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On the discrete and continuous Markov processes

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Abstract

The discrete and continuous Markov processes and Markov chains are considered. Basic concepts are introduced, and some applications of Markov processes are considered.

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1. Introduction

A random process, satisfying the so-called *Markov property*, which requires the future values of the process to depend only on the current value and not to depend on the previous values of the process (that is, on the background of the process, or equivalently, the process is independent on its prehistory), is called a *Markov process*.

The Markov process takes values of a discrete set, called the *state space*. We say that at a certain moment of time, the Markov process is in one state or in another state.

Depending on whether the transition from one state to another state can happen at arbitrary moments of time or only at fixed moments of time, there are *continuous time Markov processes* and *Markov chains*.

The concepts of Markov process, Markov property, and Markov chain are named after the Russian mathematician Andrey Markov (1856-1922).

The Markov processes and Markov chains are applicable in many areas of science and practise, for example, physics, chemistry, statistics,

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queueing theory, optimization, game theory, information theory, mathematical biology, genetics, social sciences, economics, finances, management theory, production systems, testing, speech recognition, music theory, Internet applications, etc.

The topics and problems, considered in this paper, and related to them, are discussed in references [1]-[28], etc.

In Section 2, some concepts are defined. Section 3 is dedicated to the applications of Markov processes. Section 4 summarizes the contents of the paper and defines future problems.

2. Basic concepts and definitions

Definition 1: Denote by $X(t)$ some random variable which describes the state of a random process at an arbitrary moment of time t. Consider two states y and z of the process, and the probability that the process has passed from y to z during a small amount of time between t and $t + h$. The considered process is said to be *Markovian* if

$$P[X(t+h)=z\,|\,X(s)=x(s), s\le t]=P[X(t+h)=z\,|\,X(t)=y]\ \ \forall h>0, \qquad (1)$$

where $P[...|...]$ denotes the conditional probability.

In other words, the values $X(t+h)$ of the Markov process in the future are dependent on the previous values $X(t)$, where $s < t$, only through the current value $X(t)$.

Definition 2: A Markov process is said to be *homogeneous* (*time-homogeneous*) if

$$P[X(t+h)=z\,|\,X(t)=x(t)]=P[X(h)=z\,|\,X(0)=x(0)]\quad \forall t,h>0, \qquad (2)$$

that is, if the probability of the transition from one state to another state is not dependent on the particular state. Otherwise the process of Markov is called *nonhomogeneous*.

Usually homogeneous Markov processes are simpler than the nonhomogeneous ones, and the homogeneous Markov processes constitute the most important class of Markov processes.

Definition 3: *Continuous time Markov process* is a Markov process which assumes values of a discrete set, which is said to be the *state space* (the *space of state*), and the value of this Markov process is changed at *arbitrary* moments of time.

The state space can be a finite set or an infinite, but countable, set.

The continuous time process of Markov is one of the types of a random process.

Definition 4: *Markov chain* is a Markov process which assumes values of a discrete set, which is said to be the space of state, and the value of this Markov process is changed at *fixed* moments of time.

The state space can be a finite set or an infinite, but countable, set.

The Markov chain is also one of the types of a random process.

Formally, a Markov chain can be considered as a sequence of random variables $X_1, X_2, \ldots$ with

$$P[X_{n+1} = y \mid X_1 = y_1, X_2 = y_2, \ldots, X_n = y_n] = P[X_{n+1} = y \mid X_n = y_n] \tag{3}$$

in case that $P[X_1 = y_1, \ldots, X_n = y_n] > 0.$

Although Markov chains are used for description of processes whose values are changed at particular moments of time, the model itself does not depend on the time but only on how many times the process has been activated.

Definition 5: *Markov chain of the order m* (*Markov chain with a memory*) is a process of Markov which satisfies the condition

$$\begin{aligned} &P[X_n = y_n \mid X_{n-1} = y_{n-1}, X_{n-2} = y_{n-2}, \ldots, X_1 = y_1] \\ &\quad = P[X_n = y_n \mid X_{n-1} = y_{n-1}, X_{n-2} = y_{n-2}, \ldots X_{n-m} = y_{n-m}] \quad \forall n > m, \end{aligned} \tag{4}$$

that is, if the future state of the process is dependent on the previous *m* states.

We can construct a chain $\{Y_n\}$ from the chain $\{X_n\},$ which possesses the "classical" property of Markov, where the space of state includes the ordered *m*–tuples of the values of *X*, that is, $Y_n = \{X_n, X_{n-1}, \ldots, X_{n-m+1}\}.$

3. Applications of Markov processes

In this section, some practical applications af Markov processes and Markov chains are considered.

It turns out that if the state space of a Markov process is *finite* then probability distribution of the transition can be presented through a matrix $\mathbf{A} = (p_{ij}),$ the *transition matrix*, whose elements are

$$p_{ij} = P[X_{n+1} = j \mid X_n = i]. \tag{5}$$

Example 1: Let a stock exchange be used as a market for three types of goods: animal origin products, plant origin products, and industrial products. Let the time during the year be used as follows: a time period for animal origin products, followed by another time period for animal origin products (a total of 90% of the time), a time period for plant origin products (7,5% of the time), and a time period for industrial products (2,5% of the time).

Concerning this problem, the Markov process is defined by the following transition matrix

$$\mathbf{A} = \begin{pmatrix} 0,9 & 0,075 & 0,025 \\ 0,15 & 0,8 & 0,05 \\ 0,25 & 0,25 & 0,5 \end{pmatrix}.$$

The relation, which defines the process in this case, can be presented as follows $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)}\mathbf{A}$. If at the moment of time n, the system is in the state 2, that is, $\mathbf{x} = (0,1,0)$, then after three periods of time, at the moment of time $n + 3$, the distribution will be

$$\begin{aligned} \mathbf{x}^{(n+3)} &= \mathbf{x}^{(n+2)}\mathbf{A} = (\mathbf{x}^{(n+1)}\mathbf{A})\mathbf{A} = \mathbf{x}^{(n+1)}\mathbf{A}^2 \\ &= (\mathbf{x}^{(n)}\mathbf{A})\mathbf{A}^2 = \mathbf{x}^{(n)}\mathbf{A}^3 = \\ &= (0,1,0)\begin{pmatrix} 0,9 & 0,075 & 0,025 \\ 0,15 & 0,8 & 0,05 \\ 0,25 & 0,25 & 0,5 \end{pmatrix}^3 = \\ &= (0,1,0)\begin{pmatrix} 0,7745 & 0,17875 & 0,04675 \\ 0,3575 & 0,56825 & 0,07425 \\ 0,4675 & 0,37125 & 0,16125 \end{pmatrix} = \\ &= (0.3575, 0.56825, 0.07425). \end{aligned}$$

Therefore

$$\lim_{N\to\infty} \mathbf{A}^N = \begin{pmatrix} 0,625 & 0,3125 & 0,0625 \\ 0,625 & 0,3125 & 0,0625 \\ 0,625 & 0,3125 & 0,0625 \end{pmatrix}.$$

This result can be interpreted economically as follows: during 62,5% of the time of the year (that is, during 32,5 weeks of the year) the stock exchange should be used as a market for products of animal origin, during 31,25% of the time of the year (that is, during 16,25 weeks of the year) the stock exchange should be used as a market for products of plant origin, and during 6,25% of the time of the year (that is, during 3,25 weeks of the year) the stock exchange should be used as a market for industrial products.

Definition 6: If a Markov chain is homogeneous, so that the corresponding process is presented through a matrix $\mathbf{A} = (p_{ij})$, which is time-independent, vector **d** is said to be the *stationary distribution* (or, sometimes, *invariant measure*) if for each $j \in S$ we have

$$0 \le d_j \le 1,\ \sum_{j \in S} d_j = 1,\ d_j = \sum_{i \in S} d_i p_{ij}. \tag{6}$$

Definition 7: A chain of Markov is called *reversible*, if there exists a probability distribution of the states **d**, such that

$$d_l P[X_{n+1} = m \,|\, X_n = l] = d_m P[X_{n+1} = l \,|\, X_n = m] \tag{7}$$

for all moments of time n and for all states l and m.

For a homogeneous Markov chain, the value $P[X_{n+1} = m \,|\, X_n = l]$ does not change during the time n and it can be denoted by p_{lm}. In this case, the relation (7) can be presented in the following simpler form

$$d_l p_{lm} = d_m p_{ml}. \tag{7'}$$

If for a Markov chain we have $P[X_0 = l] = d_l$ then $P[X_n = l] = d_l$ for each moment of time n, and in this case the relation (7) can be written in the following form

$$P[X_n = l \,|\, X_{n+1} = m] = P[X_{n+1} = l \,|\, X_n = m]. \tag{7''}$$

The left-hand side and the right-hand side of this equality are identical, except in the cases of reversibility of the time indexes, denoted by n and $n + 1$.

4. Conclusions

This paper is devoted to the discrete and continuous Markov processes and Markov chains. Basic concepts are introduced, and some applications of Markov processes are considered. Open problems for future research are to study and develop other applications of Markov processes and Markov chains.

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