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Numerical solution algorithm for the model of combat operations under the influence of time delay

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Abstract

This article presents a mathematical model of the process of delayed combat operations, in which the actions of two sides are delayed by a certain amount of time, based on the interaction of opposing forces, and defines a methodology for numerical solution. Differential equations with delayed arguments allow a more realistic reflection of the dynamic state of processes related to military operations. The article is to propose a finite difference method for numerically solving initial value problems for a system of linear differential equations under the influence of a constant delay time. The algorithm is based on Euler's method, and a comparative analysis of the proposed method is provided. The combat operations model is considered as a system of functional differential equations with constant delay. The applicability of the proposed approach is demonstrated using examples of conflicts between two opposing sides.

Subject Classification (2020): *34K07, 65Lxx, 93C23.*

Keywords: *Delay, Initial value problem, Euler's method, Conflict, Lanchester's quadratic law, Algorithm, Combat losses, Non-combat losses, Additional forces, Winner, Survive.*

1. Introduction

In many dynamic models of the surrounding reality, the next state of the processes is related to a certain extent not only with the current state, but also with the entire history of the process. Systems of functional differential equations, in particular, delayed or neutral differential equations, are often used to accurately model real-world processes. Functional-differentiating systems related to the previous state of the process were systematically studied in works [1], [2] and [3].

Nowadays, mathematical models, represented by differential equations with a delay argument, are considered effective in deeper research of military affairs. In this [4] work, arms race models have been studied under the influence of the time delay factor. General results were obtained for a system of functional-differential equations with a delay time argument, and these results were applied to a Richardson-type system.

In this work [5], the application of the system of differential equations with delayed arguments to combat conflict situations was considered, and the solution of the initial value problem exists and the analytical solution of the model of combat operations with delayed arguments for the Cauchy problem was found using the method of steps on the basis of uniqueness.

Also, differential equations with delayed arguments and their types are systematically studied in works [6–8], and analytical solution methods and applications are presented.

The scientific direction of differential equations with delayed arguments is rapidly being applied to many natural and technical processes. These works [9], [10] [11] and [12] show new theories, analytical and numerical solution methods, and numerical solutions of sample examples in mathematical package programs.

Additionally, in [13], numerical solutions to fractional-order integro-differential equations were obtained based on the Laguerre wavelet method. The effectiveness of the obtained results was demonstrated through illustrative examples.

In [14], numerical solution algorithms for systems of nonlinear Volterra-Fredholm type integral equations were developed using iterative methods.

In [15], an algorithm was developed to solve the problem of optimally allocating auxiliary forces at the optimal time based on Pontryagin's maximum principle. Using this algorithm, the optimal time for the allocation of auxiliary forces and the optimal strategies of the parties were determined.

In this paper [16], the control of additional forces coming to the opposite sides is studied as a piecewise-constant function based on the extended Lanchester's quadratic law, and several conflict situations are simulated.

The delayed argument, that is, the time delay effect, plays an important role in modeling combat operations more realistically and in the actions and decisions of commanders in conflict situations.

This study analyzes the interaction of opposing forces with a certain delay, using a numerical solution. The delayed start of the battle allows the opposing forces to strengthen their combat capabilities. In models of delayed combat, it is crucial not only to know the initial conditions at the beginning of the battle but also to understand the values before the battle starts. To find solutions to differential equations with delayed arguments, methods like Euler's method, the fourth-order Runge-Kutta method, and other similar techniques are used.

2. Materials and Methods

To solve an ordinary differential equation with a delay argument approximately by numerical methods, we consider the process of converting it to a discrete case. We write the initial value problem represented by an ordinary differential equation with a constant delay as follows [11]:

$$y'(t) = f(t, y(t), y(t-\tau)), \quad t_0 < t \le T. \tag{1}$$

$$y(t) = \varphi(t), \quad t_0 - \tau \le t \le t_0. \tag{2}$$

Where t is independent variable bounded above by T.

When solving problems (1) - (2) using numerical methods, the continuous function $y(t)$ is substituted by an approximated function, which is represented by a discrete argument over a set of points selected from the interval $[t_0, T]$. We divide the intersection $[t_0, T]$ into m pieces with the points $t_0,\ t_1,\ ...,\ t_M = T$. We define the distance (step) between each node point as $h_m = t_{m+1} - t_m$, where $m = 0,\ 1,\ ...,\ M$. Similarly, if h_m is constant for all m, then the step is constant, otherwise it is variable. t_m - a discrete function of a discrete argument given at a node $y_h = \{y_m = y_h(t_m)\}$ is called a lattice function. We denote the continuous approximation of the function $y_h(t)$ constructed by interpolation by $\tilde{y}_h(t)$.

The problem of numerically solving problems (1) - (2) is posed as follows. Let the initial function $y(t) = \varphi(t)$ be given on the interval $[-\tau,\ t_0]$. We choose a suitable step h_m, and we need to find the approximate values y_m of the unknown function $y(t)$ at the points t_m, where $m = 0,1,...,M$.

In the case of constant delay, the calculation step h_m should be chosen such that (constant or variable) $h_m \le \tau$, i.e. $(t_{m+1} - \tau \le t_m)$. Then at each step the value of the function $y(t-\tau)$ is known, and this value, depending on $t-\tau$, is equal to either the initial value $\varphi(t-\tau)$ or the value of the continuous approximation $\tilde{y}_h(t-\tau)$. Thus, at step $m+1$ the following subproblem for an ordinary differential equation must be solved without delay.

$$\begin{aligned} & y'(t) = f(t, y(t), y(t-\tau)), \quad t_m < t \le t_{m+1}, \\ & y(t_m) = y_m. \end{aligned} \tag{3}$$

where, $y_m = y_h(t_m)$ and

$$y(t-\tau) = \begin{cases} \varphi(t-\tau), & t \le t_0 + \tau, \\ \tilde{y}_h(t-\tau), & t_0 + \tau < t \le t_{m+1}. \end{cases}$$

As a result of numerically solving problem (3), we obtain the value y_{m+1} of the lattice function y_h and the value $\tilde{y}_h(t_{m+1}) = y_{m+1}$ of the continuous approximation of the function $\tilde{y}_h(t)$ on the interval $[t_m, t_{m+1}]$.

In general, numerical methods used to solve differential equations with a delayed argument should be stable, provide a good approximation to the problem being solved, and provide results close to the exact solution.

We write problem (1) - (2) as follows:

$$y'(t) = f(t, y(t), y(t-\tau)), \quad t_0 < t \le T,$$
$$y(t) = \varphi(t), \quad -\tau \le t \le t_0. \tag{4}$$

To solve problem (4) numerically, we use Euler's first-order approximation method and integrate the equation on the interval $[t_m, t_{m+1}]$.

$$y(t_{m+1}) = y(t_m) + \int_{t_m}^{t_{m+1}} f(t, y(t), y(t-\tau))dt. \tag{5}$$

By approximating the integral in (5) by the rectangular method, we have the formulas that give the Euler's disclose method

$$y_{m+1} = y_m + hf(t_m, y_m, y_{m-N}), \; m = 0,1,\ldots,M-1,$$
$$y_m = \varphi(t_m), \; m = -N, -N+1,\ldots,0. \tag{6}$$

The constant grid step h should be chosen based on the condition $h = \frac{\tau}{N}$, t_{m-N} point always belongs to the lattice, where $(N \in \mathbb{Z}^+)$. Then, store the last N value of the grid function y_h in the additional variable, y_{m-N} also the value will be determined.

Now we consider the variable step state h_m. In general, now $t_m - \tau \ne t_{m-N}$ i.e. $t_m - \tau$ the point does not belong to the grid, and the expressions defining Euler's method are as follows

$$y_{m+1} = y_m + hf(t_m, y_m, \tilde{y}_h(t_m - \tau)), \; m = 0,1,\ldots,M-1,$$
$$\tilde{y}_h(t) = \varphi(t), \; -\tau \le t \le t_0. \tag{7}$$

where, $\tilde{y}_h(t)$ function is a continuous approximation of the y_h.

The value $\tilde{y}_h(t_m - \tau)$ is calculated by interpolation on the section $[t_n, t_{n+1}]$, where the grid point is such that $t_n \le t_m - \tau \le t_{n+1}$. For instance, a simpler method such as piecewise-linear interpolation may be employed:

$$\tilde{y}_h(t) = \frac{t_{n+1}-t}{h_{n+1}} y_n + \frac{t-t_n}{h_{n+1}} y_{n+1}, \quad t_n \le t \le t_{n+1}. \tag{8}$$

There are continuous methods that allow you to calculate the value of the function value $\tilde{y}_h(t)$ according to special algorithms. In this context, $\tilde{y}_h(t)$ is referred to as the interpolation of the numerical method.

Euler's continuous method is defined using the following expressions.

$$y_{m+1} = \tilde{y}_h(t_m + h_{m+1}), \; m = 0,1,\ldots,M-1,$$
$$\tilde{y}_h(t_m + \theta h_{m+1}) = y_m + \theta h_{m+1} f(t_m, y_m, \tilde{y}_h(t_m - \tau)), \; 0 \le \theta \le 1,$$
$$\tilde{y}_h(t) = \varphi(t), \; -\tau \le t \le t_0.$$

Ultimately, Euler's constant-step implicit method is given by the following formulas

$$
\begin{aligned}
y_{m+1} &= y_m + hf(t_{m+1}, y_{m+1}, y_{m+1-N}), \quad m = 0,1,\ldots,M-1, \\
y_m &= \varphi(t_m), \quad m = -N, -N+1, \ldots, 0.
\end{aligned}
\tag{9}
$$

It is necessary to solve a system of algebraic equations to calculate y_{m+1} at each step. h_m variable-step continuous interpolation is performed as follows:

$$
\tilde{y}_h(t_m + \theta h_{m+1}) = y_m + \theta h_{m+1} f(t_{m+1}, y_{m+1}, \tilde{y}_h(t_{m+1} - \tau)), \; 0 \le \theta \le 1.
$$

3. Model of Delayed Combat Operations Including Auxiliary Forces

Let's examine a model of combat operations where auxiliary forces are incorporated with a constant delay. In this scenario, the delay is such that the change in the number of combat forces on both sides doesn't occur instantly, but is instead delayed. In this delayed situation, it's crucial to know not only the values of the opposing forces at the start of the battle but also the values prior to the battle.

Let's denote the warring parties as X and Y. $x(t)$ and $y(t)$ are functions representing the number of forces of the sides at time t, respectively.

We also introduce the parameter τ, which is the delay of the interaction of the opposing sides with each other. In this case, the system of differential equations with delayed arguments for the combat model is given by the following form [5; 9]:

$$
\begin{cases}
\dfrac{dx(t)}{dt} = -a_1 x(t) - b_1 y(t-\tau) + u(t), & t \ge 0, \\
\dfrac{dy(t)}{dt} = -a_2 y(t) - b_2 x(t-\tau) + \upsilon(t), & t \ge 0.
\end{cases}
\tag{10}
$$

The initial parameters for system (10) are as follows:

$$
\begin{cases}
x(t) = \varphi_1(t) = x_0, & t \in [-\tau, 0], \\
y(t) = \varphi_2(t) = y_0, & t \in [-\tau, 0].
\end{cases}
\tag{11}
$$

where $x(t)$ and $y(t)$ represents the number of forces on each side at time t; τ - is the time delay between the conflicting parties influence on each other; a_1, a_2 - coefficients of losses due to non-combat casualties; b_1, b_2 - coefficient of losses due to combat casualties; $u(t)$ and $\upsilon(t)$ are the rates of arrival for auxiliary forces, which can also be described as the control functions for the auxiliary forces.

Based on Euler's method, we provide the algorithm for determining the approximate solution to the initial value problem for the functional differential system (10) – (11) when $u(t)=0$ and $\upsilon(t)=0$.

4. Algorithm for Numerical Solution of Problem

Step 1. Giving values

1. a_1, a_2, b_1, b_2, τ assign values to parameters;
2. $\varphi_1 = x_0$ va $\varphi_2 = y_0$ give initial values;
3. Give initial values to calculation step h and maximum time $t_{\max}$;
4. Determine the number of steps of the delay $N = t_{\max} / h$.

Step 2. Create an array to hold results

1. h step by step create a time array in the range from $-\tau$ to $t_{\max}$;
2. create arrays $x(t)$ and $y(t)$.

Step 3. Set the initial values. Initialize the arrays $x(t)$ and $y(t)$ in interval $t = \overline{0:N}$:

1. $x(t) = x_0$; 2. $y(t) = y_0$.

Step 4. Calculation process. For each time step t_i from N to $t_{\max}$, calculate $(N \le t_i \le t_{\max})$:

1. Calculate the values of $x(t_i - N)$ and $y(t_i - N)$;
2. Use Euler's method to calculate the new values of arrays $x(t)$ and $y(t)$;

$$\begin{cases} x_{i+1} = x_i + h(-a_1 x_i - b_1 y(t_i - \tau)), \\ y_{i+1} = y_i + h(-a_2 y_i - b_2 x(t_i - \tau)). \end{cases}$$

3. Check $x(t)$ and $y(t)$ for zero. If the condition is satisfied, meaning if either of them is zero, we terminate the loop at this point and display the current values.

With the help of this algorithm, it is possible to create a program that finds numerical solutions to the system of differential equations with delayed arguments (10) (11) in a modern programming language, and simulate conflict situations between two opposing parties.

5. Simulation Results

We perform a numerical analysis of combat scenarios considering the impact of time delay. A simulation of a combat encounter between two sides is carried out based on the following assumptions.

- Each side's strength is no more than 7,000 soldiers;
- Combat and non-combat casualties, no more than 15%;
- The conflict happens with an ongoing delay;
- Once one side is defeated, no further changes occur in the forces;
- The arrival of supplementary forces is factored in.

To evaluate the outcome of this conflict, we use the "*Delayed combat operation model*» software created in C# using the Euler disclosure method. For simplicity, we assume that the time step in the discrete model is one day.

Conflict 1: In this conflict, we assume that the initial number of troops is 7000 for X side and 5000 for Y side. A side with more initial strength suffers 2.5 times higher combat casualties and 0.75 times higher non-combat casualties than a side with less strength. Also, if the conflict occurs with a constant time delay of $\tau = 1.5$, then which side wins, how much forces of the winning side survive, and what is the duration of the battle?

Table 1

The starting parameters of the sides

Battle parametrs	X side	Y side
Number of initial forces of the parties	7000	5000
Combat casualties (%)	3.00	1.20
Non-combat casualties (%)	0.75	1.00
Arrival of reinforcements	0	0
Collision delay time	1.5	1.5

The simulation result of the delayed argument combat model based on the battle parameters in Table 1 is shown in Figure 1. In this conflict, Side Y wins within 90 days, 690 soldiers survive, and the survival rate of 14%.

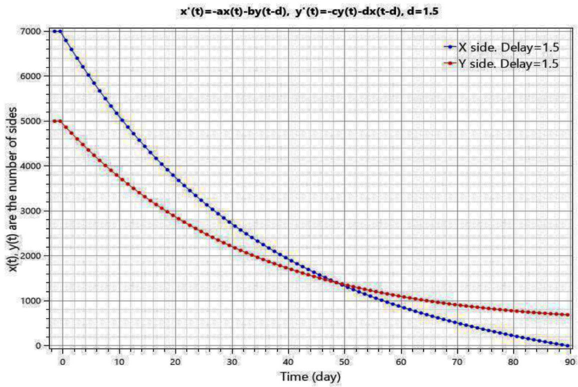

Figure 1

Graphic solution of Conflict 1

Conflict 2. If auxiliary forces join the parties in the above conflict, then the question of determining how the situation will change will arise. In this conflict, we present the simulation results based on the initial conditions of the Conflict 1, considering the speed of arrival of the additional forces coming to the sides as control functions (See Figure 2)..

Let's assume that each side has a total of 1,000 reinforcements, and the strategies for distributing them are as follows:

a) Side X directs and distributes its self-supporting forces as follows. These auxiliaries will reinforce X side by 15 from the 11th to the 70th day of the battle, and the remaining 100 auxiliaries will join on the 77th day.

b) On the 20th day of the battle, 300, 500 on the 40th, and 200 on the 60th day of the battle should join the Y side.

In this conflict, side Y wins after 88 days, with 1,224 soldiers surviving, resulting in a survival rate of 24%. This is 10% higher than the survival rate in Conflict 1. Therefore, the inclusion of reinforcements forces has a significant effect, not only on the outcome of the battle but also on the survival rate.

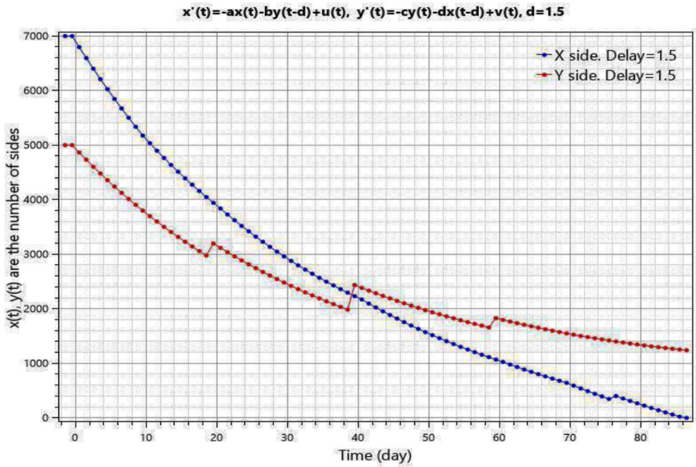

Figure 2

Graphical solution to Conflict 2

6. Discussion

Now, based on the conditions and auxiliary force allocation strategies in Conflict 2, we present the results for both delayed and non-delayed cases in a single graph.

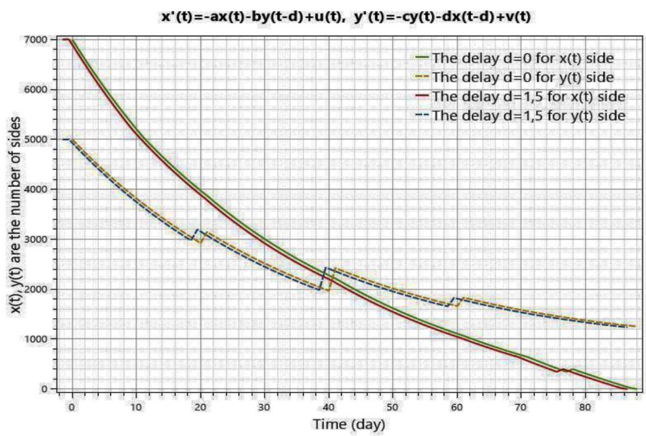

Figure 3

Auxiliary forces taken into account delay and a graph of non-delayed combat states

It is clearly observed in Figure 3 that the delayed process graph lags behind the non-delayed process graph. From this difference, we can see the impact of the delay on the process.

Incorporating delay time into a combat model for mathematically analyzing conflict situations would be useful in more accurately analyzing the outcomes of military conflicts that occur.

The results obtained for the second conflict (battle duration and the number of forces remaining on the winning side) and the comparative analysis of the solutions calculated using the Python ddeint function are presented in Tables 2 and 3.

Table 2

Comparison of results

Duration of the battle		**0**	**1**	**2**	**3**	**…**	**87**	**88**	**89**
The result obtained using the developed algorithm	***Side X***	7000	6798	6597	6401	…	3	0	
	Side Y	5000	4866	4733	4604	…	1236	1224	
Result obtained using Python ddeint function	***Side X***	7000	6797	6595	6392	…	54	16	0
	Side Y	5000	4866	4732	4598	…	1241	1227	1214

Table 3

Comparison of the time of the end of the conflict and the number of forces on the victorious side

	Our results	**Results in Python**
Battle duration time	88	89
Number of remaining forces	1224	1214

From the results presented in Tables 2 and 3, the following conclusions can be drawn:

- Although the code written in two different programming languages aims to solve the same problem, they may return different results. The reasons for this are related to the features of the software, the syntax of the language, the level of accuracy, the principles of the functions, as well as the different methods of the libraries used.
- Nevertheless, the results obtained using the program developed in C# programming based on the above algorithm are partially consistent with the results obtained using Python's 'ddeint' function. This shows that methods and functions in both programming

languages work on similar principles. However, this similarity also reveals some differences related to the specific features and methods of programming languages.

The correctness of the results obtained was compared with the results of the 'ddeint' function of the 'ddeint' library in Python. The computational results confirm the efficiency and reliability of the developed algorithm and software.

7. Conclusions

A delay argument was incorporated into Lanchester's combat operation model, and conflict situations were mathematically modeled. Using the theory of numerical solutions for Cauchy problem associated with a simple differential equation featuring constant delay, as well as Euler's first-order approximation method, an algorithm was developed for numerically solving the combat operation model with delay. Based on this algorithm, software named "Delayed Combat Operation Model" was created. This software facilitated the computer-based simulation of various combat conflict scenarios and enabled a comparative analysis of the obtained results.

References

[1] R. Bellman and K. L. Cooke, Differential-Difference Equations, *Mathematics in Science and Engineering*. New York, NY, USA: Academic Press (1963).

[2] A. D. Myshkis, Linear Differential Equations with Retarded Arguments (in Russian). Moscow, Russia: Nauka (1972).

[3] L. E. Elsgolts and S. B. Norkin, Introduction to the Theory of Differential Equations with Deviating Argument (1971).

[4] A. V. Shatyrko, D. Ya. Khusainov, B. Puza, and V. Novotna, "The dyamics of one arms race mathematical model with delay," *Int. Sci. Tech. J. Probl. Control Inform.*, vol. 6, pp. 89–99 (2020).

[5] N. M. Karimov, "About the method of solution the model of combat actions with delay argument," *Sci. Bull. Namangan State Univ.*, vol. 11, pp. 49–57 (2023).

[6] E. A. Gevorkyan, Differential Equations with Retarded Argument. Moscow, Russia, 79 p. (2004)

[7] R. D. Driver, Ordinary and Delay Differential Equations, *Applied Mathematical Sciences*, vol. 20. New York, NY, USA: Springer-Verlag, 514 p. (1977)

[8] A. Shatyrko, B. Půža, and V. Novotna, "Comparative analysis and new field of application of Lanchester's combat models," Post-conf. Proc. of Selected Papers (Extended Version), Conf. MITAV-2018, Brno, Czech Republic, pp. 118–133 (2018).

[9] A. Bellen and M. Zennaro, Numerical Methods for Delay Differential Equations. Oxford, UK: Oxford University Press, 206 p. (2003)

[10] L. F. Shampine and S. Thompson, "Numerical solution of delay differential equations," in Delay Differential Equations, Springer, Boston, MA, pp. 245–272 (2009). [Online]. Available: https://doi.org/10.1007/978-0-387-85595-0_9

[11] Z. Luo, M. Fečkan, and J. Wang, "A new method to study ILC problem for time-delay linear systems," Adv. Differ. Equ., vol. 2017, Article no. 35, pp. 1–14 (2017). [Online]. Available: https://doi.org/10.1186/s13662-017-1080-y

[12] A. D. Polyanin, V. G. Sorokin, and A. I. Zhurov, Delay Differential Equations: Properties, Methods, Solutions and Models. Moscow, Russia: Publishing "IPMex RAN", 464 p. (2022).

[13] S. Kumbinarasaiah and R. A. Mundewadi, "Numerical solution of fractional-order integro-differential equations using Laguerre wavelet method," *J. Inf. Optim. Sci.,* vol. 43, no. 4, pp. 643–662 (2022). [Online]. Available: https://doi.org/10.1080/02522667.2021.1943823

[14] A. J. Abdulqader, R. B. Abdulmaged, and B. M. Abdali, "Numerical solution for solving nonlinear Volterra-Fredholm system by using iterative techniques," *J. Interdiscip. Math.,* vol. 27, no. 4, pp. 991–1000 (2024). [Online]. Available: https://doi.org/10.47974/JIM-1962

[15] M. Mamatov and N. Karimov, "On the algorithm for solving the problem of optimal distribution of auxiliary forces in conflict situations based on the principle of maximum," AIP Conf. Proc., vol. 3244, no. 1, pp. 1–8 (Nov. 2024). [Online]. Available: https://doi.org/10.1063/5.0241480

[16] N. Karimov and T. Rakhmonov, "Method of controlling reinforcement forces to win in combat conflict situations," *Mod. Sci. Res.,* vol. 3, no. 1, pp. 1–5 (Jan. 2024). [Online]. Available: https://inlibrary.uz/index.php/science-research/article/view/28231

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