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Solving linear inverse problem via mixture regularization

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Abstract

The iterative shrinkage/thresholding algorithm (ISTA) with the regularizer gives good efficiency to solve the linear inverse problem. In fact, the maximum a posteriori (MAP) estimator with the probability density function (PDF) and ISTA with the regularizer are similar, and we imply that PDF is similar to the regularizer. In general cases, the mixture PDF created from many ordinary PDFs gives better efficiency than the ordinary PDF. Consequently, we present the simple method based on the linear combination of various

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ordinary regularizers to create the mixture regularizer, and also present the simple-mixture regularizer for ISTA to solve the linear inverse problem. Experimental results of the generated and real signals show that our method outperforms the state-of-the-art methods.

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Keywords: *Linear inverse problem, Linear combination, Non-convex regularizer (penalty function), Iterative shrinkage/thresholding algorithm (ISTA).*

1. Introduction

Many fundamental problems in signal processing are the linear inverse problem, such as the noise-suppression (denoising) and deconvolution problems [1-3]. In fact, we can covert the non-linear problem into the linear form. For example, the non-linear equation of the flow problem can be converted into the linear differential equation via the quasi linear technique, and we use the cubic spline collocation method to solve this problem [4]. Therefore, we focus on presenting the algorithm for solving the linear inverse problem in this work. In general cases, we address this problem via two well-known methods as follows. The first method is the Bayesian estimator, such as the maximum a posteriori (MAP) and minimum mean square error (MMSE) estimators, and the second method is called as the regularized linear least squares method, such as the iterative shrinkage/thresholding algorithm (ISTA). These methods are based on the proximal operator (shrinkage function) [5]. Indeed, MAP and ISTA are very similar. We use MAP with the probability density function (PDF) or ISTA with the regularizer to solve our problem. Therefore, we imply that PDF and the regularizer are similar. In fact, the mixture PDF, also called as the mixture model, that is based on the linear combination of many ordinary PDFs outperforms the non-mixture PDF for many tasks in signal processing, such as the compression, signal-enhancement and pattern recognition tasks. The mixture model has more flexibility than the non-mixture model for representing data as follows. To solve the trajectory-reconstruction problem via the new unscented Kalman filter, the Gaussian mixture model (GMM) is more suitable to represent the noise model than the simple Gaussian distribution [6]. In many denoising (noise-reduction) algorithms, the non-local similarity of natural images and the patch group prior for the prior learning are represented by GMM with good efficiency [7]. The Gaussian and Laplacian mixture models give good efficiency to represent coefficients of many sparse transforms, such as the discrete Fourier, wavelet and discrete cosine transforms, in the Bayesian estimator [8]. Moreover, other non-Gaussian mixture models, such as the beta, Dirichlet, beta-Liouville, Watson and inverted beta mixture models, are used to represent data that is not based on the support range of GMM [9]. Unfortunately, the main drawback of many mixture models is that statistical parameters of these models must be created by many complex algorithms, such as the expectation-maximization (EM) algorithm, so we must use numerous time and resources to compute these parameters [10]. On the other hand, ISTA based on the non-convex regularizer is the well-known method to solve our problem. Many research works show that the non-convex regularizer is better than the convex regularizer, such as the l_1 and l_2 norms, in many situations [5, 11-13]. One advantage of the non-convex regularizer is that we can create the condition for this regularizer to preserve the convexity of various objective functions [5, 14, 15]. Note that every non-convex regularizer in this work is based on this advantage. Now, we call the non-convex regularizer as the regularizer for easiness. The univariate (separable) regularizer has the drawback where this regularizer is similar to the convex regularizer in some cases. Therefore, many works proposed the multivariate (non-separable) regularizer, which was like the random vector in the Bayesian estimator, to solve this drawback as follows. The bivariate case was proposed in [16]. However, this case cannot address many important problems in signal processing. The general multivariate case was presented to solve this drawback[17]. However, this multivariate case is only based on small groups of non-separable regularizers. In the past, no one studied the mixture regularizer via the linear combination of various ordinary regularizers in our problem.

As mentioned above, the simple method based on the linear combination of various ordinary regularizers is proposed to create the mixture regularizer. To solve the linear inverse problem, we also present the example of the simple-mixture regularizer that is used to create the shrinkage function in ISTA.

This work is organized as follows. We present the linear combination of various ordinary regularizers to construct the mixture regularizer in Sec.2. In Sec.3, the example of the simple-mixture regularizer that is used to create the shrinkage function is present to address the linear inverse problem. In the generated and real signals, we show experimental results of the proposed method and the state-of-the-art methods in Sec.4. Finally, Sec.5 describes about the conclusion and discussion.

2. Constructing Mixture Regularizer

The algorithm to create the mixture regularizer is proposed in this section. We present the mixture regularizer $\phi(x;a)\colon \mathbb{R} \to \mathbb{R}, a > 0$ that is constructed by the linear combination of many regularizers $\phi_m(x;a)$ as $\phi(x) = \sum_m \beta_m \phi_m(x)$ where β_m is the weighted parameter based on $\sum_m \beta_m = 1$ and $\beta_m > 0$. Indeed, the regularizer must have important properties for the convexity of the shirnkage function as follows: (1) $\phi(x) = \phi(-x)$ (2) $\lim_{x\to 0}\phi(x) = 0$ (3) $\lim_{a\to 0}\phi(x) = |x|$ (4) $\lim_{x\to 0^+}\phi'(x) = 1$ and (5) $\lim_{x\to 0^+}\phi''(x) = -a$ [17, 18]. Here, $\phi_m(x)$ has these properties because this function is the regularizer. We show that $\phi(x)$ has these properties as follows:

(1) $\phi(x) = \phi(-x)$ because $\phi_m(x)$ has the symmetrical property.

(2) $\lim\limits_{x\to 0}\phi(x) = \sum_m \beta_m \lim\limits_{x\to 0}\phi_m(x) = 0$ because of $\lim\limits_{x\to 0}\phi_m(x) = 0$.

(3) $\lim\limits_{a\to 0}\phi(x) = \sum_m \beta_m \lim\limits_{a\to 0}\phi_m(x) = |x|$ because of $\lim\limits_{a\to 0}\phi_m(x) = |x|$.

(4) $\lim\limits_{x\to 0^+}\phi(x) = \sum_m \beta_m \lim\limits_{x\to 0^+}\phi'_m(x) = 1$ because of $\lim\limits_{x\to 0^+}\phi'_m(x) = 1$.

(5) $\lim\limits_{x\to 0^+}\phi''(x) = \sum_m \beta_m \lim\limits_{x\to 0^+}\phi''_m(x) = -a$ because of $\lim\limits_{x\to 0^+}\phi''_m(x) = -a$.

Therefore, our proposed function $\phi(x)$ is the regularizer.

In this research, the noise-free signal $\mathbf{x} \in \mathbb{R}^d$ is estimated from the noisy signal in the linear model $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$ where $\mathbf{H}$ is the known convolution matrix and $\mathbf{n}$ is additive white Gaussian noise (AWGN) based on the noise variance σ_n. This convolution matrix is created by the convolution filter $h[n]$, and we can find the algorithm to create the convolution matrix via the convolution filter in Sec.4. Note that we also called this convolution matrix as the dictionary that concerns about the sparse signal [19]. We show that our function $\phi(x)$ is used in ISTA to estimate $\hat{x}_i$ as follows. The ISTA iteration is defined as $\hat{x}_i^{k+1} = T(u_i^k)$ where $\mathbf{u}^k = \mathbf{x}^k + \frac{1}{\alpha}\mathbf{H}^T(\mathbf{y} - \mathbf{H}\mathbf{x}^k)$. We define the shrinkage function for ISTA as

$$T(y_i) = \hat{z}_i = \arg\min_{z_i} \left[\frac{1}{2}(y_i - z_i)^2 + \lambda\phi(z_i)\right], \lambda > 0 \tag{1}$$

where $\alpha \geq \text{maxeig}(\mathbf{H}^T\mathbf{H})$ and $\text{maxeig}(.)$ is the maximum value of eigenvalues [20].

3. Example of Simple-Mixture Regularizer and Its Shrinkage Function

In this section, we present the simple-mixture regularizer as $\phi(z) = \beta_1\phi_1(z) + \beta_2\phi_2(z)$ to solve ISTA. We use $\phi_1(z)$ and $\phi_2(z)$ as the exponential and logarithmic regularizers that are defined as $\phi_1(z) = (1 - e^{-a|z|})/a$ and $\phi_2(z) = \log(1 + a|z|)/a$ [14, 17]. We find the shrinkage function (1) via this mixture regularizer $\phi(z)$ as follows. Note that we use simple notations y and z to represent components of vectors y_i and z_i for easiness. Solving (1) using $\phi(z)$ gives

$$\frac{\partial}{\partial z}\left[\frac{1}{2}(y - z)^2 + \lambda\phi(z)\right] = \frac{\partial}{\partial z}\left[\frac{1}{2}(y - z)^2 + \lambda\beta_1\phi_1(z) + \lambda\beta_2\phi_2(z)\right] = 0 \tag{2}$$

where $\frac{\partial}{\partial z}\frac{1}{2}(y - z)^2 = z - y$, $\frac{\partial\phi_1(z)}{\partial z} = \frac{z}{|z|}e^{-a|z|}$ and $\frac{\partial\phi_2(z)}{\partial z} = \frac{z}{|z|(1+a|z|)}$. Then $z - y + \frac{\lambda\beta_1 z}{|z|}e^{-a|z|} + \frac{\lambda\beta_2 z}{|z|(1+a|z|)} = 0$. Setting $r = |z|$ gives

$$z\left(1 + \frac{\lambda\beta_1}{r}e^{-ar} + \frac{\lambda\beta_2}{r(1+ar)}\right) = y. \tag{3}$$

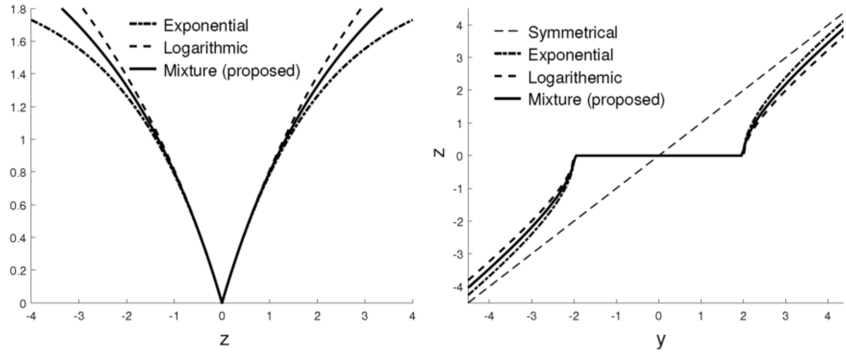
It can be written as $|z|\left|1 + \frac{\lambda\beta_1}{r}e^{-ar} + \frac{\lambda\beta_2}{r(1+ar)}\right| = |y|$. We obviously see that $1 + \frac{\lambda\beta_1}{r}e^{-ar} + \frac{\lambda\beta_2}{r(1+ar)} > 0$. Therefore,

$$1 + \frac{\lambda\beta_1}{r}e^{-ar} + \frac{\lambda\beta_2}{r(1+ar)} = \frac{|y|}{r}, \tag{4}$$

$$r + \lambda\beta_1 e^{-ar} + \frac{\lambda\beta_2}{1+ar} - |y| = 0. \tag{5}$$

Substituting (4) in (3) and using the successive method [21] in (5) for finding r give

$$T(y) = \hat{z} = r_+ sign(y) \tag{6}$$



(a) (b)

Figure 1
(a) The exponential, logarithmic and mixture (proposed) regularizers. (b) Shrinkage functions corresponding to these regularizers.

where $r^{k+1} = |y| - \lambda\beta_1 e^{-ar^k} - \frac{\lambda\beta_2}{1+ar^k}$ on $r^0 = |y|$, $g_+ = \max(0, g)$ and $sign(y)$ is the standard sign function. In this work, we set $a = 1/\lambda$ for the convex condition of this shrinkage function [22]. The exponential, logarithmic and mixture (proposed) regularizers are shown in Fig.1(a) where $a = 0.5$ and $\beta_1 = \beta_2 = 0.5$. We illustrate many shrinkage functions corresponding to these regularizers in Fig.1(b). Note that the shrinkage function of the exponential regularizer is built by setting $\beta_2 = 0$ in (6), and the closed-form shrinkage function of the logarithmic regularizer can be seen in [14].

4. Experimental Results

We used two situations in the generated and real signals to evaluate our proposed algorithm as follows. In the first situation (the generated signals), our shrinkage function (6) was only evaluated in AWGN. Therefore, the convolution matrix **H** is equal to the identity matrix **I**. The Wavelab database in [23] is the standard database in signal processing. Therefore, we used all 20 signals (1,024 and 2,048 points) in this database and various values of the noise standard deviation σ_n to evaluate our algorithm. We made sparse data via the stationary wavelet transform in [24], 7-scale decomposition and 2-vanishing moments, and set $\lambda_j = 2.375\sigma_n 2^{-j/2}$ where $1 \leq j \leq 7$ is the scale of wavelet decomposition [25]. We assumed that σ_n was known in this work. We can study many practical methods to find this parameter in many research works [26, 27]. However, this is not the main idea in this work. The root mean square error (RMSE) is used to evaluate efficiency. For rationality, we used the soft threshold (the standard function in our research topic) and the shrinkage functions based on the regularizers that relate to our proposed mixture regularization in Sec.3 (the logarithmic and exponential regularizers) for comparison. Note that we can find algorithms to create these compared methods in [21] and Sec.3. We used various values of the weighted parameter β_1 to evaluate our shrinkage function where

$\beta_2 = 1 - \beta_1$. For stability results, we used over ten runs for all denoising algorithms. Table 1 shows many experimental results on average RMSE values where the best values are star. These results show that we obtain the best efficiency in many cases where $\beta_1 = 0.5$. Therefore, we propose this value for our shrinkage function. Note that the parameter $\beta_1 = 0.45, 0.55$ can also be good choices where the efficiency is slightly reduced. We selected one signal (1,024 points), called as "Bump", based on $\sigma_n = 0.35$ to show our results. The noise-free and noisy signals of the Bump signal are shown in Fig.2, and enhanced data via the compared and proposed shrinkage functions are shown in Fig.3. We can see in this case that our shrinkage function reduces noise and preserves features of original data better than compared methods.

In the second situation (the real signal), the observed signal in the wireless communication system should be the multipath because the transmitted signal reflects off various objects, and arrives in many paths. The real signal of the multipath (the average power values in the decible (dB) scale of the wireless communication system in the big cities of Thailand: the National Broadcasting and Telecommunication Commission (NBTC), Thailand) on the signal length $d = 300$ was chosen for evaluating our method. We used $\mathbf{u}^0 = \mathbf{H}^T y$, $\lambda = 3\sigma_n \|h[n]\|_2/\alpha$ and $\alpha = \text{maxeig}(\mathbf{H}^T\mathbf{H})$. We defined these parameters similar to previous works in [14, 20] for reliability. The noise standard deviation was set as $\sigma_n = 0.02$. According to the first situation, we used weighted parameters as $\beta_1 = \beta_2 = 0.5$. We used the compared methods similar to the first situation, expected in the least square method (another standard function). In this case, we used $\mathbf{H}$ as the enhanced inverse discrete cosine transform that is based on the real values, and we did not manage the problem with the complex values in our shrinkage function. Our shrinkage function should be used in the real values because this function has the comparing step of the magnitude values. Another advantage is that the matrix $\mathbf{H}$ can be adapted from the considered signal. This further information can be seen in [28]. Note that the convolution matrix may be shown in the filter form. Therefore, we show the relationship between $\mathbf{H}$ and $h[n]$ with the filter length M [20] as follows

$$\mathbf{H} = \begin{bmatrix} h[1] & 0 & \dots & 0 \\ \vdots & h[1] & \dots & \vdots \\ h[M] & \vdots & \ddots & 0 \\ 0 & h[M] & \ddots & h[1] \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h[M] \end{bmatrix}$$

In this situation, we illustrate the noise-free and noisy signals in Fig.4, and enhanced data via the compared and proposed algorithms in Fig.5. Again, we can see that our method reduces noise and preserves the features of original data better than compared methods.

Table 1
Average RMSE values of denoising algorithms.

Noise standard deviation σ_n	0.25	0.3	0.35	0.4
Soft threshold function	0.186	0.195	0.232	0.404
Shrinkage function of logarithmic regularizer	0.081	0.088	0.105	0.153
Shrinkage function of exponential regularizer	0.084	0.090	0.110	0.170
Our proposed method on $\beta_1, \beta_2 = 1 - \beta_1$				
$\beta_1 = 0.4$	0.083	0.089	0.106	0.16
$\beta_1 = 0.45$	0.079	0.086*	0.097	0.136
$\beta_1 = 0.5$	0.075*	0.087	0.090*	0.125*
$\beta_1 = 0.55$	0.081	0.091	0.101	0.141
$\beta_1 = 0.6$	0.083	0.102	0.105	0.170

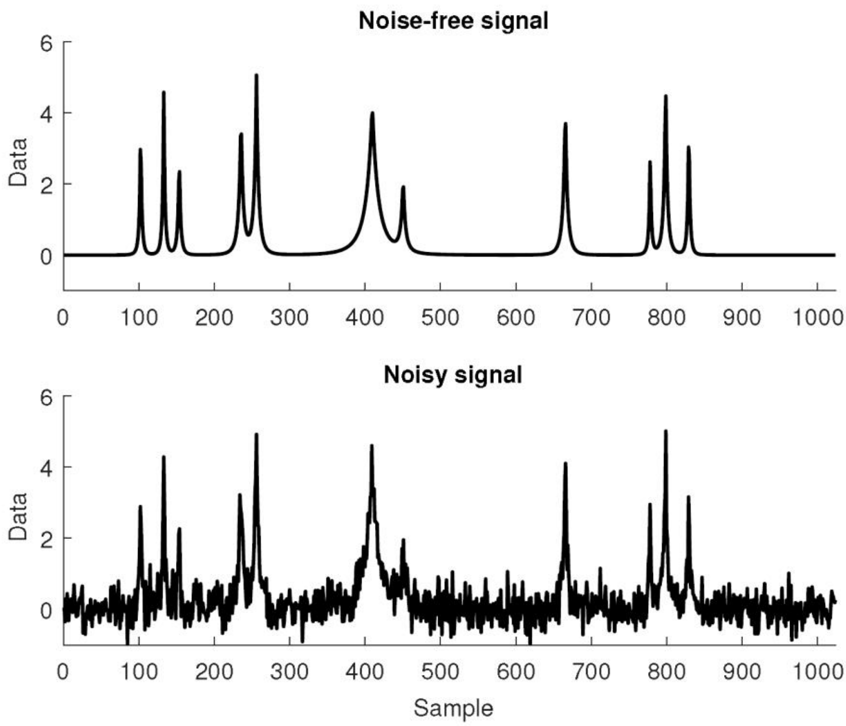


Figure 2
The noise-free and noisy signals of the generated signal (Bump).

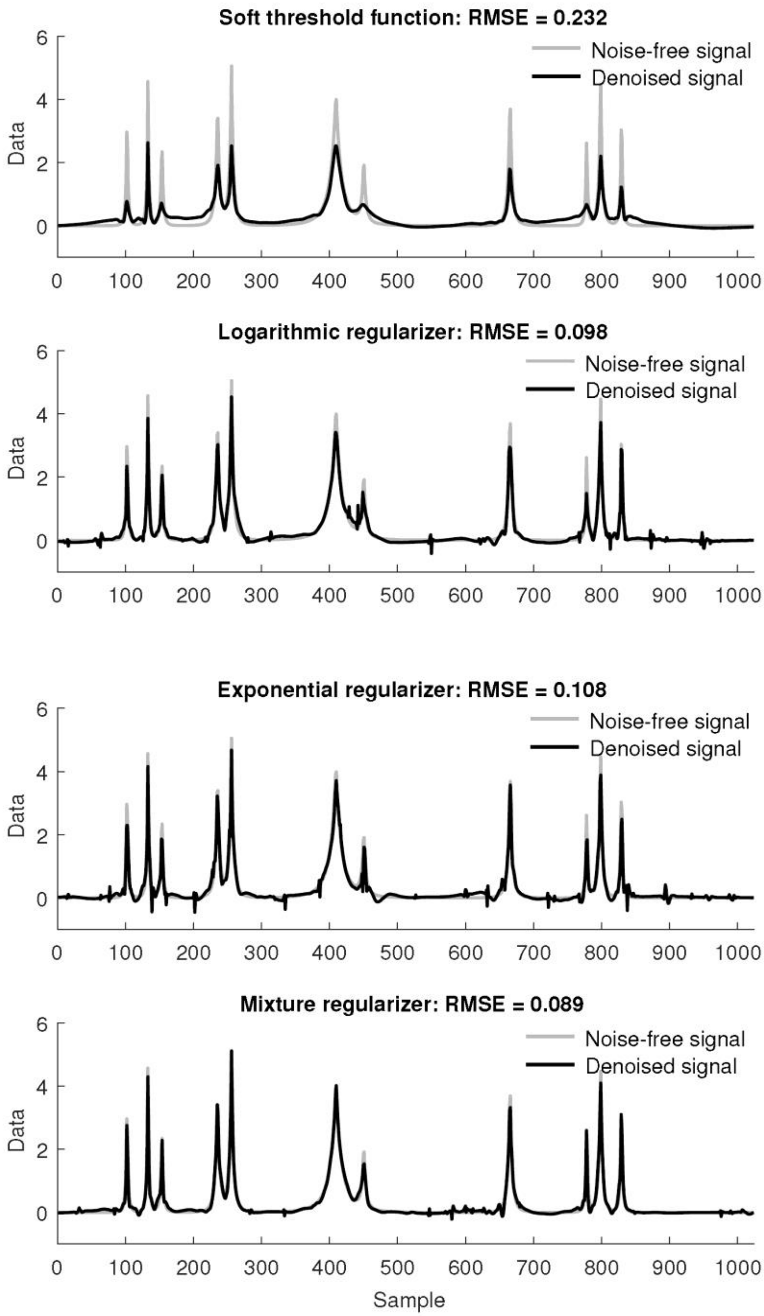


Figure 3
Enhancing the generated signal (Bump) via various shrinkage functions

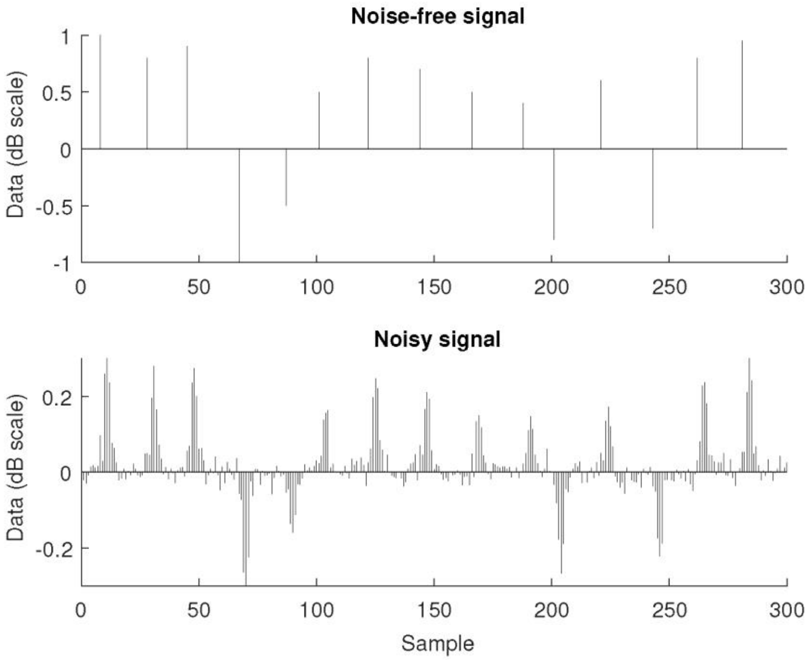
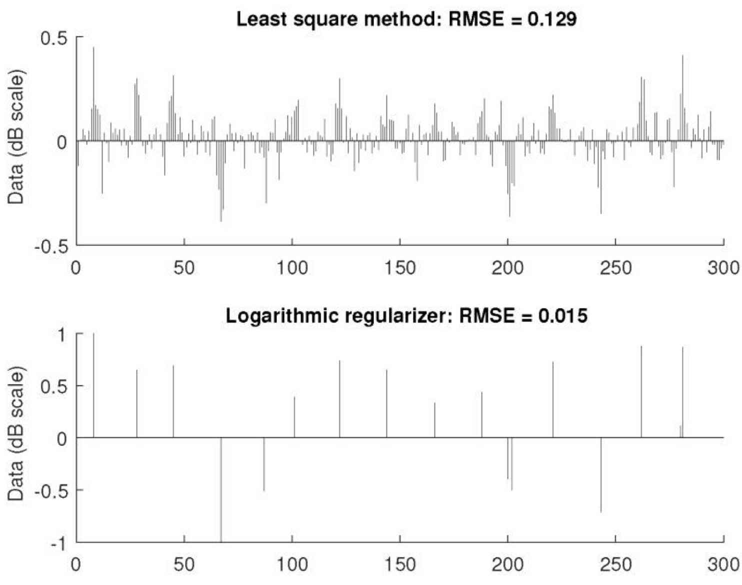


Figure 4
The noise-free and noisy signals of the real signal.



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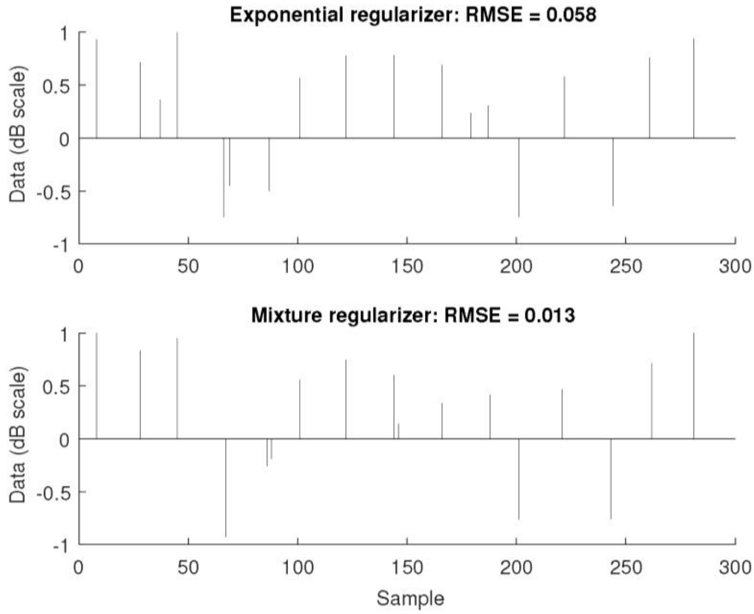

Figure 5
Enhancing the real signal via various algorithms

5. Conclusion and Discussion

In the general cases, many algorithms based on the mixture model gives better efficiency than these algorithms based on the ordinary model. Therefore, we present the simple method based on the linear combination to create the mixture regularizer, and the shrinkage function based on the simple-mixture regularizer is proposed to solve the linear inverse problem. The experimental results show that our signal-enhancing method outperforms the state-of-the-art methods.

In the future works, we can test other interesting regularizers, such as the inverse hyperbolic sine [5], trigonometric [14], Pareto [29] and double threshold value [30] regularizers, to make the mixture regularization via our proposed method for enhancing data. Our weighted parameters (β_1, β_2) were selected from efficiency of some experimental results. Therefore, if we can propose the automatic method to create these parameters, then efficiency of our method may be improved. In fact, the multivariate case of many signal-enhancing methods often gives better efficiency than the univariate case [21, 22]. Therefore, if we can make the multivariate case for our methods, then our efficiency should be improved. In many modern technologies, we know that various machine learning algorithms, such as the pattern recognition and classification algorithms, are based on the adapted regularization. Therefore, if we can apply our mixture regularization for these algorithms, then efficiency of these algorithms should be improved. These topics are very interesting.

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