}\right)\right) dt$, $k=1, 2, \dots$, and the Fourier sine series of g on $[-1, 1]$ is

$$g(x) \sim \sum_{k=1}^{\infty} b_{2k} \sin(2k\pi x) \quad (3.3)$$

where

$$b_{2k-1} = 4 \int_0^{\frac{1}{2}} g(x) \sin(2k\pi x) dx = 4 \int_0^{\frac{1}{2}} f(x(b-a)+a) \sin(2k\pi x) dx$$

for $k=1, 2, \dots$. Substituting $x = \frac{t-a}{b-a}$ into (3.3) we obtain the Fourier sine series of $f(t)$ on $[a, b]$, i.e.,

$$f(t) \sim \sum_{k=1}^{\infty} b_{2k} \sin\left(2k\pi \left(\frac{t-a}{b-a}\right)\right)$$

where $b_{2k} = \frac{4}{b-a} \int_a^{\frac{a+b}{2}} f(t) \sin\left(2k\pi \left(\frac{t-a}{b-a}\right)\right) dt$, $k=1, 2, \dots$.

Theorem 3.5: Let f be an integrable function on $[a, b]$, which satisfies the condition $f(t) = f(b+a-t)$ where $\frac{a+b}{2} < t \leq b$, then

(1) the Fourier cosine series of f is

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k} \cos \left(2k\pi \left(\frac{t-a}{b-a} \right) \right)$$

$$\text{where } a_{2k} = \frac{4}{b-a} \int_a^{\frac{a+b}{2}} f(t) \cos \left(2k\pi \left(\frac{t-a}{b-a} \right) \right) dt, \quad k=0,1,2,\dots$$

(2) the Fourier sine series of f is

$$f(t) \sim \sum_{k=1}^{\infty} b_{2k-1} \sin \left((2k-1)\pi \left(\frac{t-a}{b-a} \right) \right)$$

$$\text{where } b_{2k-1} = \frac{4}{b-a} \int_a^{\frac{a+b}{2}} f(t) \sin \left((2k-1)\pi \left(\frac{t-a}{b-a} \right) \right) dt, \quad k=1,2,\dots$$

Proof: We can prove by the same way as in Theorem 3.4.

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