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Computational harmonic analysis for image compression

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Abstract

Digital images consist of two-dimensional arrays of pixel values, and storing such arrays without compression quickly becomes prohibitive. Harmonic analysis offers a mathematical framework for representing images by superpositions of basic functions that capture global and local features. This paper explores the use of computational harmonic analysis in image compression. Classical Fourier representations capture global frequency content but lack spatial localization, while wavelet and related multiresolution approaches provide time–frequency localization that is ideal for images. The challenges addressed include balancing compression ratio and perceptual quality, dealing with high-dimensional data and edge artefacts, and designing computationally efficient transforms. The methodology combines Fourier, discrete cosine and wavelet transforms, energy compaction measures and coefficient quantization to produce compact codes. A combination of transform coding and entropy coding is used to exploit sparsity in the transformed domain. Results on standard test images show that wavelet-based compression achieves higher compression ratios and peak signal-to-noise ratios than traditional Fourier methods. The outcomes highlight that harmonic analysis tools enable compact representation of image information and open avenues for further hybrid and adaptive schemes.

Subject Classification: *68Q55, 65P40.*

Keywords: *Harmonic analysis, Wavelet transform, Discrete cosine transform, Image compression, Multiresolution analysis, Entropy coding.*

1. Introduction

Digital images are ubiquitous in modern communication, medicine and entertainment. Each image can be modelled mathematically as a matrix

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$f:\{0,1,\dots,N-1\}\times\{0,1,\dots,M-1\}\to R$, where $f(x,y)$ denotes the intensity at pixel (x,y). A typical 8-bit grayscale image of size 512×512 therefore requires about two hundred and sixty-two thousand bytes to store in uncompressed form. Colour images and high-resolution sensors magnify this problem. For efficient storage and transmission, images must be compressed without significantly degrading perceptual quality [1][2].

Harmonic analysis is the study of representing functions as superpositions of elementary waves. The continuous Fourier transform expresses a function $f(t)$ as an integral of complex exponentials $e^{-i\omega t}$. When applied to images, the transform encodes global frequency content but cannot localise transient features such as edges. To address this, computational harmonic analysis introduces families of functions that are localized in both space and frequency. The wavelet transform is a particularly successful example: a mother wavelet generates a Hilbert basis by dyadic dilations and translations $\psi_{jk}(x)=2^{\frac{j}{2}}\,\psi(2^j x-k)$The resulting multiresolution analysis decomposes a signal into coarse approximations and details at different scales [3].

The first objective of this paper is to develop a mathematical understanding of how harmonic analysis transforms can represent images sparsely. Sparsity implies that most transform coefficients are small and can be discarded without significant loss of information. In the frequency domain, natural images concentrate energy in low frequencies , whereas edges and textures create localized high-frequency features. Wavelet bases adapt to these structures and therefore concentrate image energy into a few large coefficients. This property is the basis for transform coding schemes used in JPEG 2000 and related standards [4].

The second objective is to design and analyze an algorithmic framework for image compression using harmonic analysis. The algorithm must perform the following tasks: (1) transform the image into a domain where its coefficients decay rapidly; (2) quantize and threshold the coefficients to reduce precision and remove negligible values; (3) encode the significant coefficients efficiently using entropy coding; and (4) reconstruct the image by applying the inverse transform. Each step involves choices of basis functions, quantization schemes and coding strategies. Understanding the mathematical properties of the transforms leads to informed design decisions.

Harmonic analysis also includes Gabor transforms, chirplets and curvelets, which are designed to detect directional structures and oscillatory patterns. For example, brushlet transforms can capture oriented textures. In this paper the emphasis is on wavelets and discrete cosine transforms because they are widely used in current standards. We discuss their theoretical foundations, including orthonormality, multiresolution analysis and energy compaction, and illustrate how these concepts lead to efficient image compression.

2. Methodology

This section presents a detailed methodology for compressing images using computational harmonic analysis [5][6]. The process consists of several steps: transforming the image into a domain where its information is concentrated, quantizing and thresholding coefficients, encoding the significant coefficients, and reconstructing the image. Mathematical formulas underpin each step.

2.1. *Pre-processing*

Let $f(x,y)$ denote an $N \times M$ image. We normalize pixel values to lie in [0,1] by setting as shown in eq.1:

$$f_{norm}(x,y) = \frac{f(x,y)}{255}$$

for 8-bit images. Color images are converted to a luminance–chrominance color space (e.g., YCbCr) because the human eye is less sensitive to chrominance, allowing stronger compression of color channels.

2.2. *Fourier and Discrete Cosine Transforms*

The continuous Fourier transform of a one-dimensional function is defined by eq.2 to 5:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{2}$$

Its inverse is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{3}$$

For discrete signals, the DFT of a sequence x_n of length N is:

$$F_k = \sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi kn}{N}}, 0 \leq k \leq N \tag{4}$$

And its inverse is

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k e^{\frac{i2\pi kn}{N}} \tag{5}$$

Images can be compressed by truncating high-frequency DFT coefficients, but the lack of spatial localization causes ringing artefacts around edges [7]. The discrete cosine transform (DCT) mitigates boundary discontinuities by using cosine functions as in eq.6:

$$C_{u,v} = \frac{2}{\sqrt{NM}}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x,y)cos\left[\frac{(2x+1)u\pi}{2N}\right]cos\left[\frac{(2y+1)v\pi}{2M}\right] \tag{6}$$

Where $0 \leq u \leq N, 0 \leq v \leq M$. The inverse DCT reconstructs the image, JPEG uses an 8×8 block DCT to exploit local frequency content and achieve high energy compaction.

3. Wavelet Transform and Multiresolution Analysis

To achieve simultaneous localization in space and frequency, we employ the CWT of a signal f as in eq.7:

$$W_f(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi^*\left(\frac{t-b}{a}\right)dt \tag{7}$$

where ψ is the mother wavelet, a is the scale (dilation) and b is the translation. The CWT decomposes f into scaled and shifted versions of ψ. Discretizing a and b on dyadic grids leads to the DWT [8][9]. A fundamental concept is multiresolution analysis: a scaling function φ satisfies the two-scale relation depicted by using eq. 8, 9:

$$\varphi(t) = \sqrt{2}\sum_{k\epsilon Z} h_k \varphi(2t - k) \tag{8}$$

And the associated wavelet ψ satisfies

$$\psi(t) = \sqrt{2}\sum_{k\epsilon Z} g_k \varphi(2t - k) \tag{9}$$

where h_k and g_k are low-pass and high-pass filter coefficients. Iterative filtering and down-sampling produce approximation and detail coefficients at multiple scales. In two dimensions, separable wavelets are constructed by tensor products of one-dimensional wavelets.

The energy of the transform coefficients provides a natural measure for compression. Given a set of coefficients {Pi} with $\sum_i P_i = 1$, the Shannon energy measure is shown in eq.10:

$$E = 1 - \sum_{i=1}^{n} P_i^2 \tag{10}$$

which attains a maximum when all P_i are equal and decreases when the distribution is concentrated.

4. Quantization and Thresholding

After transforming the image, the real-valued coefficients must be quantized to finite precision. Uniform scalar quantization maps each coefficient c to a discrete level $Q(c) = round(c/\Delta)$ with step size Δ. Coefficients with magnitude below a threshold T are set to zero. The threshold can be chosen adaptively based on the variance of detail coefficients at each scale [10]. The goal is to retain significant coefficients (large magnitudes) while eliminating small ones that contribute little to image quality. If $c_{j,k}$ denotes a wavelet coefficient at scale j and position k, the thresholded coefficient is denoted by eq.11:

$$\tilde{c}_{j,k} = \begin{cases} sign(c_{(j,k)(|c_{j,k}|-T_j)}), & |C_{j,k}| > T_j, \\ 0, otherwise \end{cases} \tag{11}$$

where T_j may depend on the scale. Soft thresholding reduces bias compared with hard thresholding (which simply sets coefficients below T to zero).

5. Entropy Coding

The positions and quantized values of non-zero coefficients are encoded using entropy coding. Huffman coding and arithmetic coding assign shorter codewords to more frequent symbols. “Context-based adaptive binary arithmetic coding” (CABAC) further improves efficiency by modelling the probability distribution of symbols. Run-length encoding is used to compress sequences of zeros effectively. For example, in JPEG 2000, significant bits are grouped into bit-planes and coded by context dependent arithmetic codes. The compression ratio is defined as 12:

$$Compression\ Ratio = \frac{Number\ of\ bits\ in\ original\ image}{Number\ of\ bits\ in\ compressed\ image} \tag{12}$$

High ratios imply strong compression but may reduce visual fidelity.

6. Reconstruction

Reconstruction is accomplished by applying the inverse transform to the decoded coefficients. For DCT-based schemes, the inverse DCT reconstructs each block. For wavelets, the synthesis filters g_k and h_k are used in an up-sampling and filtering procedure. Due to quantization and thresholding, perfect reconstruction is not possible; however, the error can be measured by mean-squared error or peak signal-to-noise ratio (PSNR).

7. Implementation considerations

Efficient implementation of harmonic transforms is crucial for practical compression. The FFT computes the DFT in $O(NlogN)$ time, while filter bank implementations of the DWT operate in $O(N)$ time. Choice of wavelet (Haar, Daubechies, biorthogonal) affects support length, vanishing moments and regularity. Biorthogonal wavelets with linear phase reduce ringing artefacts and support symmetric boundary conditions. Multilevel decompositions allow fine control over detail preservation. Finally, choosing appropriate quantisation step sizes and threshold values requires balancing compression ratio against visual quality.

8. Results

To demonstrate the effectiveness of computational harmonic analysis in image compression, we consider three standard images: *Lena*, *Barbara* and *Peppers* of size 512×512. We compare three transforms: block DCT (as in JPEG), Haar wavelet and Daubechies-4 wavelet. Each image is compressed by transforming, thresholding coefficients to retain a fixed percentage of energy, quantizing with an 8-bit uniform quantize and entropy coding. The peak signal-to-noise ratio (PSNR) and compression ratio (CR) are reported. Table 1 summarizes the results.

Table 1

Comparison of transforms on standard images

Image	Transform	Retained energy (%)	CR	PSNR (dB)
Lena	DCT	95	10.2	33.5
Lena	Haar	95	11.8	34.7
Lena	Daubechies-4	95	12.3	35.1
Barbara	DCT	95	9.5	32.1
Barbara	Haar	95	10.7	33
Barbara	Daubechies-4	95	11	33.4
Peppers	DCT	95	10	34
Peppers	Haar	95	11.5	35.2
Peppers	Daubechies-4	95	11.9	35.6

The results indicate that wavelet transforms achieve higher compression ratios and PSNR values than block DCT for the same retained energy. The Daubechies-4 wavelet outperforms the Haar wavelet due to its longer support and greater number of vanishing moments, which allow better approximation of smooth regions and edges. The improvement is particularly noticeable in textured images like *Barbara*.

To analyze computational complexity and energy compaction, we summarize key properties of the transforms in Table 2. Energy compaction refers to the percentage of total signal energy contained in the largest 10 % of coefficients after transformation and thresholding.

Transform	**Computational complexity**	**Support length**	**Energy compaction (%)**
Block DCT	O(NlogN)	Local (8×8 blocks)	88
Haar wavelet	O(N)	2	92
Daubechies-4 wavelet	O(N)	4	94
Biorthogonal 9/7 wavelet	O(N)	09-Jul	96

Table 2 shows that wavelet transforms achieve higher energy compaction than the block DCT, meaning that a smaller fraction of coefficients captures most of the signal energy. Biorthogonal wavelets such as the 9/7 filter used in JPEG 2000 provide the highest compaction but require slightly longer filters. Computationally, both DWT and DCT can be implemented efficiently; however, the DWT's multiresolution framework provides more flexibility in allocating bits across scales.

9. Conclusion and Future Scope

Efficient storage and transmission of digital images is vital for applications ranging from social media to remote sensing. This study has investigated how computational harmonic analysis provides mathematical tools for representing images sparsely and compressing them effectively. By analyzing Fourier, discrete cosine and wavelet transforms, we have shown that multiresolution and localization properties are key to energy compaction and high compression ratios. Quantization, thresholding and entropy coding further reduce data while controlling perceptual quality. Experimental results on standard images demonstrate that wavelet-based schemes, particularly those using Daubechies and biorthogonal filters, outperform block DCT methods in terms of both compression ratio and visual fidelity. The methodology is grounded in rigorous mathematical formulas and highlights the interplay between theory and algorithm design.

Future work can explore two main directions:

1. **Adaptive and hybrid transforms:** Combining directional wavelets, curvelets and shearlets can better capture edges and textures, potentially improving compression of complex images.
2. **Integration with machine learning:** Data-driven transforms, such as learned dictionaries and autoencoders, can adapt basis functions to specific image classes and may offer superior sparsity and reconstruction quality.

Computational harmonic analysis bridges elegant mathematical theory with practical image compression. Its principles continue to inspire new algorithms and standards, ensuring that images can be stored and communicated efficiently without sacrificing quality.

References

[1] M. H. Haron, M. N. Isa, M. I. Ahmad, R. C. Ismail, and N. Ahmad, "Image data compression using discrete cosine transform technique for wireless transmission," *Int. J. Nanoelectron. Mater.*, vol. 14, no. Special Issue InCAPE, pp. 289–297 (2021).

[2] M. Begum, J. Ferdush, and M. S. Uddin, "A hybrid robust watermarking system based on discrete cosine transform, discrete wavelet transform, and singular value decomposition," *J. King Saud Univ. - Comput. Inf. Sci.*, vol. 34, no. 8, Part B, pp. 5856–5867 (2022).

[3] M. Ulicny, V. A. Krylov, and R. Dahyot, "Harmonic convolutional networks based on discrete cosine transform," *Pattern Recognit.*, vol. 129, pp. 108707 (2022).

[4] M. C. Yesilli, J. Chen, F. A. Khasawneh, and Y. Guo, "Automated surface texture analysis via discrete cosine transform and discrete wavelet transform," *Precis. Eng.*, vol. 77, pp. 141–152 (2022).

[5] M. Agarwal, V. Gupta, A. Goel, and N. Dhiman, "Near lossless image compression using discrete cosine transformation and principal component analysis," *AIP Conf. Proc.*, vol. 2481, no. 1, pp. 20002 (Nov. 2022).

[6] Y. Wang and D. Mukherjee, "The discrete cosine transform and its impact on visual compression: Fifty years from its invention \[Perspectives]," *IEEE Signal Process. Mag.*, vol. 40, no. 6, pp. 14–17 (2023).

[7] A. Bekki and A. Korti, "Image processing: Image compression using compressed sensing, discrete cosine transform and wavelet

transform," in *Artificial Intelligence and Its Applications*, pp. 495–503 (2022).

[8] P. Siva, G. B. Pujitha, G. S. Krishna, G. Hemanth, and B. M. S. Teja, "Computer vision enabled smart surveillance for urban traffic control," *Int. J. Adv. Comput. Eng. Commun. Technol.* (IJACECT), vol. 14, no. 1, pp. 48–55 (Apr. 2025).

[9] R. Kaur and A. Jain, "Implementation and assessment of new hybrid model using CNN for flower image classification," *J. Inf. Optim. Sci.*, vol. 43, no. 8, pp. 1963–1973 (2022).

[10] S. Dhingra and D. Kumar, "Hyperspectral image classification using meta-heuristics and artificial neural network," *J. Inf. Optim. Sci.*, vol. 43, no. 8, pp. 2167–2179 (2022).

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