Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 5, pp. 2005–2010
DOI : 10.47974/JIM-2228

Solutions of the Diophantine equation $n^x + (7p)^y = z^2$

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Abstract

Our purpose of this paper is to find integer solutions n, p, x, y and z for the Diophantine equation $n^x + (7p)^y = z^2$ where n is positive, p is prime and x, y . z are non-negative.

Subject Classification (2010): *11D72.*

Keywords: *Diophantine equations, Integer solutions, Congruences.*

1. Introduction and Preliminaries

A Diophantine equation which have a rich history is an equation relating integers. In [1], Ahmadi and Janfada showed that for some specific choices of a, b and c, the Diophantine equation in the form $ax^4 + by^4 = cz^2$ has only trivial solutions in Gaussian integers. In [2], some Diophantine equations were applied to study prime numbers in Lucas sequences. Recently, the Diophantine equation $n^x + (5p)^y = z^2$, where n, p, x, y and z are integers such that $n > 0$, p is prime, $x \geq 0, y \geq 0$ and $z \geq 0$, was verified its solutions in [4]. Catalan's conjecture was conjectured by Eugène Charles Catalan [3] in 1844. Later, this conjecture was proven by Preda Mihăilescu [5] in 2002. For this reason, Catalan's conjecture is called Mihăilescu's Theorem.

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Theorem 1.1: (Mihăilescu's Theorem) *For any two integers* $a, b > 1$, *the Diophantine equation* $x^a - y^b = 1$ *has no solutions in positive integers* x *and* y, *other than* $3^2 - 2^3 = 1$.

In this paper, we will use Mihăilescu's Theorem to find some solutions of the Diophantine equation $n^x + (7p)^y = z^2$.

2. Main Results

In this section, we will find the non-negative integers x, y and z that satisfy the Diophantine equation $n^x + (7p)^y = z^2$ where n is a positive integer and p is a prime number.

Theorem 2.1: *If* $n \equiv 1 \pmod 4$ *and* $p \equiv 3 \pmod 4$, *then the Diophantine equation* $n^x + (7p)^y = z^2$ *has no solution in the case of* $x \geq 0, y \geq 0$ *and* $z \geq 0$.

Proof: Suppose that (n, p, x, y, z) is a solution of the Diophantine equation $n^x + (7p)^y = z^2$. By assumption, we have

$$z^2 = n^x + (7p)^y \equiv 1^x + (7 \cdot 3)^y \equiv 1^x + 1^y \equiv 2 \pmod 4.$$

This is impossible since $z^2 \equiv 0 \pmod 4$ or $z^2 \equiv 1 \pmod 4$.

Theorem 2.2: *Assume that* $n = 2$, $p \equiv 1 (mod 4)$ *and* $x \neq 1$. *Then* (n, p, x, y, z) *is a solution for the Diophantine equation* $n^x + (7p)^y = z^2$ *if and only if*

$$(n, p, x, y, z) \in \left\{(2, p, 0, 1, \sqrt{7p+1}) | \sqrt{7p+1} \in \mathbb{Z}\right\} \cup \{(2, p, 3, 0, 3), (2, 73, 11, 2, 513)\}.$$

Proof: Assume that (n, p, x, y, z) is a solution of the Diophantine equation $n^x + (7p)^y = z^2$. Then $2^x + (7p)^y = z^2$.

Case 1: $x = 0$. Then we have $(7p)^y = z^2 - 1$. We consider the following subcases:

If $y = 0$, then $z^2 = n^x + (7p)^y = 2$. This implies that $z^2 = 2$ which is impossible.

If $y = 1$, then $z^2 = n^x + (7p)^y = 7p + 1$. Thus $z = \sqrt{7p+1}$ where $\sqrt{7p+1} \in \mathbb{Z}$.

If $y = 2$, then we have $(7p)^2 = z^2 - 1$. We get $7p = 0$ which is impossible.

If $y \geq 3$, then by Mih ă ilescu's Theorem, we get $7p = 2$ which is impossible.

By all subcases, we have that $(n, p, x, y, z) \in \{(2, p, 0, 1, \sqrt{7p+1}) | \sqrt{7p+1} \in \mathbb{Z}\}$.

Case 2: $x \geq 2$. Then we have $4 | 2^x$, so that $2^x \equiv 0 \pmod 4$. Since 2^x is even and $(7p)^y$ is odd, we obtain $z^2 = 2^x + (7p)^y$ is odd. Since $z^2 \equiv 0 \pmod 4$

or $z^2 \equiv 1 \pmod 4$, this implies $z^2 \equiv 1 \pmod 4$. We get $2^x + (7p)^y \equiv 0 + (-1)^y \pmod 4$ because $p \equiv 1 \pmod 4$. If y is odd, then $z^2 = 2^x + (7p)^y \equiv 0 + (-1) \equiv 3 \pmod 4$, this is impossible. Thus, y is even.

Case 2.1: $y = 0$. Then we have $z^2 - 1 = 2^x$. We consider the following subcases:

Subcase $x = 0$. We get $z^2 = 2$, this is impossible.

Subcase $x \geq 2$. By Mih ă ilescu's Theorem, we get $(n, p, x, y, z) = (2, p, 3,0,3)$.

Case 2.2: $y \geq 2$. Then $y = 2k$ for some positive integer k. It follows that $z^2 - (7p)^{2k} = 2^x$ and so $(z - (7p)^k)(z + (7p)^k) = 2^x$. Then we have an integer $w \geq 0$ that satisfies the following equations

$$z - (7p)^k = 2^w \tag{1}$$

and

$$z + (7p)^k = 2^{x-w}. \tag{2}$$

From Equation (1) and Equation (2), we get $2^{x-w} = z + (7p)^k > z - (7p)^k = 2^w$. We get $x - w > w$ and so $x - 2w > 0$. Subtract Equation (1) from Equation (2), we get

$$2(7p)^k = 2^{x-w} - 2^w$$
$$2(7p)^k = 2^w(2^{x-2w} - 1)$$
$$(7p)^k = 2^{w-1}(2^{x-2w} - 1).$$

Since $(7p)^k$ is odd, we have 2^{w-1} must be odd. Thus, $2^{w-1} = 1$. This implies $w = 1$. Then $2^{x-2} - (7p)^k = 1$. We consider the following subcases:

Case 2.2.1: $k > 1$. We consider the following subcases:

If $x = 2$, then $(7p)^k = 0$. This is impossible.

If $x = 3$, then $(7p)^k = 1$ .This implies $k = 0$ which is contradict.

If $x > 3$, By Mihăilescu's Theorem, we get $7p = 2$ which is impossible.

Case 2.2.2: $k = 1$. Then we have $2^{x-2} - 1 = 7p$. We consider this Equation by using modulo 7, we have $2^{x-2} \equiv 1 \pmod 7$.

If $x - 2 \equiv 1 \pmod 3$, then we get $2^{x-2} - 1 \equiv 2 - 1 \equiv 1 \pmod 7$ which is contradict.

If $x - 2 \equiv 2 \pmod 3$, then we get $2^{x-2} - 1 \equiv 4 - 1 \equiv 3 \pmod 7$ which is a contradiction.

If $x - 2 \equiv 0 \pmod 3$, then we get $2^{x-2} - 1 \equiv 1 - 1 \equiv 0 \pmod 7$. Thus, $x - 2 \equiv 0 \pmod 3$. Let $x - 2 = 3l$ for some positive integer l. It follows that $2^{3l} - 1 = 7p$. Thus $(2^l - 1)(2^{2l} + 2^l + 1) = 7p$. We consider the following subcases.

Subcase $2^l - 1 = 1$ and $2^{2l} + 2^l + 1 = 7p$. We have that $l = 1$ and $p = 1$, which is impossible.

Subcase $2^l - 1 = 7$ and $2^{2l} + 2^l + 1 = p$. We have $l = 3$, $p = 73$, $x = 3l + 2 = 3(3) + 2 = 11$, $y = 2k = 2(1) = 2$ and $z^2 = n^x + (7p)^y = 2^{11} + (7 \cdot 73)^2 = 2048 + 261121 = 263169$

This implies $z = 513$. Thus, $(n, p, x, y, z) = (2,73,11,2,513)$.

Subcase $2^l - 1 = p$ and $2^{2l} + 2^l + 1 = 7$. We get $2^{2l} + 2^l - 6 = 0$ and so $(2^l - 2)(2^l + 3) = 0$. Then we get $2^l - 2 = 0$ or $2^l + 3 = 0$, but $2^l + 3 > 0$. Thus, $2^l - 2 = 0$, this implies $l = 1$ and $p = 2^l - 1 = 1$ which is impossible.

Subcase $2^l - 1 = 7p$ and $2^{2l} + 2^l + 1 = 1$. Since $2^{2l} + 2^l + 1 \geq 2^2 + 2^1 + 1 = 7 > 1$, we get this case is impossible.

Remark: In Case 1 of the proof of Theorem 2.2, if we choose $p = 5$, we have that $(n, p, x, y, z) = (2,5,0,1,6)$ is a solution for the Diophantine equation $n^x + (7p)^y = z^2$. However, if $p = 13$, we cannot find a solution in this case for the Diophantine equation $n^x + (7p)^y = z^2$.

Theorem 2.3: *Suppose that* $p \equiv 1 \pmod 3$ *and* $n \equiv 1 \pmod 3$. *The Diophantine equation* $n^x + (7p)^y = z^2$ *has no solution in the case of* $x \geq 0, y \geq 0$ *and* $z \geq 0$.

Proof: Assume that non-negative integers x, y and z satisfy the Diophantine equation $n^x + (7p)^y = z^2$. By assumption, we get

$$z^2 = n^x + (7p)^y \equiv 1^x + (7 \cdot 1)^y \equiv 1^x + 1^y \equiv 1 + 1 \equiv 2 \pmod 3.$$

This is impossible since $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$.

Theorem 2.4: *Assume that* $n \equiv 2 \pmod 3$, $p \equiv 2 \pmod 3$, x *is odd and* y *is even. Then* (n, p, x, y, z) *is a solution of the Diophantine equation* $n^x + (7p)^y = z^2$ *if and only if*

$$(n, p, x, y, z) \in \left\{(n, p, 1,0, \sqrt{n+1}) | \sqrt{n+1} \in \mathbb{Z}\right\} \cup \{(2, p, 3,0,3), (2,2,7,2,18)\}.$$

Proof: Suppose that three non-negative integers x, y and z satisfy the Diophantine equation $n^x + (7p)^y = z^2$. Since y is even, we get $y = 2m$ for some integer $m \geq 0$. It follows that $n^x + (7p)^{2m} = z^2$. We consider the following cases.

Case 1: $m = 0$. Then we have $z^2 - n^x = 1$. We consider the following subcases.

If $x = 0$, then we have $z^2 = 2$. This is impossible.

If $x = 1$, then we have $z^2 = 1 + n$ and so $z = \sqrt{1+n}$.

Thus, we get that $(n, p, x, y, z) = (n, p, 1,0, \sqrt{n+1})$ where $\sqrt{n+1}$ is an integer.

If $x > 1$. By Mihăilescu's Theorem, we get $(n, p, x, y, z) = (2, p, 3,0,3)$.

Case 2: $m > 0$. So $n^x = z^2 - (7p)^{2m}$, we get $n^x = (z - (7p)^m)(z + (7p)^m)$. Since n is prime, we have a non-negative integer v such that

$$z - (7p)^m = n^v \tag{3}$$

and

$$z + (7p)^m = n^{x-v}. \tag{4}$$

From Equation (3) and Equation (4), we get $n^{x-v} = z + (7p)^m > z - (7p)^m = n^v$. So we have $x - v > v$ and so $x - 2v > 0$. Subtract Equation (3) from Equation (4), we get

$$2(7p)^m = n^v(n^{x-2v} - 1). \tag{5}$$

Since n^v and $n^{x-2v} - 1$ have opposite parity and $2|n^v(n^{x-2v} - 1)$,we have $2|n^v$ or $2|n^{x-2v} - 1$.

Case 2.1: $2|n^v$. Since n is prime, we obtain $n = 2$. From Equation (5), we have

$$(7p)^m = 2^{v-1}(2^{x-2v} - 1). \tag{6}$$

Case 2.1.1: p is odd. Similar to Case 2.2 in Theorem 2.2, we obtain $(n, p, x, y, z) = (2,73,11,2,513)$. But p is odd and $p \equiv 2(\mathrm{mod}3)$, this implies that $p \neq 73$. Thus, there has no non-negative integer solution in this case.

Case 2.1.2: p is even. Hence, $p = 2$. From Equation (6), we have

$$(7 \cdot 2)^m = 2^{v-1}(2^{x-2v} - 1)$$
$$7^m \cdot 2^m = 2^{v-1}(2^{x-2v} - 1)$$
$$7^m = 2^{v-1-m}(2^{x-2v} - 1). \tag{7}$$

Since 7^m is odd, we get 2^{v-1-m} is odd. This implies $v - 1 - m = 0$ and so $m = v - 1$.

From Equation (7), it follows that $7^{v-1} = 2^{x-2v} - 1$ and so

$$2^{x-2v} - 7^{v-1} = 1. \tag{8}$$

If $v = 1$, then $2^{x-2} - 1 = 1$. This implies $x = 3$, $z^2 = n^x + (7p)^y = 2^3 + (7 \cdot 2)^0 = 9$. Thus, $(n, p, x, y, z) = (2, 2, 3, 0, 3)$.

If $v = 2$, then $2^{x-4} - 7 = 1$. This implies $x = 7, z^2 = n^x + (7p)^y = 2^7 + (7 \cdot 2)^2 = 324$. Thus, $(n, p, x, y, z) = (2, 2, 7, 2, 18)$.

If $v > 2$, from Equation (8), we consider $x - 2v$ in each case. Since x is odd, we get $x - 2v$ is odd.

Subcase $x - 2v = 1$. Then $7^{v-1} = 1$ and so $v = 1$. This is contradict.

Subcase $x - 2v > 1$. By Mihăilescu's Theorem, we get $2 = 3$. This is impossible.

Case 2.2: $2|n^{x-2v} - 1$. It follows that n must be odd. From Equation (5), $2(7p)^m = n^v(n^{x-2v} - 1)$, we get that $n^v|(7p)^m$ and hence $n|7p$. Thus, $n|7$ or $n|p$. Since n is an odd prime such that $n \equiv 2(\mathrm{mod}3)$, we get that $n = 7$. This is impossible. Thus, $n = p$. It follows that

$$2(7p)^m = p^v(p^{x-2v} - 1)$$
$$2 \cdot 7^m \cdot p^m = p^v(p^{x-2v} - 1)$$
$$2 \cdot 7^m = p^{v-m}(p^{x-2v} - 1).$$

We have that $p^{v-m} | 2 \cdot 7^m$. Since $p \nmid 7$, we get $v - m = 0$.

This implies $m = v$. Hence $p^{x-2v} - 1 = 2 \cdot 7^m$. Since $p \equiv 2 \equiv -1 \pmod 3$ and $x - 2v$ are odd, we get $p^{x-2v} - 1 \equiv -1 - 1 \equiv 1 \pmod 3$. This implies that $2 \cdot 7^m \equiv 1 \pmod 3$. This is impossible since $2 \cdot 7^m \equiv 2 \cdot 1^m \equiv 2 \pmod 3$.

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Received November, 2024