Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 6, pp. 2247–2253
DOI : 10.47974/JIM-2366

Differential geometry approaches to curvature and shape analysis

Bipin Sule *
Department of Engineering, Science, and Humanities
Vishwakarma Institute of Technology
Pune 411037
Maharashtra
India

Vimal Bibhu †
School of Engineering & Technology
Noida International University
Noida 203201
Uttar Pradesh
India

Satyendra Kumar §
Department of Mechanical Engineering
Tulsiramji Gaikwad Patil College of Engineering and Technology
Nagpur 441108
Maharashtra
India

Shivanshu Katoch ‡
Department of Computer Science and Engineering
Sri Sai University
Palampur
Himachal Pradesh
India

* *E-mail:* bipin.sule@vit.edu (Corresponding Author)
† *E-mail:* vimal.bhibu@niu.edu.in
§ *E-mail:* satyendra.mechanical@tgpcet.com
‡ *E-mail:* katochshiva1997@gmail.com

© Taru Publications

Dinesh Kumar [@]
Department of Computer Science and Engineering
Sri Sai College of Engineering and Technology
Badhani-Pathankot
Punjab
India

Lalita Kiran Wani [#]
Department of Electronics & Telecommunication
Bharati Vidyapeeth's College of Engineering
Lavale
Pune 412115
Maharashtra
India

Abstract

Curvature provides a quantitative measure of how a geometric object bends in space. Differential geometry offers a rigorous framework for defining and computing curvature on curves, surfaces and manifolds. This paper develops a mathematical approach to curvature and shape analysis using differential geometry. Challenges include handling complex surface parametrizations, computing principal curvatures from noisy data, and distinguishing intrinsic and extrinsic curvature. The proposed methodology formulates surfaces as parametric maps, derives the 1st and 2nd fundamental forms, constructs the shape operator and computes Gaussian and mean curvature. Analytical formulas for principal curvatures and curvature classification support algorithmic shape analysis. Results on canonical shapes such as spheres, cylinders and saddles demonstrate how curvature signatures classify geometric structures. The outcomes underscore that differential geometry not only provides elegant theoretical tools but also practical algorithms for shape analysis in computer graphics and geometric modelling.

Subject Classification: *65N06, 65N20.*

Keywords: *Differential geometry, Curvature, Shape analysis, Principal curvature, Fundamental forms, Shape operator.*

1. Introduction

Understanding how objects bend and curve is fundamental in mathematics and its applications. Differential geometry studies smooth curves and surfaces using calculus and linear algebra. At its core is the notion of curvature, which measures how rapidly a "*curve deviates from a straight line or how a surface*

[@] *E-mail:* **dineshgarg82@gmail.com**
[#] *E-mail:* **lalita.wani@gmail.com**

departs from being flat". For plane curves parameterized by arc length s, the curvature $\kappa(s)$ is the "reciprocal of the radius of the osculating circle at each point". This concept generalizes to surfaces through the study of normal sections and principal curvatures [1]. The maximum and minimum curvatures of normal sections at a point form the principal curvatures $k_1 \; and \; k_2$, providing a local signature of how the surface bends in different directions.

Curvature enters many scientific and engineering fields. In architecture, Gaussian curvature helps classify surfaces into elliptic, parabolic or hyperbolic types, informing structural stability. In computer graphics, curvature guides shading, smoothing and feature detection [2][3]. In medical imaging, mean curvature flow is used to smooth surfaces of anatomical structures. Intrinsic curvature (Gaussian curvature) is determined entirely by distances on the surface and does not depend on embedding, while extrinsic curvature (mean curvature) depends on how the surface sits in space. Distinguishing between these curvatures is essential for applications such as shape matching and surface optimization [4].

The 1st objective of this paper is to present a rigorous mathematical framework for computing curvature of surfaces. We derive the fundamental forms and shape operator that encode metric and curvature information. Principal curvatures emerge as eigenvalues of the shape operator and satisfy relationships with Gaussian and mean curvature. The 2nd objective is to demonstrate how these differential geometric tools enable practical shape analysis. By computing curvature signatures for different surfaces, we classify them and extract shape features that can be used in surface processing tasks [5].

The remainder of the paper is organized as follows. The methodology section develops the mathematical foundations needed to calculate curvature. It introduces parametric surfaces, tangent spaces, normal vectors, the 1st and 2nd fundamental forms and the shape operator. Formulas for Gaussian curvature, mean curvature and principal curvatures are derived. Two tables summaries curvature values for canonical surfaces and illustrate curvature-based classification. The conclusion highlights the significance of differential geometry in shape analysis and suggests future research directions.

2. Methodology

This section outlines the steps for computing curvature and applying it to shape analysis. Let $f: U \subset R^2 \rightarrow R^3$ be a smooth parametrization of a regular surface "$S = f(U).\; f(u,v) = (x(u,v), y(u,v), z(u,v))$". We denote partial derivatives by subscripts.

2.1. *Tangent vectors and the 1st fundamental form*

The partial derivatives $fu = \partial f/\partial uf$ and $fv = \partial f/\partial vf$ span the tangent plane at the point $p = f(u,v)$. The *1st fundamental form* measures inner products of tangent vectors and encodes the metric induced on the surface. It is represented by eq.1:

$$E = f_u \cdot f_u, F = f_u \cdot f_v, G = f_v \cdot f_v \tag{1}$$

The line element on the surface is depicted in eq.2:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{2}$$

The coefficients E, F, G depend on the parametrization. For example, the unit sphere can be parameterized by spherical coordinates (θ, ϕ) with $f(\theta, \phi) = (cos\theta cos\phi, sin\theta cos\phi, sin\phi)$. A computation yields $E = \cos^2\phi\,, F = 0, G = 1.$

2.2. *Normal vector and 2nd fundamental form*

A unit normal vector field on S is given by eq.3:

$$n(u, v) = \frac{f_u \times F_v}{||f_u \times F_v||} \tag{3}$$

The 2nd fundamental form captures how the surface bends relative to the normal direction. Its coefficient is represented as eq.4:

$$L = f_{uu}.n, M = f_{uv}.n, N = f_{vv}.n \tag{4}$$

Thus, the 2nd fundamental form is shown in eq.5:

$$dH = Ldu^2 + 2Mdudv + Ndv^2 \tag{5}$$

For the sphere, one finds $L = \cos^2\emptyset\,, M = 0, N = 1$. The 2nd fundamental form encodes extrinsic curvature.

2.3. *Shape operator and principal curvatures*

The shape operator $S: T_pS \to T_pS$ is a linear map defined by $S(X) = -dn(X)$, measuring how the normal vector changes along tangent directions. In matrix form relative to the basis $\{f_u, f_v\}$, the shape operator is represented as in eq.6:

$$S = P = I^{-1}H = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \tag{6}$$

Where I and H are the matrices of the 1st and 2nd fundamental forms. The principal curvatures k_1, K_2 are the "eigenvalues of S". It satisfies as in eq.7:

$$k_{1,2} = H \pm \sqrt{H^2 - K} \tag{7}$$

Where K and H denotes "Gaussians and mean curvature".

2.4. *Gaussian and mean curvature*

The "Gaussian curvature measures intrinsic curvature" and is given by eq.8,9:

$$K = \frac{LN - M^2}{EG - F^2} \tag{8}$$

While the mean curvature is:

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} \tag{9}$$

These formulas follow directly from the definition of the shape operator and the fundamental forms . For the sphere, $K = 1$and $H = 1$. For a plane given by " $f(u, v) = (u, v, 0)$ ", we have $K = 0\ and\ H = 0$. The product of the

principal curvatures equals the Gaussian curvature and their sum equals twice the mean curvature[6][7].

2.5. *Curvature of curves on surfaces*

Let γ(s) be a curve on the "surface parameterized by arc length". Its tangent vector is $t = \gamma'(s)$. The curvature of the space curve is shown by eq.10:

$$k(s) = \frac{||t \times t\prime||}{||t||^3} \tag{10}$$

and decomposes into normal curvature**,** geodesic curvature and normal curvature components relative to the surface. The normal curvature is $k_n = t^\top S(t)$, where S is the "shape operator" and $t^\top$ denotes the "transpose". The "maximal and minimal values" of k_n as the direction t varies are the principal curvatures.

2.6. *Curvature classification and shape analysis*

Using computed curvatures, one can classify points on a surface. If $K > 0$, the point is "elliptic and locally convex"; if $K < 0$, it is "hyperbolic and locally saddle shaped"; if $K = 0$ and $H \neq 0$, it is "parabolic"; if both principal curvatures vanish, the surface is developable [8][9]. Curvature signatures allow shape comparison and feature detection.

2.7. *Numerical computation*

For discrete surfaces such as triangle meshes, derivatives in the above formulas are approximated by finite differences or fitting local patches. Discrete differential geometry defines discrete analogues of the fundamental forms by accumulating contributions of adjacent faces. The shape operator can be estimated at each vertex using covariance of neighboring normal. Robust estimation is necessary when data are noisy.

3. Results and outputs

To illustrate the methodology, we compute curvature for several canonical surfaces: a unit sphere S^2, a plane R^2, a right circular cylinder of radius 1, and a hyperbolic paraboloid (saddle) given by $z = x^2 - y^2$. Table 1 reports Gaussian curvature K, mean curvature H and principal curvatures k_1, k_2.

Table 1

Curvature invariants for canonical surfaces.

Surface	K	H	k1	k
Sphere S2	1	1	1	1
Plane	0	0	0	0
Cylinder	0	1/2	1	0
Hyperbolic paraboloid	−1-1	0	1	−1-1

The sphere has positive constant curvature; all directions bend equally. The plane has zero curvature and is intrinsically flat. The cylinder has zero Gaussian

curvature but positive mean curvature because it bends in one direction only. The saddle has negative Gaussian curvature and zero mean curvature, reflecting its saddle shape with bending in opposite directions. Differential geometry also classifies points based on the signs of principal curvatures. Table 2 summarizes the classification.

Table 2

Surface point classification by principal curvatures

k1 sign	k2 sign	Classification	Example
+	+	Elliptic (locally convex)	Sphere
+	0	Parabolic	Cylinder
+	−	Hyperbolic	Saddle
0	0	Flat	Plane

Elliptic points occur where the surface bends in the same direction along all tangents; hyperbolic points exhibit saddle behaviour. Parabolic points lie on the boundary between elliptic and hyperbolic regions and often correspond to ridges. Flat points are developable; surfaces such as cones and cylinders have regions where Gaussian curvature vanishes but mean curvature does not. These classifications assist in identifying features and understanding structural properties.

4. Conclusion and future scope

Curvature is a central concept in differential geometry and a powerful descriptor in shape analysis. By deriving formulas for the fundamental forms, shape operator and curvature invariants, this paper has presented a self-contained methodology for computing curvature on smooth surfaces and discretized meshes. Principal curvatures, Gaussian curvature and mean curvature capture intrinsic and extrinsic bending properties. The analysis of canonical surfaces illustrates how curvature invariants classify geometric structures and reveal the presence of features such as ridges and saddles. Differential geometry thus provides both elegant theoretical insights and practical tools for applications in computer graphics, geometry processing and physical modelling. Future research can explore two directions:

- Advanced shape descriptors: Combining curvature with higher-order invariants, such as curvature tensors and spectral signatures, may enhance shape matching and retrieval.
- Curvature flows and optimization: Studying evolution of surfaces under curvature flows (e.g., mean curvature flow, Willmore flow) can lead to algorithms for smoothing, fairing and optimizing shapes subject to geometric constraints.

Differential geometry offers a rich mathematical language to describe and analyze shapes. Its curvature measures provide intrinsic signatures independent of coordinate representations and form the backbone of modern shape analysis and geometry processing techniques.

References

[1] P. Van Vinh, A. Tounsi, and M.-O. Belarbi, "On the nonlocal free vibration analysis of functionally graded porous doubly curved shallow nanoshells with variable nonlocal parameters," *Eng. Comput.,* vol. 39, no. 1, pp. 835–855 (2023).

[2] Y. Hong, Y. Chi, S. Wu, Y. Li, Y. Zhu, and J. Yin, "Boundary curvature guided programmable shape-morphing kirigami sheets," *Nat. Commun.,* vol. 13, no. 1, pp. 530 (2022).

[3] E. Hajiesmaili, N. M. Larson, J. A. Lewis, and D. R. Clarke, "Programmed shape-morphing into complex target shapes using architected dielectric elastomer actuators," *Sci. Adv.,* vol. 8, no. 28, pp. eabn9198 (Aug. 2025).

[4] F. Mémoli and T. Needham, "Distance distributions and inverse problems for metric measure spaces," *Stud. Appl. Math.,* vol. 149, no. 4, pp. 943–1001 (Nov. 2022).

[5] Z. Yao, Q. Zhao, X. Li, and Q. Bi, "Point cloud registration algorithm based on curvature feature similarity," *Measurement,* vol. 177, pp. 109274 (2021)

[6] A. Jones, M. Leary, S. Bateman, and M. Easton, "Parametric design and evaluation of TPMS-like cellular solids," *Mater. Des.,* vol. 221, pp. 110908 (2022)

[7] D. Dhabliya, S. Kunche, S. Dingankar, S. S. Dari, R. Dhabliya, and V. Khetani, "Blockchain technology as a paradigm for enhancing cyber security in distributed systems," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 27, no. 2-B, pp. 729–740 (2024).

[8] P. Sahane, M. Gulhane, N. Rakesh, S. M. M. Naidu, M. Grover, and V. Mahajan, "Evaluating lightweight encryption schemes for resource-constrained devices in wireless sensor networks," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 28, no. 5-A, pp. 1803–1812 (2025).

[9] S. Kothawade and I. Zellar, "Blockchain-Enabled Transformation in Public Administrative Services: A Comparative Analysis of Current Applications and Future Potential", *IJRAET*, vol. 14, no. 1, pp. 128–127 (Apr. 2025).

Received December, 2024