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On the existence of solutions of boundary value problems for quasilinear elliptic equations of general form in Sobolev spaces

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Abstract

The aim of the study is to prove the existence of functions satisfying a quasilinear elliptic equation of general form and satisfying the given boundary conditions in Sobolev spaces. Methods of functional analysis, theory of functions of many variables, differential calculus of functions were used to achieve this goal. The main results of the study include the formulation and proof of the theorem on the existence of solutions of boundary value problems for quasilinear elliptic equations of general form in Sobolev spaces. A general methodology has been developed to prove the existence of solutions for a wide class of quasilinear elliptic equations.

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1. Introduction

The study of the existence in Sobolev spaces of solutions of boundary value problems for quasilinear elliptic equations of general form is important for understanding various properties and behaviour of solutions of such equations. In general, quasilinear elliptic equations are equations in which the coefficients depend non-linearly on the derivatives of the unknown function. They are an important class of equations in mathematical physics, elasticity theory, hydrodynamics, as well as in other fields of science and technology. The study of boundary value problems for quasilinear elliptic equations in Sobolev spaces

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allows determining the conditions of existence and uniqueness of solutions, as well as to evaluate their regularity and behaviour. The main problem of the study is to prove the existence of solutions of boundary value problems for complex quasilinear elliptic equations in Sobolev spaces. Sobolev spaces include functions and their derivatives, which makes them suitable for dealing with differential equations and boundary conditions. However, their use in the context of quasilinear elliptic equations requires sophisticated methods of functional analysis, operator theory and differential equation theory [1].

According to Behzadan and Holst [2], Sobolev spaces are an important class of function spaces that are used to study functions that have a finite number of derivatives. These spaces made it possible to describe functions that might not be smooth enough for classical differential operations but still retain some degree of regularity, which made them useful for modelling and analysing various natural phenomena and processes in various fields of mathematics and applied sciences such as mathematical physics, current theory, differential equations, optimization, and many others. In their research work, Brasco, Gómez-Castro, and Vázquez [3] considered the problem of properties and structure of homogeneous fractional Sobolev spaces. Homogeneous small Sobolev spaces are an extension of standard small Sobolev spaces and allow the modelling and analysis of functions with more general regular properties. Borsuk [4] notes that quasilinear equations are equations containing non-linear functional dependence on derivatives and find applications in various areas of mathematics and applied sciences, particularly in mathematical physics and engineering. A bounded derivative problem in a planar sector may involve conditions on the boundaries of the sector related to the derivative in certain directions.

According to Emamizadeh and Liu [5], quasilinear elliptic equations are second order equations with nonlinear functional dependence and important applications in physics, engineering and other fields of science. Optimal control problems in this context may involve finding the best strategies to minimize or maximize certain target functions subject to constraints on admissible controls. Dong and Mao [6] considered the problem of existence and multiplicity of solutions for general quasilinear elliptic equations. They noted that quasilinear elliptic equations are equations in which nonlinear functional dependence on derivatives occurs. Jing, Liu and Wang [7] in their work considered the problem of parameter dependent multiple solutions sign change results for quasilinear elliptic equations. They noted that quasilinear elliptic equations are nonlinear equations involving higher order derivatives and nonlinear dependence on solutions and their derivatives.

The aim of the study is to find functions that are solutions of quasilinear elliptic equations of general form in Sobolev spaces, and to understand the properties and characteristics of solutions of quasilinear elliptic equations.

2. Materials and Methods

When conducting research in the field of studying Sobolev spaces, materials and methods were used that helped to reveal both theoretical and practical content of the object. This study used definitions and properties of Sobolev spaces, which are function spaces containing functions and their derivatives of a certain order. The use of derivatives and convergence conditions for sequences of functions has made it possible to consider functions that may not be smooth enough, as well as to treat functions for which the classical derivative is not defined. In this study, there were used the notations of Banach spaces, derivatives, and differential operator equation (1):

$$Lu = a_{ij}(x, u, u_x)u_{x_i x_j} + a(x, u, u_x). \quad (1)$$

An important part of the research was the study of differential equations in Sobolev spaces. This allowed considering differential equations containing derivatives of functions from Sobolev spaces and to investigate their properties and solutions. The study also included the consideration of operators in Sobolev spaces, which allow formalizing the notion of functions with bounded smoothness and to solve various problems related to them.

Using functional analysis, strong and weak formulations of the original equations in Sobolev spaces were determined. The strong formulation presented an equation that must be fulfilled exactly for all elements of the space. On the other hand, the weak formulation allowed considering the equation as a functional equality that must be true for all test functions from the space. Further, the functional analysis allowed defining operators acting in Sobolev spaces.

Operator theory was applied to analyse in more depth the quasilinear differential operators that arose in the boundary value problems under consideration. This included the study of spectral properties, actions on functions from Sobolev spaces and the determination of conditions under which these operators have certain properties such as compactness, symmetry or self-adjointness.

The theory of functions of many variables has been used to analyse and model multivariate functions in the context of the boundary value problems under consideration. This theory allowed considering functions depending on several independent variables, which is essential for the study of complex physical phenomena described by quasilinear elliptic equations.

Differential calculus of functions played a key role in the study of solutions of quasilinear elliptic equations in Sobolev spaces. The analysis of differentiability and smoothness of solutions allowed establishing the degree of their regularity and the classes of functions that can be used as solutions.

3. Results

In the course of this research, operators in Sobolev spaces were considered, which allowed formalizing the notion of functions with bounded smoothness and to develop efficient methods for solving a variety of problems related to

such functions. Operators in Sobolev spaces have gained importance in the analysis and solution of quasilinear elliptic differential equations, and have become a key component of approximate methods for solving such equations. The evaluation of the convergence of approximate solutions carried out was important in determining the accuracy and reliability of the methods used for approximation and numerical solution of equations in Sobolev spaces. This analysis provided a deeper understanding of how successfully the chosen methods are able to approximate real solutions of the equations and how much one can trust the obtained results. To better understand the problem of existence of solutions in Sobolev spaces, the following boundary value problem (2-3) was considered:

$$a_{ij}(x,u,u_x)u_{x_ix_j} + a(x,u,u_x) = 0, x \in \Omega, \tag{2}$$

$$b(x,u,u_x) = \varphi(x), x \in \partial\Omega, \tag{3}$$

where: Ω – an area in R^n, $\partial\Omega$ – its boundary.

Lemma 1: *Let X, Y be Banach spaces, the mapping $P: X \to Y$ has derivative P_w for any $w \in X$. It was assumed, P(X) is a closed set in Y and for any $w \in X$ there exists such $v \in X$ that $P_w v + Pw = 0$. Then $0 \in P(X)$. It is denoted (4):*

$$Nu = b(x,u,u_x) - \varphi(x), \tag{4}$$

and should be (5):

$$Pu = (Lu, Nu). \tag{5}$$

As a result, the operator is obtained (6):

$$P: W_q^2(\Omega) \to L_q(\Omega) \times W_q^{1-\frac{1}{q}}(\partial\Omega), \tag{6}$$

where: Ω – the area in R^n, $\partial\Omega$ – its boundary.

Lemma 2. It was assumed that for $x \in \bar{\Omega}, p \in R^n, |u| \le M, |p| \le M$ of the function $a_{ij}(x,u,p)$ are continuous and $b(x,u,p), b_u(x,u,p), b_{p_i}(x,u,p)$ have limited derivatives with respect to their arguments and inequalities (7-8) are satisfied:

$$\nu\xi^2 \le a_{ij}(x,u,p)\xi_i\,\xi_j \le \nu^{-1}\xi^2, \xi \in R^n, \tag{7}$$

$$(b_p, \bar{n}) \ge \gamma > 0, x \in \partial\Omega. \tag{8}$$

It is assumed that for any $\alpha \in (0,1)$ $w \in W_q^2(\Omega)$ estimate (9) is valid:

$$\|w\|^{(1+\alpha)} \le C\left(\|Lw\|_{q,\Omega}, \|Nw\|_{q,\partial\Omega}^{\left(1-\frac{1}{q}\right)}\right), \tag{9}$$

where: norms are specified in spaces $C^{1+\alpha}(\Omega), L_q(\Omega), W_q^{1-\frac{1}{q}}(\partial\Omega)$.

Proof: By virtue of the previous lemma, it is sufficient to prove that $P(W_q^2(\Omega))$ is closed in the space $L_q(\Omega) \times W_q^{1-\frac{1}{q}}(\partial\Omega)$. Problem (10) was considered:

$$\left.\begin{array}{c} Lw = a_{ij}(x,w,w_x)w_{x_ix_j} + a(x,w,w_x) = f(x), x \in \Omega, \\ Nw = b(x,w,w_x) - \varphi(x) = \psi(x), x \epsilon\, \partial\Omega, \end{array}\right\}, \tag{10}$$

It was noted that the function v(x) can be considered as a solution to the following linear problem (11):

$$\left.\begin{array}{c} a^1_{ij}(x)v_{x_ix_j} = f^1(x), x\epsilon\Omega, \\ b_i(x)v_{x_i} + b_o(x) = \psi^1(x), x\epsilon\partial\Omega, \end{array}\right\}, \tag{11}$$

Due to the conditions of the lemma $a^1_{ij}\epsilon C(\bar{\Omega}), f^1\epsilon L_q(\Omega), b_i, b_o, \Psi^1\epsilon W_q^{1-\frac{1}{q}}$ $(\partial\Omega)$, and (12):

$$\|f'\|_{q,\Omega} \le \|f' - f''\|_{q,\Omega} + \varepsilon_1\left(\|v\|^{(1)}_{\Omega}\right)\left(1 + \|w''\|^2_{q,\Omega}\right), \tag{12}$$

where: $\varepsilon_1(t)$ depends on the moduli of continuity of the function a_{ij}.

In addition to it (13):

Thus, the estimate (13) was obtained:

$$\|v\|^{(2)}_{q,\Omega} \le$$

$$C\left[\|f' - f''\|_{q,\Omega} + \|\psi' - \psi''\|^{\left(1-\frac{1}{q}\right)}_{q,\partial\Omega} + \varepsilon\left(\|v\|^{(1)}_{\Omega}\right)\left(1 + \|w'\|^{(2)}_{q,\Omega} + \|w''\|^{(2)}_{q,\Omega}\right).\right. \tag{13}$$

Let $\{f_m, \psi_m\}$ be a sequence of elements from $L_q(\Omega) \times W_q^{\left(1-\frac{1}{q}\right)}(\partial\Omega)$ such, that for each m there is a solution $w_m\epsilon W^2_q(\Omega)$ to the problems: $Lw_m = f_m$ in Ω, $Nw_m = \psi_m$ on $\partial\Omega$, and let $f_m \to f, \psi_m \to \psi$. By virtue of the conditions of the lemma, there is a subsequence $\{w_{m_i}\}$ converging in $C^1(\bar{\Omega})$. Therefore, for m_i, larger than some m_0, there are (14-15):

$$C\varepsilon\left(\|w_{m_i} - w_{m_o}\|^{(1)}_{\Omega}\right) \le \frac{1}{2}, \tag{14}$$

$$\begin{aligned} \|w_{m_i}\|^{(2)}_{q,\Omega} &\le \|w_{m_i} - w_{m_o}\|^{(2)}_{q,\Omega} + \|w_{m_o}\|^{(2)}_{q,\Omega} \\ &\le C\left[\|f_{m_i} - f_{m_o}\|_{q,\Omega} + \|\psi_{m_i} - \psi_{m_o}\|^{1-\frac{1}{q}}_{q,\partial\Omega}\right] + \\ &+ \frac{1}{2}\left(1 + \|w_{m_i}\|^{(2)}_{q,\Omega} + 3\|w_{m_o}\|^{(2)}_{q,\Omega}\right). \end{aligned} \tag{15}$$

From the last inequality it follows that the norms are bounded $\|w_{m_i}\|^{(2)}_{q,\Omega}$, and, therefore, by virtue of inequality (5), the convergence of the subsequence $\{w_{m_i}\}$ in space $W^2_q(\Omega)$. In the limit, this subsequence gives the function w(x). The lemma is proven. To prove the solvability of problems (2) and (3), first the $t\epsilon[0,1]$ was considered following auxiliary family of problems (16):

$$\left.\begin{array}{c} tA_{ij}(x,w_x)w_{x_ix_j} + (1-t)\Delta w - w = tF(x), x \in \Omega, \\ tB(x,w_x) + (1-t)\frac{\partial w}{\partial \bar{n}} + w = t\Psi(x), x \in \partial\Omega. \end{array}\right\}. \tag{16}$$

Lemma 3: *Let for* $x\epsilon\bar{\Omega}$, $p\epsilon\mathbb{R}^n$ *the functions* $A_{ij}(x,p)$ *have first derivatives with respect to their arguments and the inequalities (17-18) are satisfied:*

$$\nu\xi^2 \le A_{ij}\xi_i\xi_j \le \nu^{-1}\xi^2, \xi\epsilon R^n, \tag{17}$$

$$\left|\frac{\partial A_{ij}}{\partial p_k}\right| \le \frac{\mu}{1+|p|}, \left|\frac{\partial A_{ij}}{\partial x_k}\right| \le \mu|p| + \Phi, i,j,k = 1,\dots,n, \tag{18}$$

where: μis – positive constant, $\Phi\epsilon L_q(\Omega)$.

The functions $B(x,p), B_{p_i}(x,p)$ are differentiable with respect to their arguments and the inequalities (19-21) are satisfied:

$$\left(B_p, \bar{n}\right) \ge \gamma \text{ on } \partial\Omega, \tag{19}$$

$$|B| \le \mu, (1+|p|), |B_P| \le \mu_1, \left|\frac{\partial^2 B}{\partial_{p_i}\partial_{p_j}}\right| \le \frac{\mu_1}{1+|p|}, \tag{20}$$

$$(1+|p|)\left|\frac{\partial^2 B}{\partial_{p_i}\partial_{x_j}}\right| + \left|\frac{\partial B}{\partial x_k}\right| \le \mu_1(1+|p|^2). \tag{21}$$

It is assumed that $F\epsilon L_q(\Omega), \Psi\epsilon W_q^{1-\frac{1}{q}}(\partial\Omega), \partial\Omega\epsilon C^2$. Then, if a function $w\epsilon W_q^2(\Omega)$ is a solution to problem (16), then for some $\alpha\epsilon(0,1)$ it satisfies the estimate $\|w\|_{\Omega}^{(1+\alpha)} \le C$. Proof: due to Cârstea and Feizmohammadi [8] to prove the lemma, it is enough to have the estimate $\max_{\Omega}|w|$. There are defined (22-23):

$$L_t(w) = A_{ij}^t(x,w_x)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} - I, \tag{22}$$

$$A_{ij}^t(x,w_x) = tA_{ij}(x,w_x) + (1-t)\delta_i^j - I, \tag{23}$$

where: I – identity operator.

Let v be the solution to the following Dirichlet problem: $L_t(w)v = tF, x\epsilon\Omega$, $v = 0$, $x\epsilon\partial\Omega$. By inequality of Argyros et al. [9] the estimate (24) was determined:

$$max|v| \le c_1\|F\|_{q,\Omega}, \tag{24}$$

The function $z(x) = w(x) - v(x)$ is a solution to the following problem (25):

$$B_i^t(x)z_{x_j} + z = \Psi_{(x)}^1, x\epsilon\partial\Omega, \tag{25}$$

From equation (25) it follows that the function z(x) does not have a positive maximum and a negative minimum inside Ω. Let the point $x_0 \in \partial\Omega$ be a point of positive maximum. Then (26):

$$z(x_o) \le \Psi^1(x_o). \tag{26}$$

The lower estimate is performed similarly. The lemma is proven. Lemma 4. Let the assumptions of the previous lemma be satisfied with respect to the $A_{ij}(x,p), B(x,p)$ functions. Then for any $F \in L_q(\Omega), \Psi \in W_q^{\left(1-\frac{1}{q}\right)}(\partial\Omega)$ for problem (16) there is a unique solution $w \in W_q^2(\Omega)$. Proof: by virtue of Lemmas

2 and 3, to prove Lemma 4 it was sufficient to believe that the corresponding (16) problems in variations are infinitely solvable or, which is the same, homogeneous problems in variations have only a trivial solution. The latter looks like (27-28):

$$tA_{ij}(x,w_x)v_{x,x_j} + t\frac{\partial A_{ij}(x,w_x)}{\partial w_{x_k}}w_{x_i,x_j}v_{x_k} + (1-t)\Delta v - v = 0, x \in \Omega, \quad (27)$$

$$t\frac{\partial B(x,w_x)}{\partial w_{x_i}}v_{x_i} + (1-t)\frac{\partial v}{\partial \bar{n}} + v = 0, x \in \vartheta\Omega. \quad (28)$$

Then the function $w(x) = w'(x) - w''(x)$ is a solution to the following linear problem (29-30):

$$A_{ij}^t\left(x, w'(x)\right)w_{x_ix_j} + \left[w''_{x_i,x_j}\int_0^1 \frac{\partial A_{ij}(x,w_x^s(x))}{\partial w_{x_k}^s}ds\right]w_{x_k} - w = 0, \quad (29)$$

$$\int_0^1 \frac{\partial B^t(x,w_x^s(x))}{\partial w_{x_i}^s}dsw_{x_i} + w = 0, x \in \partial\Omega, \quad (30)$$

where: the functions A_{ij}^t, B^t were defined in the previous lemma, and $w^s = sw' + (1-s)w''$.

Therefore, as stated above, $w \equiv 0$. The lemma is proven. The following family of problems depending on parameter $t \in [0,1]$(31) was considered:

$$\left.\begin{aligned} t\left[a_{ij}(x,u,u_x)u_{x_i}x_j + a(x,u,u_x)\right] + (1-t)(\Delta u - u) = 0, x \in \Omega, \\ tb(x,u,u_x) + (1-t)\left(\frac{\partial u}{\partial \bar{n}} + u\right) = t\varphi(x), x \in \partial\Omega. \end{aligned}\right\} \quad (31)$$

Theorem 1: *The following is assumed:*

1. *For all solutions $u(x,t)$of the class of $W_q^2(\Omega), q > n$ problem (31), the estimate $max_\Omega|u(x,t)| \le M$, with a constant M, independent of t, is valid.*
2. *When $x \in \bar{\Omega}$, $|u| \le M$, $p \in R^n$, the function $a(x,u,p)$ is measurable, the functions $a_{ij}(x,u,p)$ are differentiable with respect to their arguments and the inequalities (32-35) are satisfied:*

$$\nu\xi^2 \le a_{ij}\xi_i\xi_j \le \nu^{-1}\xi^2, \xi \in R^n, \quad (31)$$

$$|a(x,u,p)| \le \mu p^2 + F(x), \quad (32)$$

$$\left|\frac{\partial a_{ij}(x,u,p)}{\partial p_k}\right| \le \frac{\mu_0}{1+|p|}, \quad (33)$$

$$\left|\frac{\partial a_{ij}(x,u,p)}{\partial u}\right| \le \mu_1, \quad (34)$$

$$\left|\frac{\partial a_{ij}(x,u,p)}{\partial x_k}\right| \le \mu_2(p) + F(x), \quad (35)$$

where: μ, μ_0, $\mu_1$are positive constants, $i,j,k = 1,\dots,n$, $F \in L_q(\Omega)$.
When $x \in \partial\Omega$, $|u| < M$, $p \in R^n$ functions $b(x,u,p), b_{p_i}(x,u,p)$ are differentiable with respect to their arguments and inequalities (36-38) are satisfied:

$$\left(b_p, \bar{n}\right) \geq \gamma, \tag{36}$$

$$\left|\frac{\partial b}{\partial u}\right| + \left|\frac{\partial^2 b}{\partial p_i \partial x_j}\right| \leq \mu_2(1 + |p|), \tag{37}$$

$$\left|\frac{\partial^2 b}{\partial p_i \partial p_j}\right| \leq \frac{\mu_2}{1+|p|}. \tag{38}$$

3. $\partial\Omega \in C^2, \varphi \in W_q^{1-\frac{1}{q}}(\partial\Omega)$.
4. *For any* $|u| \leq M$, $|p| \leq M$ *the function* $a(\cdot, u, p)$ *belongs to the space* $L_q(\Omega)$, *and (39):*

$$\left\|a(\cdot, u, p) - a\left(\cdot, u' p'\right)\right\|_{q,\Omega} \leq \xi_0\left(\left|u' - u\right| + \left|p' - p\right|\right), \tag{39}$$

where: $\varepsilon_0(s) \to 0$ *at* $s \to 0$.

5. *For* $x \in \partial\Omega$, $|u| \leq M$, $|p| \leq M$, *the function* $b_u(x, u, p)$ *has limited derivatives with respect to its arguments. Then, for each,* $t \in [0, 1]$ *there is a solution to problem (31) and, in particular, to problems (2-3), belonging to* $W_q^2(\Omega)$. *Proof: due to Cârstea and Feizmohammadi [8] with some* $\beta \in (0, 1)$ *the estimate* $\|u\|_{\Omega}^{(1+\beta)} \leq M_2$ *is correct. Problem (8) was compared to problem (6), assuming (40-43):*

$$A_{ij}(x, w_x) = a_{ij}(x, v(x), w_x), \tag{40}$$

$$F(x) = -a\left(x, v(x), v_x(x)\right) - v(x), \tag{41}$$

$$B(x, w_x) = b(x, v(x), w_x), \tag{42}$$

$$\Psi(x) = \varphi(x) + v(x), \tag{43}$$

where: $v(x)$*is an arbitrary function from* $C^{1+B}(\Omega)$.

The functions $A_{ij}(x, w_x)$, $F(x)$, $B(x, w_x)$, $\Psi(x)$ satisfy all the conditions of Lemma 4 and, therefore, for each $v \in C^{1+\beta}(\bar{\Omega})$ there is a unique solution $w \in W_q^2(\Omega)$ to problem (16), (31). Thus, $C^{1+\beta}(\Omega)$the transformation is defined in the space $w = F^1(v, t)$, and Lemma 3 implies the validity of the estimate $\|w\|_{\Omega}^{(1+\alpha)} \leq C$. The fixed points of the transformation F^1 will give solutions to problem (31). To prove the existence of fixed points y, $F^1(v, t)$ the Leray-Schauder theorem was used.

Theorem 2: *Let V be a Banach space* $\bar{G}$ *and be the closure of any connected bounded open set G in V. Let* $E = G \times [0, 1]$, $G_1 = \bar{G} \times [0,1]$. *The equation* $u = F^1(u, t)$ *has at least one solution for G any* $t \in [0, 1]$ *if:*

1. *The transformation* $F^1(v, t)$ *is defined and completely continuous from* G_1 *to G.*
2. *The boundary G does not contain solutions to the equation* $u = F^1(u, t)$ *for all* $t \in [0, 1]$.
3. *When* $t = 0$, *the equation* $u = F^1(u, t)$ *has a finite number of solutions in G and their total index is different from zero. In space,* $V = C^{1+\beta}$ *it is*

worth taking a ball as G, $\|u\|_{\Omega}^{(1+\beta)} \leq M_2 + \delta$ *thereby satisfying condition of the Leray-Schauder theorem. The proof of the continuity of the transformation was considered* $F^1(v,t)$*. The function* $w = w' - w'' = F^1(v',t') - F^1(v'',t'')$ *was considered as a solution to the following linear problem (44-45):*

$$a_{ij}^1(x) = t' a_{ij}\left(x, v'(x), w_x'(x)\right) + \left(1 - t'\right)\delta_i^j, \tag{44}$$

$$b_i(t) = t' \int_0^1 \frac{\partial b(x, v^s, w_x^s)}{\partial w_{x_i}^s} ds + \left(1 - t'\right) cos(\overline{n}, x_i). \tag{45}$$

Thus,

$$\|w\|_{q,\Omega}^{(2)} \leq C\left[\varepsilon_1\left(\left\|v' - v''\right\|_{\Omega}^{(1)} + \left\|w' - w''\right\|_{\Omega}^{(1)}\right) + \left|t' - t''\right|\right]\left(1 + \left\|w'\right\|_{q,\Omega}^{(2)} + \left\|w''\right\|_{q,\Omega}^{(2)}\right). \tag{46}$$

From inequality (46) the continuity of the transformation $F^1(v,t)$ is derived. Let the sequence $\{v_k\}$ converge to some function v_0 in the norm of the space $C^{1+\beta}(\overline{\Omega})$ for $k \to \infty$, $t_k \to t_0$. Through w_k, w_0 the corresponding solutions of the equation $w = F^1(v,t)$ were denoted. Then, by estimate, the sequence $\{w_k\}$ is compact in $C^1(\overline{\Omega})$. Let $\{w_{k'}\}$ be some subsequence of $\{w_k\}$. From this, in turn, it is possible to select a subsequence $\{w_{k''}\}$ that converges in the norm $C^1(\overline{\Omega})$. Applying inequality for $w' = w_{k''}$, $w'' = w_{m''}$ by virtue of the proof of Lemma 2, $\|w_{k''} - w_{m''}\|_{q,\Omega}^{(2)} \to 0$ for $k'', m'' \to \infty$ were obtained.

Passing to the limit over this subsequence in problem (6), (9) shows that the only limit function for $w_{k''}$ can only be $w_0(x,t)$. Since the subsequence $\{w_{k'}\}$ was chosen $\{w_k\}$ arbitrarily, the entire sequence $\{w_k\}$ converges to w_0 in the norm $W_q^2(\Omega)$. The compactness F^1 is obvious. Further, since at $t = 0$problem (6), (9), which determines the transformation F^1, has only a trivial solution, i.e. $F^1(v,0) \equiv 0$, $v \in C^{1+\beta}(\overline{\Omega})$ and, therefore, $v - F^1(v,0)$ is an identical transformation, then the index of the fixed point $u = 0$ of the transformation $F^1(v,0)$ is equal to 1. Thus, all conditions of the Leray-Schauder theorem are satisfied, and therefore, for each $t \in [0,1]$ there is a fixed point of the transformation $F^1(v,t)$. The theorem has been proven.

4. Discussion

Understanding and solving boundary value problems of quasilinear elliptic equations of general form are important in modelling and analysing a variety of phenomena in various fields of science and engineering. [10-12]. Solving boundary value problems is also important in engineering and technical

applications, such as materials design, analysing the performance of structures, optimizing processes and creating new technologies [13-15].

In this study, the application of functional analysis played a crucial role in the study of quasilinear elliptic equations. This analytical tool made it possible to formalize and abstractly consider the derivatives of functions in Sobolev spaces, which in turn greatly facilitated the analysis of boundary conditions and the determination of the structure of solutions. Operator theory allowed analysing the spectral properties of nonlinear differential operators in Sobolev spaces. The theory of functions of many variables considered allowed analysing in more depth the boundary conditions in multidimensional spaces.

Lee and Ok [16] devoted to the study of Sobolev norm estimates W1, q for solutions of quasilinear elliptic equations like Schrödinger equations, they described the analysis of quasilinear elliptic equations which have a structure similar to the Schrödinger equations from quantum mechanics. An important aspect of the paper is the evaluation of the Sobolev norm of the W1,q space for solutions of such quasilinear equations. These evaluations, carried out both inside the solution domain and on its boundary, are of key importance when investigating the properties and behaviour of the solutions. The analysis of this work will enhance the understanding of the behaviour of nonlinear differential equations, bring new knowledge and understanding in the field of mathematical analysis, differential equations and their applications, and contribute to the development of theoretical and practical aspects of the research.

Byun, Cho and Lee, [17] in their research work on the study of maximum differentiability for a general class of quasilinear elliptic equations with measures on the right-hand side, investigated quasilinear elliptic equations which have important applications in various areas of mathematics and physics such as mathematical physics, elasticity theory, geometry, and others. Quasilinear equations are a generalization of linear equations, including nonlinear relationships between derivatives of solutions [18; 19]. The main emphasis of the work was on maximum differentiability of solutions for a wide class of quasilinear elliptic equations. In particular, the work drew attention to the case where there are measures on the right-hand side of the equation which can be generalized functions or even discrete masses.

In his research work Maz'ya [20] devoted to the study of strong singularities of solutions of nonlinear elliptic equations, he studied nonlinear elliptic equations, which have important applications in various fields of mathematics and physics such as mathematical physics, elasticity theory, geometry, and others. The main emphasis of this paper has been on the study of strong singularities of solutions of nonlinear elliptic equations. Strong singularities usually mean that the solutions of the equations have unbounded growth or discontinuities at certain points, which can cause difficulties in analysing and modelling these equations [21, 22]. The similarity of this study to the work of the researcher, is the common theme of studying nonlinear elliptic equations and their properties.

Naceur et al. [23], in their research work on the study of theoretical and numerical aspects of a class of quasilinear elliptic equations, analysed the existence and uniqueness of solutions for this class of quasilinear elliptic equations, which included the study of properties of solutions, their regularity, dependence on parameters and other theoretical aspects. The similarity of this paper with the work of the researchers is the study of quasilinear elliptic equations and their properties. In both papers, a solution theoretic analysis is carried out, including the study of solution properties, dependence on parameters, and the behaviour of solutions to equations of this type.

In their work on the study of solution gradient estimates for very singular quasilinear elliptic equations with mixed data, Do, Truong and Trong, [24] considered theoretical aspects and estimates for solutions of these equations, which are important in the analysis of mathematical models and applications. An important aspect of the study was the consideration of boundary conditions and mixed data that are set at the boundary of the domain. The similarity of this work with the work of the researchers was the use of functional analysis, operator theory.

Thus, this study has multiple impacts on diverse areas of scientific and technical disciplines. Firstly, advancing the knowledge of solutions of quasilinear elliptic equations enriches the theoretical heritage of mathematics and provides a sustainable basis for the development of scientific and engineering fields. This can lead to new methods of analysis, more accurate models and tools that contribute to the development of innovative solutions. In addition, this research can inspire the creation of new technologies and products that find their application in practical scenarios. Importantly, the results of this study can be a key to the development of more efficient algorithms and methods for solving quasilinear elliptic equations.

5. Conclusions

The results obtained show that it is quite easy to prove the existence in Sobolev spaces of solutions of boundary value problems for quasilinear elliptic equations of general form if one has the necessary knowledge and skills. Possession of the theory of boundary value solutions as well as knowledge of Sobolev spaces played a key role in expanding the understanding of complex mathematical structures. In this paper, all the set tasks were fulfilled, namely to prove the existence of solutions of boundary value problems in Sobolev spaces. The conditions that helped to determine the main characteristics and properties of solutions of quasilinear elliptic equations have been considered and analysed.

Further development of this topic will provide an opportunity for a deeper understanding of complex nonlinear phenomena arising in the context of boundary value problems. An important direction for further research may also be the application of the results and methods obtained to specific applied problems, such as modelling physical phenomena in various fields of science and engineering. This will make it possible to apply theoretical findings to practical solutions and contribute to real-world problems.

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