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Solving stochastic delay differential equation by using Runge-Kutta method and method of lines

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Abstract

In this study, we have examined a stochastic delay differential equation in one dimension. We employed the Runge-Kutta method along with the method of lines to address the issue, establishing stability analysis and studying the spread of discontinuities. Additionally, we derived a semi-discretization in the temporal domain. Subsequently, the Runge-Kutta method of order four and cubic Hermite Interpolation is derived and numerical examples are illustrated for the theoretical result.

Subject Classification: *37H10, 60G20, 35R60.*

Keywords: *Stochastic delay differential equation, Runge-kutta method, Method of lines and partial discretization.*

1. Introduction

In recent times, nature exhibits numerous mechanisms involving time delays, where the forthcoming condition of a system depends upon its preceding events. Stochastic Delay Differential Equations (SDDE), emerge as the optimal approach for simulating such systems. The literature [15], [16] has extensively explored numerical methods and convergence analysis for ODDE as well as hyperbolic PDE. Implementing such a technique requires a substantial understanding of computational methods as outlined in [10]. A fully implicit method with unconditional stability has been investigated in [11]. Implicit Runge-Kutta (IRK) methods represent sophisticated methods for discretizing

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time discussed, offering high levels of precision [12], [13], [14]. The methods used leverage structured formulations derived from carefully chosen discretization formulae in time [7], [8], [9]. Given that systems resulting from discretization of spatial dimensions in time-varying PDEs can be exceedingly large, specialized approaches are required to exploit the structural characteristics stemming from spatial discretization [4], [5], [6]. Additionally, a randomized two-stage Runge–Kutta scheme for delay differential equations (DDEs) is proposed, and its upper error bound is analyzed in the $Lp(\Omega)$ - norm for $p \in [2, +\infty)$ [18]. The sufficient and necessary conditions for mean square stability of a general class of stochastic Runge–Kutta methods via predictor-corrector approaches (SRK-PCMs) are established, demonstrating superior performance [17]. Moreover, exponential time distinguishing Runge–Kutta (ETDRK) integrators, up to second order, are analyzed by redefining ETDRK methods as a type of parametric Runge–Kutta integrator [1], [2]. The research evaluates the stabilization-induced lag effect and addresses delayed convergence using a relaxation approach [3], [19].

This paper is structured as follows: Initially, the problem is stated. Then, the maximum principles along with associated attributes are demonstrated. Consequently, this discussion explicit the techniques of solving the partial-discretization problem through the backward Euler scheme. Subsequently, we concentrate on the Runge-Kutta method's fourth-order discretization. The next couple of sections presents the final observations along with numerical examples. Eventually, we complete it by providing the conclusion.

2. Problem Statement

Considering the problem,

$$\mathcal{J}\,\psi := \frac{\partial\,\psi}{\partial\,s} + \alpha\,\frac{\partial\,\psi}{\partial\,y} + \beta\,\psi(y,s) + F\psi(y-\delta,s)ds + G\psi(y-\delta,s)dM(s) = f(y,s) \qquad (1)$$

where, s is the time variable.

$$\psi(y,s) = \phi^*(y,s) \qquad (2)$$

$$\psi(y,0) = v_0^*(y) \qquad (3)$$

The equation (1):

$$\mathcal{J}\,\psi := \begin{cases} \dfrac{\partial\,\psi}{\partial\,s} + \alpha\,\dfrac{\partial\,\psi}{\partial\,y} + \beta\,\psi = f - F\phi^*(y-\delta,s)ds - G\phi^*(y-\delta,s)dM(s) \\ \dfrac{\partial\,\psi}{\partial\,s} + \alpha\,\dfrac{\partial\,\psi}{\partial\,y} + \beta\,\psi = f - F\psi^*(y-\delta,s)ds - G\psi^*(y-\delta,s)dM(s) \end{cases} \qquad (4)$$

3. Main Results

Theorem 3.1: *Let $\mathcal{J}$ be a function satisfying $\mathcal{J}\ \rho \geq 0, (y,s) \in D$ then, $\rho(y,t) \geq 0, \forall\,(y,s) \in \overline{D}$.*

The above Maximum principle theorem leads to the subsequent stability result.

Theorem 3.2: (Stability Result) *Let* $\mathcal{J}$ *be a function, then* $|\rho(y,s)| \leq C\ max\{max_s|\rho(0,s)|, max_y|\rho(y,0)|, sup_{(y,s)\in\bar{D}}\ |\mathcal{J}\rho(y,s)|\ \}, \forall(y,s) \in (\bar{D})$

Proof: Solving the discontinuous propagation, s is fixed and considered positively and negatively from the equation (1) – (3).

$$\lim_{y\to\delta^-} \alpha\psi_{yy} = f_y(\delta^-,s) - \psi_{ys}(\delta^-,s) - \alpha_y(\delta^-,s)\psi_y(\delta^-,s) - \beta_y(\delta^-,s)\psi_y(\delta^-,s) - F_y(\delta^-,s)\psi(0^-,s)ds - F(\delta^-,s)\psi(0^-,s)ds - G_y(\delta^-,s)\psi(0^-,s)dM(s) - G(\delta^-,s)\psi_y(0^-,s)dM(s)$$

and

$$\lim_{y\to\delta^+} \alpha\psi^*_{yy} = f_y(\delta^+,s) - \psi_{ys}(\delta^+,s) - \alpha_y(\delta^+,s)\psi_y(\delta^+,s) - \beta_y(\delta^+,s)\psi_y(\delta^+,s) - F_y(\delta^+,s)\psi(0^+,s)ds - F(\delta^+,s)\psi(0^+,s)ds - G_y(\delta^+,s)\psi(0^+,s)dM(s) - G(\delta^+,s)\psi_y(0^+,s)dM(s)$$

Since $\psi_y(0^-,s) \neq \psi_y(0^+,s)$, then

$$\alpha(\delta^-,s)\psi_{yy}(\delta^-,s) \neq \alpha_y(\delta^+,s)\psi_{yy}(\delta^+,s)$$

These point $\delta, 2\ \delta, 3\delta, \ldots$ are primary discontinuities.

4. Partial Discretization along the temporal axis

The solution $\psi^{j*}(y)$ be

$$\mathcal{J}^j\psi^j(y) := D_s\psi^j(y,s_j) + \alpha(y,s_j)\psi^j_y(y,s_j) + \beta(y,s_j)\psi^j_y(y,s_j) + F(y,s_j)\psi^j(y-\delta,s_j)dt + G(y,s_j)\psi^j(y-\delta,s_j)dM(s) = f(y,s_j) \quad (5)$$

where $D_s\psi^j(y,s_j) = \frac{\psi(y,s_j)-\psi(y,s_{j-1})}{\Delta s}$

Taking $s = s_j$ as fixed, the equation (5) can be written as,

$$\Delta s\alpha(y,s_j)\frac{d\psi^j}{dy}(y) + \left(1 + \Delta s\beta(y,s_j)\right)\psi^j(y,s_j) + \Delta sF(y,s_j)\psi^j(y-\delta,s_j)dt + \Delta sG(y,s_j)\psi^j(y-\delta,s_j)dM(t) = \Delta sf(y,s_j) + \psi^{j-1}(y,s_{j-1}) \quad (6)$$

Lemma 4.1: *Let* ψ *be the solution of (1) – (3), when* $s = s_j, \psi^j(x)$ *be the solution of partial discretization along the temporal axis, then* $\left\||\psi - \psi^j|\right\| \leq C\Delta S$.

Proof: Let $E_j(y) = \psi(y,s_j) - \psi^j(y)$, and taking y as fixed, then

$$\begin{aligned}\mathcal{J}E_j(y) &= D_sE_j(y) + \alpha(y,s_j)E_j(y) + \beta(y,s_j)E_j(y) \ + F(y,s_j)E_j(y-\delta,s_j)dt \\ &\quad + G(y,s_j)E_j(y-\delta,s_j)dM(s) \\ &= D_s\psi(y,s_j) - D_s\psi^j(y,s_j) + \alpha(y,s_j)\left(\psi_y'(y,s_j) - \psi^j(y,s_j)\right) \\ &\quad + \beta(y,s_j)\left(\psi(y,s_j) - \psi^j(y,s_j)\right) \\ &\quad + F(y,s_j)\left(\psi_y'(y-\delta,s_j) - \psi^j(y-\delta,s_j)\right)dt \\ &\quad + G(y,s_j)\left(\psi_y'(y-\delta,s_j) - \psi^j(y-\delta,s_j)\right)dM(s) \\ &\quad \Rightarrow ||E_j(y) \le C(\Delta s)\end{aligned}$$

5. Runge-Kutta Method

From theorem 3.2, We know that $\delta, 2\delta, 3\delta, \ldots$ are primary unsteady points, and applying the Runge-Kutta method of order four to the problem (6), we get,

$$\begin{aligned}f^*(y,\psi^j,\psi^{j-1},s_j) &= \frac{1}{\Delta s\alpha(y,s_j)}[\Delta s f(y,t_j) - (1+\beta(y,s_j)\Delta s)\psi^j(y,s_j) \\ &\quad + \psi^{j-1}(y,s_{j-1}) - F(y,s_j)\Delta s\psi^{j,I}(y)ds \\ &\quad - G(y,s_j)\Delta s\psi^{j,I}(y)dM(s)]\end{aligned} \tag{7}$$

Applying a fourth-order Runge-Kutta method of order four, we get, ψ_{i+1}^j where,

$$\begin{aligned}K_1 &= \frac{1}{\alpha(y_i,s_j)\Delta s}\{\Delta s f(y_i,s_j) + \psi^{*(j-1)}(y_i,s_{j-1}) \\ &\quad -(1+\beta(y_i,s_j)\Delta s)\psi^{*j}(y_i,s_j) \\ &\quad -F(y_i,s_j)\Delta s\psi_i^{*(j,I)}(y_i) - G(y_i,s_j)\Delta s\psi_i^{*(j,I)}(y\_i)\} \\ K_2 &= \frac{1}{\alpha(y_i+\frac{h_i}{2},s_j)\Delta s}\left\{\Delta s f\left(y_i+\frac{h_i}{2},s_j\right) + \left(\psi_i^{*(j-1)} + \frac{K_1}{2}\right)\right. \\ &\quad -\left(1+\beta\left(y_i+\frac{h_i}{2},s_j\right)\Delta s\right)\left(\psi_i^{*j}+\frac{K_1}{2}\right) \\ &\quad -F\left(y_i+\frac{h_i}{2},s_j\right)\Delta s\left(\psi_i^{*(j,I)}\left(y_i+\frac{h_i}{2}\right)\right) \\ &\quad \left.-G\left(y_i+\frac{h_i}{2},s_j\right)\Delta s\ \psi_i^{*(j,I)}\left(y_i+\frac{h_i}{2}\right)\right\} \\ K_3 &= \frac{1}{\alpha(y_i+\frac{h_i}{2},s_j)\Delta s}\left\{\Delta s f\left(y_i+\frac{h_i}{2},s_j\right) + \left(\psi_i^{*(j-1)} + \frac{K_2}{2}\right)\right. \\ &\quad -\left(1+\beta\left(y_i+\frac{h_i}{2},s_j\right)\Delta s\right)\left(\psi_i^{*j}+\frac{K_2}{2}\right) \\ &\quad -F\left(y_i+\frac{h_i}{2},s_j\right)\Delta s\left(\psi_i^{*(j,I)}\left(y_i+\frac{h_i}{2}\right)\right) \\ &\quad \left.-G\left(y_i+\frac{h_i}{2},s_j\right)\Delta s\ \psi_i^{*(j,I)}\left(y_i+\frac{h_i}{2}\right)\right\}\end{aligned}$$

$$K_4 = \frac{1}{\alpha(y_i+h_i,s_j)\Delta s}\{\Delta s f(y_i + h_i, s_j) + \left(\psi_i^{*(j-1)} + K_3\right)$$
$$-(1 + \beta(y_i + h_i, s_j)\Delta s)(\psi_i^{*j} + K_3)$$
$$-F(y_i + h_i, s_j)\Delta s \left(\psi_i^{*(j,I)}(y_i + h_i)\right)$$
$$-G(y_i + h_i, s_j)\Delta s \ \psi_i^{*(j,I)}(y_i + h_i)\}$$

$$\psi^{*(j,I)}(y) = \begin{cases} \Phi(y_i - \delta, s_j) & , if\ (y_i - \delta, s_j) \leq 0, \\ \psi_r^* A_r(y) + \psi_{r+1}^{*j} A_{r+1}(y) + B_r(y) f^* \left(y_r, \psi_r^{*j}, \psi_r^{*(j-1)}, s_j\right) & \\ +B_{r+1}(y) f^* \left(y_{r+1}, \psi_{r+1}^{*j}, \psi_{r+1}^{*(j-1)}, s_j\right), if\ (y_i - \delta, s_j) < 0, & \end{cases}$$

and r is an integer such that $x_i - \delta \in (y_r, y_{r+1})$

$$A_r(y) = \left[1 - \frac{2(y-y_r)}{y_r-y_{r+1}}\right]\left[\frac{y-y_{r+1}}{y_r-y_{r+1}}\right]^2$$
$$A_{r+1}(y) = \left[1 - \frac{2(y-y_{r+1})}{y_{r+1}-y_r}\right]\left[\frac{y-y_r}{y_{r+1}-y_r}\right]^2$$
$$B_r(y) = \frac{(y-y_r)(y-y_{r+1})^2}{(y_r-y_{r+1})^2}$$
$$B_{r+1}(y) = \frac{(y-y_r)(y-y_{r+1})^2}{(y_r-y_{r+1})^2}$$
$$r = i - N$$

Theorem 5.1: *Let* ψ_i^{*j} *be the solution of fourth-order Runge-Kutta method problem and* $\psi(y_i, s_j)$ *be the solution of partial discretize problem then* $\left\|\psi(y_i, s_j) - \psi_i^{*j}\right\| \leq C(\overline{h^4})$.

Proof: The subsequent theorem provides a calculation of error for the above method.

Theorem 5.2: *Let* ψ_i^j *be the exact solution corresponding to the point* (y_i, s_j) *and* ψ_i^j *be the solution of fourth-order Runge-Kutta method problem, then* $\left\|\psi(y_i, s_j) - \psi_i^j\right\| \leq C(\Delta s + \overline{h^4})$.

Proof: Using 5 and 7, we get,

$$\left\|\psi - \psi_i^{*j}\right\| = \left\|\psi - \psi_i^j + \psi_i^j - \psi_i^{*j}\right\| \leq \left\|\psi - \psi_i^j\right\| + \left\|\psi_i^j - \psi_i^{*j}\right\|$$
$$\leq C(\Delta\, t + \overline{h^4})$$

6. Discontinuous Initial Data

Suppose y^* represents a point where the function ψ_0 exhibits a jump discontinuity.

$$\mathcal{J}\,\psi := \frac{\partial\,\psi}{\partial\,s} + \alpha\,\frac{\partial\,\psi}{\partial\,y} + \beta\,\psi(y,s) + F\psi(y-\delta,s)ds$$
$$+G\psi(y-\delta,s)dM(s) = f(y,s) \tag{8}$$

For $j = 1$, then the equation becomes

$$\Delta s\alpha\,\frac{d\psi^j}{dy} + \beta\psi^j + F\psi(x-\delta) + G\psi^j(y-\delta) = \Delta s f(y,s_j) + \psi^0(y)$$

At $y = y^*$, the function y^* is discontinuous in equation (8) and the function $\psi(y,s_j)$ is asymmetrical at $y = y^*$. And, $\psi(y)$ is asymmetrical at $y^* + \delta, y^* + 2\delta, y^* + 3\delta, \ldots$ these points $y^*, y^* + \delta, y^* + 2\delta, y^* + 3\delta, \ldots.$ are to be considered as mesh points.

7. Numerical Example

Consider the following 1st order SDDE,

$$\frac{\partial\,\psi}{\partial\,s} + \alpha\,\frac{\partial\,\psi}{\partial\,y} + \beta\,\psi(y,s) + F\psi(y-\delta,s)ds + G\psi(y-\delta,s)dM(s)$$
$$= 0, (y,s) \in (0,4] \times (0,4]$$
$$\psi(y,s) = 0, (y,s) \in [-\delta,0] \times [0,4]$$
$$\psi(y,0) = \begin{cases} y\exp\left(-\frac{(4y-2)^2}{2}\right) \times (2-y) & [0,4] \\ (y-2)\times(4-y)\times\exp\left(-\frac{(4y-2)^2}{10}\right) & [2,4] \end{cases}$$
$$a(y,s) = \frac{3+y^2+s^2}{1+2sy+2y^2}, b(y,s) = 1, c(y,s) = -2$$

The numerical solution for divergent time levels of the curves is plotted in Figure 1 and assuming numerical method for the equation at different grid resolutions for M = 64, 128, 256, 512, 1024 and N = 64, 128, 256, 512, 1024 are given in Table 1.

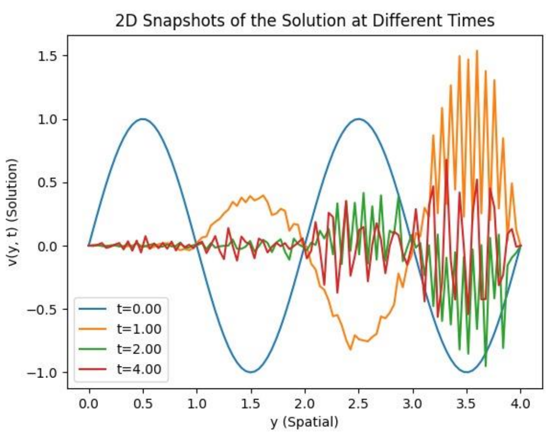

Figure 1
Numerical solution of the above example at different time level

Table 1

Maximum error by using conditional method

M↓ /N →	**64**	**128**	**256**	**512**	**1024**
64	2.5421e-03	1.2457e-03	6.3221e-04	3.1710e-04	1.5892e-04
128	4.1023e-03	2.0311e-03	1.0023e-03	5.0219e-04	2.5134e-04
256	6.8997e-03	3.4124e-03	1.6721e-03	8.3411e-04	4.1580e-04
512	1.0792e-02	5.3241e-03	2.6153e-03	1.3027e-03	6.4859e-04
1024	1.6478e-02	8.1230e-03	4.0019e-03	1.9930e-03	9.9231e-04

The maximum error for the stochastic delay differential equation was computed using a conditional numerical method for varying grid resolutions $(M \text{ and } N)$. As M and N increase, the error consistently decreases, confirming convergence. The finest grid $(M = 1024,\ N = 1024)$ yields the smallest error $(9.9231e - 04)$. And the resolutions of $M, N \geq 512$ achieve a good balance of accuracy and computational cost. These results validate the stability and effectiveness of the numerical method.

8. Conclusion

In this paper, we explored SDDE, a versatile model, and used time domain partial discretization for numerical solution, achieving $O(\Delta s)$ accuracy. Further, refined with 4^{th} order Runge-Kutta and cubic Hermite interpolation. This study analyzed the numerical solution of a first-order SDDE using a conditional numerical method. The computational results demonstrated that the method is stable and convergent, with the maximum error consistently decreasing as the spatial (M) and temporal (N) resolutions increased. For the finest grid $M = 1024, N = 1024$, the method achieved the smallest error and confirmed its accuracy. The results highlight the effectiveness of the method in handling stochastic delay differential equations, making it a reliable approach for practical applications. The error decreases as both M and N increase, indicating convergence of the method as the grid is refined.

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