

Using triple maximal absolute young function determined by paranormed triple sequence spaces

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Abstract

This study presents an illustration of the triple maximal absolute young function as determined by paranormed triple sequence spaces. It also discusses several properties, such as $(\mathbb{U}^3)[\psi, \gamma, \rho]$, $(\mathbb{U}_0^3)[\psi, \gamma, \rho]$, $(\mathbb{U}_\infty^3)[\psi, \gamma, \rho]$ are linear spaces, $(\mathbb{U}^3)[\psi, \gamma, \rho]$, $(\mathbb{U}_0^3)[\psi, \gamma, \rho]$ are paranormed spaces and $(\mathbb{U}^3)[\gamma, \rho] \subset (\mathbb{U}^3)[\psi, \gamma, \rho]$.

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1. Introduction

The triple maximal absolute young function described by paranormed triple sequence spaces is presented in this study. Orlicz functions were the basis for the work of Lindenstauss and Tzafriri [5]. In a subsequent phase, we offer analyses of the many spaces of Orlicz sequences by other authors, including Lupaş [1], Parashar and Choudhary [6], Esi, Isik and Esi [3], Esi and Et [2] and many more [7-14]. We present the classes of spaces of triple sequences spaces $(\mathbb{U}^3)[\psi, \gamma, \rho]$, $(\mathbb{U}_0^3)[\psi, \gamma, \rho]$, $(\mathbb{U}_\infty^3)[\psi, \gamma, \rho]$ in this study. It fulfills the subsequent axioms:

1) $\mathcal{G}(0)$ is equal to 0. 2) If $\mathfrak{a}$ in $\mathbb{V}$, $\mathcal{G}(\mathfrak{a}) = \mathcal{G}(-\mathfrak{a})$. 3) $\mathcal{G}(\mathfrak{a} + \mathfrak{b}) \leq \mathcal{G}(\mathfrak{a}) + \mathcal{G}(\mathfrak{b})$, where $\mathfrak{a}, \mathfrak{b}$ are in $\mathbb{V}$. 4) Given a sequence of scalars (α_n) and a sequence of vectors $(\mathfrak{A}_n)$ with $\mathcal{G}(\mathfrak{A}_n - \mathfrak{A}) \rightarrow 0$ as $n \rightarrow \infty$, $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$ then

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$\mathcal{G}(\alpha_n \mathfrak{X}_n - \alpha \mathfrak{X}) \rightarrow 0$ as $n \rightarrow \infty$. The final characteristic is known as continuity of multiplication by scalars and it tends to the function $\mathcal{G}: \mathbb{V} \rightarrow \mathbb{R}$, where $\mathcal{G}$ is a paranorm and $\mathbb{V}$ is a paranormed space. A continuous, non-decreasing, convex function with $\varphi(0) = 0, \varphi(\mathfrak{Z}) > 0$ as $\mathfrak{Z} > 0, \varphi(\mathfrak{Z}) \rightarrow \infty$ as $\mathfrak{Z} \rightarrow \infty$ is represented by φ this implies that $\varphi: [0, \infty) \rightarrow [0, \infty)$ is an Orlicz function.

2. Definitions and Preliminaries

Let $\mathfrak{Y}: [0, \infty) \rightarrow [0, \infty) \ni \mathfrak{Y}(\mathfrak{X}) = \frac{\theta(\mathfrak{X})}{|\mathfrak{X}|^2}$ and φ is an Orlicz function leads to $\mathfrak{Y}$ is a maximal absolute young function. A triple maximum absolute young functions is a function

$$\psi: [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \times [0, \infty) \times [0, \infty) \ni$$

$$\psi(\mathfrak{Q}, \mathfrak{W}, \mathfrak{M}) = (\psi_1(\mathfrak{Q}), \psi_2(\mathfrak{W}), \psi_3(\mathfrak{M})),$$

where $\psi_1: [0, \infty) \rightarrow [0, \infty) \ni \psi_1(\mathfrak{X}) = \frac{\theta_1(\mathfrak{X})}{|\mathfrak{X}|^2}, \Theta_2: [0, \infty) \rightarrow [0, \infty) \ni \psi_2(\mathfrak{S}) = \frac{\theta_2(\mathfrak{S})}{|\mathfrak{S}|^2}, \psi_3: [0, \infty) \rightarrow [0, \infty) \ni \psi_3(\mathfrak{R}) = \frac{\theta_3(\mathfrak{R})}{|\mathfrak{R}|^2}$. These functions are even, convex, continuous and non-decreasing, that hold the following conditions:

- i) $\psi_1(0) = 0, \psi_2(0) = 0, \psi_3(0) = 0 \Rightarrow \psi(0, 0, 0) = (\psi_1(0), \psi_2(0), \psi_3(0)) = (0, 0, 0)$
- ii) $\psi_1(\mathfrak{Q}) > 0, \psi_2(\mathfrak{W}) > 0, \psi_3(\mathfrak{M}) > 0 \Rightarrow \psi(\mathfrak{Q}, \mathfrak{W}, \mathfrak{M}) = (\psi_1(\mathfrak{Q}), \psi_2(\mathfrak{W}), \psi_3(\mathfrak{M})) > (0, 0, 0), \forall \mathfrak{Q} > 0, \mathfrak{W} > 0, \mathfrak{M} > 0$ by which we say $(\mathfrak{Q}, \mathfrak{W}, \mathfrak{M}) > (0, 0, 0)$ as $\psi_1(\mathfrak{Q}) > 0, \psi_2(\mathfrak{W}) > 0, \psi_3(\mathfrak{M}) > 0$.
- iii) $\psi_1(\mathfrak{Q}) \rightarrow \infty, \psi_2(\mathfrak{W}) \rightarrow \infty, \psi_3(\mathfrak{M}) \rightarrow \infty$ as $\mathfrak{Q} \rightarrow \infty, \mathfrak{W} \rightarrow \infty, \mathfrak{M} \rightarrow \infty \Rightarrow \psi(\mathfrak{Q}, \mathfrak{W}, \mathfrak{M}) = (\psi_1(\mathfrak{Q}), \psi_2(\mathfrak{W}), \psi_3(\mathfrak{M})) \rightarrow (\infty, \infty, \infty)$ as $(\mathfrak{Q}, \mathfrak{W}, \mathfrak{M}) \rightarrow (\infty, \infty, \infty)$ by which we say $\psi(\mathfrak{Q}, \mathfrak{W}, \mathfrak{M}) \rightarrow (\infty, \infty, \infty)$ as $\psi_1(\mathfrak{Q}) \rightarrow \infty, \psi_2(\mathfrak{W}) \rightarrow \infty, \psi_3(\mathfrak{M}) \rightarrow \infty$. for more we ref [4].

Let $\gamma = (\gamma_{\text{nmui}})$ be a non-decreasing triple sequence of positive numbers and $\beta_{111} = 1, \beta_{\text{nmui}+1} \leq \beta_{\text{nmui}} + 1, \mathcal{P} = (\mathcal{P}_{\text{sre}}), \mathfrak{X} = (\mathfrak{X}_{\text{sre}})$ be a triple sequence and $\rho > 0, \mathbb{L}_{\text{nmui}} = [\text{nmui} - \beta_{\text{nmui}} + 1, \text{nmui}]$ and $\mathfrak{X}_0$ is any number.

$$(\mathbb{W}^3)[\psi, \gamma, \mathcal{P}] = \{(\mathfrak{X}_{\text{sre}}) = ((\mathfrak{X}_1)_{\text{sre}}, (\mathfrak{X}_2)_{\text{sre}}, (\mathfrak{X}_3)_{\text{sre}}):$$

$$\lim_{\gamma_{\text{nmui}}} \frac{1}{\gamma_{\text{nmui}}} \sum_{\mathfrak{S} \in \mathbb{L}_{\text{nmui}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nmui}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nmui}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{X}_1)_{\text{sre}}, (\mathfrak{X}_1)_0)}{\rho} \right) \gamma \right. \\ \left. \psi_2 \left(\frac{\bar{d}((\mathfrak{X}_2)_{\text{sre}}, (\mathfrak{X}_2)_0)}{\rho} \right) \gamma \psi_3 \left(\frac{\bar{d}((\mathfrak{X}_3)_{\text{sre}}, (\mathfrak{X}_3)_0)}{\rho} \right) \right]^{\mathcal{P}_{\text{sre}}} = (0, 0, 0)\}.$$

$$(\mathbb{W}_0^3)[\psi, \gamma, \mathcal{P}] = \{(\mathfrak{X}_{\text{sre}}) = ((\mathfrak{X}_1)_{\text{sre}}, (\mathfrak{X}_2)_{\text{sre}}, (\mathfrak{X}_3)_{\text{sre}}):$$

$$\lim_{\gamma_{\text{nmui}}} \frac{1}{\gamma_{\text{nmui}}} \sum_{\mathfrak{S} \in \mathbb{L}_{\text{nmui}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nmui}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nmui}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{X}_1)_{\text{sre}}, 0)}{\rho} \right) \gamma \psi_2 \left(\frac{\bar{d}((\mathfrak{X}_2)_{\text{sre}}, 0)}{\rho} \right) \gamma \right. \\ \left. \psi_3 \left(\frac{\bar{d}((\mathfrak{X}_3)_{\text{sre}}, 0)}{\rho} \right) \right]^{\mathcal{P}_{\text{sre}}} = (0, 0, 0)\}. (\mathbb{W}_{\infty}^3)[\psi, \gamma, \mathcal{P}] = \{(\mathfrak{X}_{\text{sre}}) = ((\mathfrak{X}_1)_{\text{sre}}, (\mathfrak{X}_2)_{\text{sre}}, (\mathfrak{X}_3)_{\text{sre}}):$$

$$\left\{ \sup_{\gamma_{\text{nm}u}} \frac{1}{\gamma_{\text{nm}u}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}u}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}}, 0)}{\rho} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}}, 0)}{\rho} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}}, 0)}{\rho} \right) \right]^{\mathcal{P}_{\text{sre}}} < (\infty, \infty, \infty) \right\}.$$

3. Main Results

Theorem 3.1: $(\mathbb{U}^3)[\psi, \gamma, \mathfrak{p}]$, $(\mathbb{U}_0^3)[\psi, \gamma, \mathfrak{p}]$, $(\mathbb{U}_\infty^3)[\psi, \gamma, \mathfrak{p}]$ are linear spaces over the field complex numbers.

Proof: Consider the space $(\mathbb{U}_0^3)[\psi, \gamma, \mathfrak{p}]$. Let $\mathfrak{A} = (\mathfrak{A}_{\text{sre}}) = ((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_3)_{\text{sre}})$, $\mathfrak{S} = (\mathfrak{S}_{\text{sre}}) = ((\mathfrak{S}_1)_{\text{sre}}, (\mathfrak{S}_2)_{\text{sre}}, (\mathfrak{S}_3)_{\text{sre}}) \in (\mathbb{U}_0^3)[\psi, \gamma, \mathfrak{p}]$ and $\alpha, \beta \in \mathbb{C}$. To prove this theorem we need to find some

$$\rho_3 > 0 \ni \lim_{\gamma_{\text{nm}u}} \frac{1}{\gamma_{\text{nm}u}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}u}} \left[\psi_1 \left(\frac{\bar{d}(\alpha(\mathfrak{A}_1)_{\text{sre}} + \beta(\mathfrak{S}_1)_{\text{sre}}, 0)}{\rho_3} \right) \vee \psi_2 \left(\frac{\bar{d}(\alpha(\mathfrak{A}_2)_{\text{sre}} + \beta(\mathfrak{S}_2)_{\text{sre}}, 0)}{\rho_3} \right) \vee \psi_3 \left(\frac{\bar{d}(\alpha(\mathfrak{A}_3)_{\text{sre}} + \beta(\mathfrak{S}_3)_{\text{sre}}, 0)}{\rho_3} \right) \right]^{\mathcal{P}_{\text{sre}}} = (0, 0, 0).$$

Since $\mathfrak{A} = (\mathfrak{A}_{\text{sre}}) = ((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_3)_{\text{sre}})$, $\mathfrak{S} = (\mathfrak{S}_{\text{sre}}) = ((\mathfrak{S}_1)_{\text{sre}}, (\mathfrak{S}_2)_{\text{sre}}, (\mathfrak{S}_3)_{\text{sre}}) \in (\mathbb{U}_0^3)[\psi, \gamma, \mathfrak{p}]$, then there exist $\rho_1 > 0, \rho_2 > 0 \ni$

$$\lim_{\gamma_{\text{nm}u}} \frac{1}{\gamma_{\text{nm}u}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}u}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}}, 0)}{\rho_1} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}}, 0)}{\rho_1} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}}, 0)}{\rho_1} \right) \right]^{\mathcal{P}_{\text{sre}}} = (0, 0, 0),$$

And

$$\lim_{\gamma_{\text{nm}u}} \frac{1}{\gamma_{\text{nm}u}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}u}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{S}_1)_{\text{sre}}, 0)}{\rho_2} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{S}_2)_{\text{sre}}, 0)}{\rho_2} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{S}_3)_{\text{sre}}, 0)}{\rho_2} \right) \right]^{\mathcal{P}_{\text{sre}}} = (0, 0, 0).$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$.

Since ψ is convex and non-decreasing, we obtain

$$\begin{aligned} & \frac{1}{\gamma_{\text{nm}u}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}u}} \left[\psi_1 \left(\frac{\bar{d}(\alpha(\mathfrak{A}_1)_{\text{sre}} + \beta(\mathfrak{S}_1)_{\text{sre}}, 0)}{\rho_3} \right) \vee \psi_2 \left(\frac{\bar{d}(\alpha(\mathfrak{A}_2)_{\text{sre}} + \beta(\mathfrak{S}_2)_{\text{sre}}, 0)}{\rho_3} \right) \vee \psi_3 \left(\frac{\bar{d}(\alpha(\mathfrak{A}_3)_{\text{sre}} + \beta(\mathfrak{S}_3)_{\text{sre}}, 0)}{\rho_3} \right) \right]^{\mathcal{P}_{\text{sre}}} \\ & \leq \frac{1}{\gamma_{\text{nm}u}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}u}} \frac{1}{2^{\mathcal{P}_{\text{sre}}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}}, 0)}{\rho_1} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}}, 0)}{\rho_1} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}}, 0)}{\rho_1} \right) \right]^{\mathcal{P}_{\text{sre}}} \\ & \quad + \left(\psi_1 \left(\frac{\bar{d}((\mathfrak{S}_1)_{\text{sre}}, 0)}{\rho_2} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{S}_2)_{\text{sre}}, 0)}{\rho_2} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{S}_3)_{\text{sre}}, 0)}{\rho_2} \right) \right]^{\mathcal{P}_{\text{sre}}} \\ & \leq \frac{\mathbb{K}}{\gamma_{\text{nm}u}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}u}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}}, 0)}{\rho_1} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}}, 0)}{\rho_1} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}}, 0)}{\rho_1} \right) \right]^{\mathcal{P}_{\text{sre}}} \\ & \quad + \frac{\mathbb{K}}{\gamma_{\text{nm}u}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}u}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}u}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{S}_1)_{\text{sre}}, 0)}{\rho_2} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{S}_2)_{\text{sre}}, 0)}{\rho_2} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{S}_3)_{\text{sre}}, 0)}{\rho_2} \right) \right]^{\mathcal{P}_{\text{sre}}} \rightarrow (0, 0, 0) \text{ as } n, m, u \rightarrow \infty, \text{ where} \end{aligned}$$

$\mathbb{K} = \max(1, 2^{\mathbb{H}-1})$, $\mathbb{H} = \sup_{\text{sre}} \mathcal{P}_{\text{sre}}$. Therefore $\alpha\mathfrak{A} + \beta\mathfrak{S} \in (\mathbb{U}_0^3)[\psi, \gamma, \mathfrak{p}]$. Other cases are similar.

Theorem 3.2: $\forall p = (p_{\text{sre}})$ then $(\mathbb{U}^3)[\psi, \gamma, p]$, $(\mathbb{U}_0^3)[\psi, \gamma, p]$ are paranormed spaces under the para-norm:

$$\begin{aligned} \mathcal{G}(\mathfrak{A}) &= \inf \left\{ \rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}}, \rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}}, \rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}} : \right. \\ &\left(\frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}, 0})}{\rho} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}, 0})}{\rho} \right) \vee \right. \right. \\ &\left. \left. \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}, 0})}{\rho} \right) \right] \right)^{\frac{1}{\mathcal{M}}} \leq (1, 1, 1), \forall n, m, u \geq 1 \Big\}, \mathcal{M} = \max(1, \mathbb{E}), \mathbb{E} = \\ &\sup_{\text{sre}} \mathcal{P}_{\text{sre}}. \end{aligned}$$

Proof: It well know $\mathcal{G}(0) = 0$ and $\mathcal{G}(\mathfrak{A}) = \mathcal{G}(-\mathfrak{A})$. As $\bar{d}$ is a translation invariant, it can be demonstrate easily, $\mathcal{G}(\mathfrak{A}_1 + \mathfrak{S}_1) = \mathcal{G}(\mathfrak{A}_1) + \mathcal{G}(\mathfrak{S}_1)$, $\mathcal{G}(\mathfrak{A}_2 + \mathfrak{S}_2) = \mathcal{G}(\mathfrak{A}_2) + \mathcal{G}(\mathfrak{S}_2)$, $\mathcal{G}(\mathfrak{A}_3 + \mathfrak{S}_3) = \mathcal{G}(\mathfrak{A}_3) + \mathcal{G}(\mathfrak{S}_3)$, $\forall \mathfrak{A} = (\mathfrak{A}_{\text{sre}}) = ((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_3)_{\text{sre}})$, $\mathfrak{S} = (\mathfrak{S}_{\text{sre}}) = ((\mathfrak{S}_1)_{\text{sre}}, (\mathfrak{S}_2)_{\text{sre}}, (\mathfrak{S}_3)_{\text{sre}}) \in (\mathbb{U}_0^3)$ $[\psi, \gamma, p]$. As $\psi_1(0) = 0, \psi_2(0) = 0, \psi_3(0) = 0$, we obtain $\inf \left\{ \left(\rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}}, \rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}}, \rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}} \right) : \right\} = (0, 0, 0)$, $\forall \mathfrak{A} = (\mathfrak{A}_{\text{sre}}) = ((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_3)_{\text{sre}}) = (0, 0, 0)$.

On the other hand, Assume that $\mathcal{G}(\mathfrak{A}_1) = \mathcal{G}((\mathfrak{A}_1)_{\text{sre}}) = 0, \mathcal{G}(\mathfrak{A}_2) = \mathcal{G}((\mathfrak{A}_2)_{\text{sre}}) = 0, \mathcal{G}(\mathfrak{A}_3) = \mathcal{G}((\mathfrak{A}_3)_{\text{sre}}) = 0$, then

$$\begin{aligned} \inf \left\{ \left(\rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}}, \rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}}, \rho^{\frac{\mathcal{P}_{\text{nm}}}{\mathbb{E}}} \right) : \left(\frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}, 0})}{\rho} \right) \vee \right. \right. \right. \\ \left. \left. \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}, 0})}{\rho} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}, 0})}{\rho} \right) \right] \right)^{\frac{1}{\mathcal{M}}} \leq (1, 1, 1), \forall n, m, u \geq 1 \Big\} = (0, 0, 0). \end{aligned}$$

This signifies that,

$$\begin{aligned} \forall \varepsilon > 0, \exists 0 < \rho_\varepsilon < \varepsilon \exists \left(\frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}, 0})}{\rho_\varepsilon} \right) \vee \right. \right. \\ \left. \left. \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}, 0})}{\rho_\varepsilon} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}, 0})}{\rho_\varepsilon} \right) \right] \right)^{\frac{1}{\mathcal{M}}} \leq (1, 1, 1). \end{aligned}$$

Thus, $\forall n, m, u$, we obtain,

$$\begin{aligned} \left(\frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}, 0})}{\varepsilon} \right) \vee \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}, 0})}{\varepsilon} \right) \vee \right. \right. \\ \left. \left. \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}, 0})}{\varepsilon} \right) \right] \right)^{\frac{1}{\mathcal{M}}} < \left(\frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}, 0})}{\rho_\varepsilon} \right) \vee \right. \right. \\ \left. \left. \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}, 0})}{\rho_\varepsilon} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}, 0})}{\rho_\varepsilon} \right) \right] \right)^{\frac{1}{\mathcal{M}}} \leq (1, 1, 1). \end{aligned}$$

Suppose that

$$\bar{d}((\mathfrak{A}_1)_{\text{nmupqa}}, 0) \neq 0, \bar{d}((\mathfrak{A}_2)_{\text{nmupqa}}, 0) \neq 0, \bar{d}((\mathfrak{A}_3)_{\text{nmupqa}}, 0) \neq 0,$$

for some $p, q, a \in \mathbb{L}_{\text{nm}}$.

Let $\varepsilon \rightarrow 0$ then $\left(\frac{((\mathfrak{A}_1)_{\text{nmupqa}}, 0)}{\varepsilon}\right) \rightarrow \infty, \left(\frac{((\mathfrak{A}_2)_{\text{nmupqa}}, 0)}{\varepsilon}\right) \rightarrow \infty, \left(\frac{((\mathfrak{A}_3)_{\text{nmupqa}}, 0)}{\varepsilon}\right) \rightarrow \infty$.

This mean that,

$$\left(\frac{1}{\gamma_{\text{nmuu}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nmuu}}} \left[\psi_1 \left(\frac{\tilde{d}((\mathfrak{A}_1)_{\text{nmupqa}}, 0)}{\varepsilon} \right) \vee \psi_2 \left(\frac{\tilde{d}((\mathfrak{A}_2)_{\text{nmupqa}}, 0)}{\varepsilon} \right) \vee \right. \right. \\ \left. \left. \psi_3 \left(\frac{\tilde{d}((\mathfrak{A}_3)_{\text{nmupqa}}, 0)}{\varepsilon} \right) \right] \right]^{\mathcal{P}_{\text{sre}}} \frac{1}{\mathcal{M}} \rightarrow (\infty, \infty, \infty),$$

so we arrive at a contradiction. Thus $(\mathfrak{A}_1)_{\text{nmupqa}} \neq 0, (\mathfrak{A}_1)_{\text{nmupqa}} \neq 0, (\mathfrak{A}_1)_{\text{nmupqa}} \neq 0$. Let λ be any complex number. By definition

$$\mathcal{G}(\lambda \mathfrak{A}) = \inf \left\{ \rho^{\frac{\mathcal{P}_{\text{nmuu}}}{\mathbb{E}}}, \rho^{\frac{\mathcal{P}_{\text{nmuu}}}{\mathbb{E}}}, \rho^{\frac{\mathcal{P}_{\text{nmuu}}}{\mathbb{E}}} : \right. \\ \left. \left(\frac{1}{\gamma_{\text{nmuu}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nmuu}}} \left[\psi_1 \left(\frac{\tilde{d}(\lambda(\mathfrak{A}_1)_{\text{sre}}, 0)}{\rho} \right) \vee \psi_2 \left(\frac{\tilde{d}(\lambda(\mathfrak{A}_2)_{\text{sre}}, 0)}{\rho} \right) \vee \right. \right. \right. \\ \left. \left. \left. \psi_3 \left(\frac{\tilde{d}(\lambda(\mathfrak{A}_3)_{\text{sre}}, 0)}{\rho} \right) \right] \right]^{\mathcal{P}_{\text{sre}}} \frac{1}{\mathcal{M}} \leq (1, 1, 1), \forall \mathfrak{n}, \mathfrak{m}, \mathfrak{u} \geq 1 \right\}.$$

Then

$$\mathcal{G}(\lambda \mathfrak{A}) = \inf \left\{ (|\lambda| \mathfrak{t})^{\frac{\mathcal{P}_{\text{nmuu}}}{\mathbb{E}}} : \left(\frac{1}{\gamma_{\text{nmuu}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nmuu}}} \left[\psi_1 \left(\frac{\tilde{d}((\mathfrak{A}_1)_{\text{sre}}, 0)}{\mathfrak{t}} \right) \vee \right. \right. \right. \\ \left. \left. \left. \psi_2 \left(\frac{\tilde{d}((\mathfrak{A}_2)_{\text{sre}}, 0)}{\mathfrak{t}} \right) \vee \psi_3 \left(\frac{\tilde{d}((\mathfrak{A}_3)_{\text{sre}}, 0)}{\mathfrak{t}} \right) \right] \right]^{\mathcal{P}_{\text{sre}}} \frac{1}{\mathcal{M}} \leq (1, 1, 1), \forall \mathfrak{n}, \mathfrak{m}, \mathfrak{u} \geq 1 \right\}, \text{ where}$$

$\mathfrak{t} = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{\mathcal{P}_{\text{sre}}} \leq \max(1, |\lambda|^{\mathbb{E}})$, we have

$$\mathcal{G}(\lambda \mathfrak{A}) \leq (\max(1, |\lambda|^{\mathbb{E}}))^{\frac{1}{\mathcal{M}}}. \\ \inf \left\{ (\mathfrak{t})^{\frac{\mathcal{P}_{\text{nmuu}}}{\mathbb{E}}} : \left(\frac{1}{\gamma_{\text{nmuu}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nmuu}}} \left[\psi_1 \left(\frac{\tilde{d}((\mathfrak{A}_1)_{\text{sre}}, 0)}{\mathfrak{t}} \right) \vee \psi_2 \left(\frac{\tilde{d}((\mathfrak{A}_2)_{\text{sre}}, 0)}{\mathfrak{t}} \right) \vee \right. \right. \right. \\ \left. \left. \left. \psi_3 \left(\frac{\tilde{d}((\mathfrak{A}_3)_{\text{sre}}, 0)}{\mathfrak{t}} \right) \right] \right]^{\mathcal{P}_{\text{sre}}} \frac{1}{\mathcal{M}} \leq (1, 1, 1), \forall \mathfrak{n}, \mathfrak{m}, \mathfrak{u} \geq 1 \right\}.$$

Thus, the above inequality implies that a scalar multiplication is continuous.

Theorem 3.3: Let $0 < \mathfrak{h} = \inf_{\text{sre}} \mathcal{P}_{\text{sre}} \leq \sup_{\text{sre}} \mathcal{P}_{\text{sre}} = \mathfrak{H} < \infty, \psi$ which satisfies the Δ_2 -condition, we have $(\mathbb{U}^3)[\gamma, \mathcal{P}] \subset (\mathbb{U}^3)[\psi, \gamma, \mathcal{P}]$, where

$$(\mathbb{U}^3)[\gamma, \mathcal{P}] = \{(\mathfrak{A}_{\text{sre}}) = ((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_3)_{\text{sre}}) : \\ \lim_{\gamma_{\text{nmuu}}} \frac{1}{\gamma_{\text{nmuu}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nmuu}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nmuu}}} \left[\psi_1 \left(\frac{\tilde{d}((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_1)_0)}{\rho} \right) \vee \right. \\ \left. \psi_2 \left(\frac{\tilde{d}((\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_2)_0)}{\rho} \right) \vee \psi_3 \left(\frac{\tilde{d}((\mathfrak{A}_3)_{\text{sre}}, (\mathfrak{A}_3)_0)}{\rho} \right) \right]^{\mathcal{P}_{\text{sre}}} = (0, 0, 0) \},$$

for some $\rho > 0$.

Proof: Let $(\mathfrak{A}_{\text{sre}}) = ((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_3)_{\text{sre}}) \in (\mathbb{U}^3)[\gamma, \rho]$ so that,

$$\lim_{\gamma_{\text{nm}} \rightarrow 0} \frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} \left[\psi_1 \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_1)_0)}{\rho} \right) \vee \right. \\ \left. \psi_2 \left(\frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_2)_0)}{\rho} \right) \vee \psi_3 \left(\frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}}, (\mathfrak{A}_3)_0)}{\rho} \right) \right]^{\rho_{\text{sre}}} = (0, 0, 0),$$

for some $\rho > 0$. Let $\varepsilon > 0$ and choose $\mathcal{S}$ with $0 < \mathcal{S} < 1$: $[\psi_1(\mathfrak{t}) \vee \psi_2(\mathfrak{t}) \vee \psi_3(\mathfrak{t})] < (\varepsilon, \varepsilon, \varepsilon), \forall 0 \leq \mathfrak{t} \leq \mathcal{S}$.

Suppose that $\mathfrak{S}_{\text{sre}} = \left(\frac{\bar{d}((\mathfrak{A}_1)_{\text{sre}}, (\mathfrak{A}_1)_0)}{\rho} \vee \frac{\bar{d}((\mathfrak{A}_2)_{\text{sre}}, (\mathfrak{A}_2)_0)}{\rho} \vee \frac{\bar{d}((\mathfrak{A}_3)_{\text{sre}}, (\mathfrak{A}_3)_0)}{\rho} \right)$ and considered $\lim_{\gamma_{\text{nm}} \rightarrow 0} \frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} [\psi(\mathfrak{S}_{\text{sre}})]^{\rho_{\text{sre}}} = 0 < \beta_{\text{nm}} \max(\varepsilon, \varepsilon^{\mathbb{H}})$. Based on the continuity of ψ , we have $\mathfrak{S}_{\text{sre}} \leq \frac{\mathfrak{S}_{\text{sre}}}{\mathcal{S}} \leq 1 + \frac{\mathfrak{S}_{\text{sre}}}{\mathcal{S}}$. Given that ψ is convex and does not decrease, it should follow that, $\psi(\mathfrak{S}_{\text{sre}}) < \psi\left(1 + \frac{\mathfrak{S}_{\text{sre}}}{\mathcal{S}}\right) \leq \frac{1}{2}\psi(2) + \frac{1}{2}\psi\left(\frac{2\mathfrak{S}_{\text{sre}}}{\mathcal{S}}\right)$. As ψ satisfies Δ_2 -condition, on it we can write, $\psi(\mathfrak{S}_{\text{sre}}) \leq \frac{\mathbb{L} \mathfrak{S}_{\text{sre}}}{2 \mathcal{S}} \psi(2) + \frac{\mathbb{L} \mathfrak{S}_{\text{sre}}}{2 \mathcal{S}} \psi(2) = \mathbb{L} \frac{\mathfrak{S}_{\text{sre}}}{\mathcal{S}} \psi(2)$. We'll receive the following approximations

$$\sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}, \mathfrak{S}_{\text{sre}} > \mathcal{S}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}, \mathfrak{S}_{\text{sre}} > \mathcal{S}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}, \mathfrak{S}_{\text{sre}} > \mathcal{S}} [\psi(\mathfrak{S}_{\text{sre}})]^{\rho_{\text{sre}}} \\ \leq \max(1, [\mathbb{L}\mathbb{M}(2)\mathcal{S}^{-1}]^{\mathbb{H}}) \gamma_{\text{nm}} \frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} [(\mathfrak{S}_{\text{sre}})]^{\rho_{\text{sre}}}, \\ \frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} [(\mathfrak{S}_{\text{sre}})]^{\rho_{\text{sre}}} \\ \leq \max(\varepsilon, \varepsilon^{\mathbb{H}}) + \max\left(1, \left[\frac{\mathbb{L}}{\mathcal{S}} \psi(2)\right]^{\mathbb{H}}\right) \frac{1}{\gamma_{\text{nm}}} \sum_{\mathfrak{s} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{r} \in \mathbb{L}_{\text{nm}}} \sum_{\mathfrak{e} \in \mathbb{L}_{\text{nm}}} [(\mathfrak{S}_{\text{sre}})]^{\rho_{\text{sre}}}.$$

We take it limits $\varepsilon \rightarrow 0$ and $\mathfrak{n}, \mathfrak{m}, \mathfrak{u} \rightarrow \infty$, consequently $\mathfrak{A} = (\mathfrak{A}_{\text{sre}}) \in (\mathbb{U}^3)[\psi, \gamma, \rho]$. Consequently $(\mathbb{U}^3)[\gamma, \rho] \subset (\mathbb{U}^3)[\psi, \gamma, \rho]$.

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