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# Numerical simulation of nonlinear equations by converting quadrature rule to iterative technique

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**Abstract**

In computational mathematics, numerical integration is an essential method that yields effective approximations of definite integrals. The trapezoidal technique is one of the most popular numerical integration techniques because of its ease of use and efficiency. Unfortunately, when working with complicated functions or intervals that exhibit erratic behavior, its applicability is restricted. The goal of this research study is to examine the effectiveness and practical implications of converting the trapezoidal rule to iterative procedures through the use of fundamental theorem of calculus for numerical integration. In addition, we provide case studies and numerical experiments to show the usefulness and efficiency of the proposed iterative techniques in comparison with the conventional Trapezoidal Rule. These experiments demonstrate situations in which iterative approaches perform better than the conventional method, indicating the potential of these approaches in practical integration problems. The research findings provide insights into the transformation of simple methods such as the Trapezoidal Rule into advanced iterative approaches, hence contributing to the improvement of numerical integration methods. This study provides

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more dependable numerical solutions in a variety of scientific and engineering applications by increasing the accuracy and efficiency of integration techniques.

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***Keywords:*** *Convergence analysis, Number of iterations, Nonlinear problems, Quadrature rules.*

## 1. Introduction

Numerous nonlinear equation-related problems can be found in literature, but probabilistic modeling is increasingly being used in every field of research and innovation. Real-world physical problems are frequently simulated by nonlinear equations and their solutions [12-13]. Even in cases where they are not directly applicable, these solutions might draw attention to the equations' general properties. The sine-cosine technique is one of several new accurate methods for solving nonlinear equations that have been developed in recent years [17–18]. The homotopy analysis method [1], the tanh technique [10].

Quadrature computations use values of the integrand at finite number of points to approximate definite integrals. It is frequently necessary to compute the definite integral of functions without an explicit anti-derivative in some circumstances, while in other cases, the function is directly provided by a sequence of measured or tabular values rather than an explicit function. Using a new proving methodology proposed by Chou, Y.L., the classical convergence theorem—which says that convergence of measurable functions on a limited measure space often always implies convergence in measure—is demonstrated in a broader context. They extend the theorem to the case where codomain is a separable metric space and the limiting map is constant, as well as to the case where the codomain is an arbitrary topological space [20]. These quadrature formulae are known as Newton-Cotes quadrature formulae [8, 15]. They are based on equally spaced points, or abscissas. Since numerical integration solutions are approximations, they are usually associated with the error of approximation, which measures the degree to which the approximate response differs from the exact value [5].

$I = \int_a^b w(x) f(x) dx$ is a typical quadrature formula in which the weight function w(x)>0 in a closed interval [a, b] is assumed to be integrable in Riemann sense in [a, b], and limits of integration are finite [6, 9].

It is sometimes impossible to evaluate definite integrals using analytical techniques This necessitates the development of numerical integration methods [4, 15]. To increase accuracy, the errors of all numerical approaches must be examined, as they only provide approximations. For practical applications, computer programs are required because most numerical approaches cannot be employed manually due to the accompanying errors [5, 6]. Numerous techniques exist for estimating the integral in numerical integration to the required precision [3, 5, 7]

Numerical integration is the process of applying a collection of integrand numerical valuecs to determine value of a specified integral. A function of single variable's integration is evaluated using a technique called mechanical quadrature. Rao [15] and Sastry [16] define mechanical Cubature as the process of calculating a function of two independent variables' double integral.

A collection of numerical integration formulas known as the Newton-Cotes formulae includes the trapezoidal rule. The trapezoid rule and the midpoint rule of this family are similar. Simpson rule, another rule in same family, converges faster than the trapezoidal rule for twice continuously differentiable functions, but not often [2, 19]. However, the trapezoidal rule usually converges faster than Simpson's rule for many classes of rougher functions (those with lower smoothness criterion) and, also when periodic functions are integrated throughout their periods, which can be analyzed in a variety of ways, the trapezoidal rule also tends to become very accurate [11, 14].

Therefore, if integrand is concave up the trapezoidal approach overestimates the true value and the error is negative. This idea is further demonstrated by the geometric illustration, where the trapezoids encompass the whole area underneath the curve. Burg [4] and Weidman [19] propose that a concave-down function also leads to an underestimation since area is computed above the curve rather than below it.

This paper is set up as follows: Section 2 present new technique. Section 3 present convergence analysis. Section 4 presents findings of the numerical experiments. Section 5 presents result discussions and conclusion.

**Preliminaries**

**Definition 1:** Let $s \in R$, $\{x_n\} \in R$, $n = 0,1,2\ldots$ then the sequence $\{x_n\}$ is said to converge to $s$ if $\lim_{n\to\infty}|x_n - s| = 0$.

**Definition 2:** Approximated computational local order of convergence of sequence $\{x_n\}$

$$\widetilde{\lambda_n} = \frac{\log|\widetilde{e_n}|}{\log|\tilde{e}_{n-1}|} \text{ where } \tilde{e}_n = |x_n - x_{n-1}|$$

**Definition 3:** The notation $e_n = x_n - \alpha$ is error in $n$th iteration. The equation $e_{n+1} = ce_n{}^r + 0(e_n{}^{r+1})$ is known as error equation. The value of $r$ obtained is order of convergence of the technique.

## 2. Methodology

Well known quadrature rules of integration which provides the base for the development of new proposed method for solving nonlinear equations. In this part attention is devoted to the trapezoidal rule to derive new iterative technique.

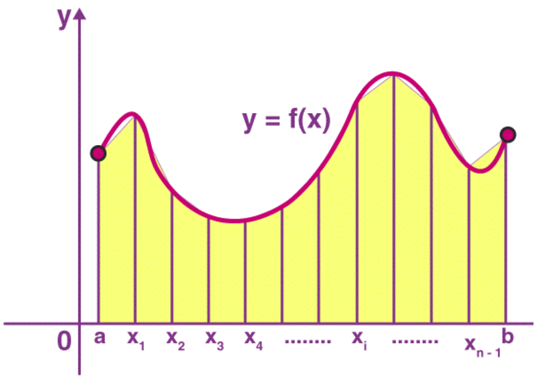
Consider the noninear equation

$$f(x) = 0 \tag{1}$$

The Fundamental Theorem of Calculus states that if f is any antiderivative of F on $[a, b]$ i.e. $f' = F$ and $F$ is continuous at every point of $[a,b]$, then

$$\int_a^b f'(t)\,dt = f(b) - f(a) \tag{2}$$

Assume that a continuous function $y = f(x)$ defined on $[a, b]$. The interval $[a, b]$ divided into $n$ equal sub – intervals each of width $h$ s.t. $a = x_0 < x_1 < x_2 < \cdots < x_n = x = b$.




Apply the Trapezoidal rule in the strip $x_0$ to $x_1$, we get

$$\int_{x_0}^{x_1} f'(t)\,dt = \frac{x_1 - x_0}{2}\left[f'(x_0) + f'(x_1)\right] \tag{3}$$

Apply, the Fundamental theorem of calculus on equation (3), we get

$$f(x_1) - f(x_0) = \frac{x_1 - x_0}{2}\left[f'(x_0) + f'(x_1)\right]$$

$$x_1 - x_0 = \frac{2\,[f(x_1) - f(x_0)]}{[f'(x_0) + f'(x_1)]}$$

$$x_1 = x_0 + \frac{2\,[f(x_1) - f(x_0)]}{[f'(x_0) + f'(x_1)]} \tag{4}$$

Similarly, apply the Trapezoidal rule in the strip $x_1$ to $x_2$, we get

$$\int_{x_1}^{x_2} f'(t)\,dt = \frac{x_2 - x_1}{2}\left[f'(x_1) + f'(x_2)\right] \tag{5}$$

Apply, the Fundamental theorem of calculus on equation (5), we get

$$f(x_2) - f(x_1) = \frac{x_2 - x_1}{2}\left[f'(x_1) + f'(x_2)\right]$$

$$x_2 - x_1 = \frac{2[f(x_2) - f(x_1)]}{[f'(x_1) + f'(x_2)]}$$

$$x_2 = x_1 + \frac{2\,[f(x_2) - f(x_1)]}{[f'(x_1) + f'(x_2)]} \tag{6}$$

Continuing and so on, apply the Trapezoidal rule in the strip $x_{n-1}$ to $x$, we get

$$\int_{x_{n-1}}^{x} f'(t)\,dt = \frac{x-x_{n-1}}{2}\left[f'(x_{n-1}) + f'(x)\right] \tag{7}$$

Apply, the Fundamental theorem of calculus on equation (7), we get

$$f(x) - f(x_{n-1}) = \frac{x-x_{n-1}}{2}\left[f'(x_{n-1}) + f'(x)\right]$$

$$x - x_{n-1} = \frac{2[f(x) - f(x_{n-1})]}{[f'(x_{n-1}) + f'(x)]}$$

$$x = x_{n-1} + \frac{2[f(x) - f(x_{n-1})]}{[f'(x_{n-1}) + f'(x)]} \tag{8}$$

$$x = x_{n-1} + \frac{-2f(x_{n-1})}{[f'(x_{n-1}) + f'(x)]} \tag{9}$$

$[\because f(x) = 0]$

$$x = x_n + \frac{-2\,f(x_n)]}{[f'(x_n) + f'(x)]}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(x_n) + f'(x_{n+1})]} \tag{10}$$

$$x_{n+1} = x_n - \frac{2\,f(x_n)}{f'(x_n) + f'(y_n)} \quad n = 0,1,2 \tag{11}$$

where we consider

$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ on the right-hand side to remove implicitness, and thus

Hence equation (11) is the required proposed technique.

## 3. Convergence Analysis of the Proposed Method

**Theorem 1:** *For an open interval I, let $\alpha \in I$ be the simple zero of a suitably differentiable function $f : R \to R$. If $x_0$ is sufficiently near to $\alpha$ then the iterative procedure provided by (11) is of order 3 and accomplishes the following error equation*

$$e_{n+1} = \left(c_2{}^2 + \frac{1}{2}c_3\right) e_n{}^3 + o(e_n{}^4)$$

***Proof:*** Let $\alpha$ be simple zero of $f$, and $e_n = x_n - \alpha$. Using Taylor expansion around $x=\alpha$, taking into account $f(\alpha)=0$, we get,

$$f(x_n) = f(e_n + \alpha)$$

$$= f(\alpha) + e_n f'(\alpha) + \frac{e_n{}^2}{2!} f''(\alpha) + \frac{e_n{}^3}{3!} f'''(\alpha) + \frac{e_n{}^4}{4!} f''''(\alpha) + \ldots$$

$$f(x_n) = f(e_n + \alpha) = e_n f'(\alpha) + \frac{e_n{}^2}{2!} f''(\alpha) + \frac{e_n{}^3}{3!} f'''(\alpha) + \frac{e_n{}^4}{4!} f''''(\alpha) + \ldots$$

$$f(x_n) = f(e_n + \alpha) = f'(\alpha)\left[ e_n + \frac{e_n{}^2}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n{}^3}{3!}\frac{f'''(\alpha)}{f'(\alpha)} + \frac{e_n{}^4}{4!}\frac{f''''(\alpha)}{f'(\alpha)} + \ldots \right]$$

$$f(x_n) = f(e_n + \alpha) = f'(\alpha)\left[ e_n + c_2 e_n{}^2 + c_3 e_n{}^3 + c_4 e_n{}^4 + \ldots \right]$$

Where $c_k = \frac{f^k(\alpha)}{k!f'(\alpha)} k = 2, 3, 4 \ldots$

$$2f(x_n) = 2f\left(e_n + \alpha\right) = 2f'(\alpha)\,[\, e_n + c_2 e_n{}^2 + c_3 e_n{}^3 + c_4 e_n{}^4 + \ldots\,] \tag{12}$$

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2 e_n + 3c_3 e_n{}^2 + 4c_4 e_n{}^3 + \ldots\right] \tag{13}$$

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n{}^2 + 2(c_2{}^2 - c_3)e_n{}^3 + \left(7c_2 c_3 - 4c_2{}^3 - 3c_4\right)e_n{}^4 + \ldots$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = e_n + \alpha - [e_n - c_2 e_n{}^2 + 2(c_2{}^2 - c_3)e_n{}^3 + (7c_2 c_3 - 4c_2{}^3 - 3c_4)e_n{}^4 + \ldots]$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n{}^2 - 2(c_2{}^2 - c_3)e_n{}^3 - \left(7c_2 c_3 - 4c_2{}^3 - 3c_4\right)e_n{}^4 + \ldots$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n{}^2 - 2(c_2{}^2 - c_3)e_n{}^3 - \left(7c_2 c_3 - 4c_2{}^3 - 3c_4\right)e_n{}^4 + \ldots$$

$$f(y_n) = f(\alpha + c_2 e_n{}^2 - 2(c_2{}^2 - c_3)e_n{}^3 - \left(7c_2 c_3 - 4c_2{}^3 - 3c_4\right)e_n{}^4 + \ldots)$$

$$f(y_n) = f(\alpha) + f'(\alpha)\left[\, c_2 e_n{}^2 - 2(c_2{}^2 - c_3)e_n{}^3 - \left(7c_2 c_3 - 4c_2{}^3 - 3c_4\right)e_n{}^4 + \ldots\right] + \frac{f''(\alpha)}{2!}\,[c_2 e_n{}^2 - 2(c_2{}^2 - c_3)e_n{}^3 - \left(7c_2 c_3 - 4c_2{}^3 - 3c_4\right)e_n{}^4 + \ldots]^2 + \ldots$$

$$f'(y_n) = f'(\alpha)[1 + 2c_2{}^2 e_n{}^2 + 4(c_2 c_3 - c_2{}^3)e_n{}^3 + (-11c_2{}^3 c_3 + 6c_2 c_4 + 8c_2{}^4)e_n{}^4 + \ldots] \tag{14}$$

Adding equation (13) and (14), we get

$$f'(x_n) + f'(y_n) = 2f'(\alpha)[1 + c_2 e_n + c_2{}^2 e_n{}^2 + \tfrac{3}{2}c_3 e_n{}^2 + 2c_4 e_n{}^3 + 2c_2 c_3 e_n{}^3 - 2c_2{}^3 e_n{}^3 + \ldots]\ldots \tag{15}$$

From equation (11), we get

$$x_{n+1} = x_n - \frac{2f'(\alpha)\,[\, e_n + c_2 e_n{}^2 + c_3 e_n{}^3 + c_4 e_n{}^4 + \ldots\,]}{2f'(\alpha)[1 + c_2 e_n + c_2{}^2 e_n{}^2 + \frac{3}{2}c_3 e_n{}^2 + 2c_4 e_n{}^3 + 2c_2 c_3 e_n{}^3 - 2c_2{}^3 e_n{}^3 + \ldots]}$$

$$x_{n+1} = x_n - \frac{[\, e_n + c_2 e_n{}^2 + c_3 e_n{}^3 + c_4 e_n{}^4 + \ldots\,]}{[1 + c_2 e_n + c_2{}^2 e_n{}^2 + \frac{3}{2}c_3 e_n{}^2 + 2c_4 e_n{}^3 + 2c_2 c_3 e_n{}^3 - 2c_2{}^3 e_n{}^3 + \ldots]}$$

$$x_{n+1} = x_n - [\, e_n + c_2 e_n{}^2 + c_3 e_n{}^3 + c_4 e_n{}^4 + \ldots\,]\,[1 + c_2 e_n + c_2{}^2 e_n{}^2 + \tfrac{3}{2}c_3 e_n{}^2 + 2c_4 e_n{}^3 + 2c_2 c_3 e_n{}^3 - 2c_2{}^3 e_n{}^3 + \ldots]^{-1}$$

$$e_{n+1} + \alpha = e_n + \alpha - [\, e_n - c_2 e_n{}^2 - \tfrac{3}{2}c_3 e_n{}^3 + c_2 e_n{}^2 - c_2{}^2 e_n{}^3 + c_3 e_n{}^3 + o(e_n{}^4)\ldots]$$

$$e_{n+1} = \left(c_2{}^2 + \tfrac{1}{2}c_3\right)e_n{}^3 + o(e_n{}^4)$$

Hence, order of convergence of the proposed technique is 3.

## 4. Numerical Examples

The proposed technique is tested using some benchmark problems in this section. Both the number of iterations and approximated computational local order of convergence are compared between proposed technique and the Newton Raphson technique. Table 1 displays details of the chosen benchmark problems as well as the outcomes of executing the iterative approaches' algorithms.

**Table 1**

**Comparison between the newly created approach and the Newton approach in terms of iterations and approximated computational local order of convergence.**

| Function | Initial point | Root | Number of iterations | | Approximated computational local order of convergence | |
|---|---|---|---|---|---|---|
| | | | Newton Raphson technique | Proposed technique | Newton Raphson technique | Proposed technique |
| $x^4 - x - 10$ | 2 | 1.85558452864 | 7 | 5 | 2.02011916226 | 3.02624941129 |
| $\cos x - 4x + 2$ | 0.5 | 0.74997858312 | 6 | 5 | 1.29470999026 | 1.49433500725 |
| $x^3 + 3x + 1$ | 3 | −0.3221853546 | 9 | 5 | 2.1594690545 | 3.30011308942 |
| $x^2 - e^x - 3x + 2$ | 2 | 0.25753028544 | 6 | 4 | 2.46895140033 | 3.77407258373 |

## 5. Discussion and Conclusion

Numerical integration techniques have demonstrated encouraging outcomes in terms of increased accuracy and efficiency since the trapezoidal rule has been converted into an iterative method. This method provides a more accurate approximation of the integral than a single application of trapezoidal technique by iteratively improving estimation of the integral. The iterative technique has been found to converge quickly through analysis and testing, especially when used to functions that exhibit smooth and well-behaved behavior.

Finally, this study extends to our knowledge of iterative conversion methods for numerical integration and helps practitioners choose the best approach based on the properties of the integrand and the required precision level. In order to improve integration efficiency and resilience even further, future research directions can include expanding the analysis to higher-dimensional integration issues and investigating hybrid approaches that combine the best features of several iterative strategies. Comparing the performance of the Newton Raphson and modified method from the table 1, we observe that newly developed technique performing very well in terms of number of iteration and approximated computational local order of convergence.

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## References

[1] S. Abbasbandy, "Homotopy analysis method for the Kawahara-equations," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 307–312 (2010).

[2] M. W. Alomari and S. S. Dragomir, "Various error estimations for several Newton-Cotes Quadrature formulae in terms of at most first derivative and applications in numerical integration," *Jordan Journal of Mathematics and Statistics*, vol. 7, no. 2, pp. 89–108 (2014).

[3] G. Amos and V. Subramanian, Numerical Methods for Engineers and Scientists, 3rd ed., Wiley (2013), ISBN-10: 1118554930, ISBN-13: 978-1118554937.

[4] C. O. E. Burg, "Derivative based closed Newton Cotes numerical quadrature," *Applied Mathematics and Computation*, vol. 218, no. 13, pp. 7052–7065 (2012).

[5] C. S. Chapra, Applied Mathematical Methods with MATLAB for Engineers and Scientists, 3rd ed., McGraw-Hill (2006), ISBN 978-0-07-340110-2, MHID 0-07-340110-2.

[6] M. K. Jain, S. R. K. Iyengar, and R. K. Jain, Numerical Methods, 2nd ed., New Age International Publisher (2004), ISBN: 9788122424263.

[7] A. Kharab and R. B. Guenther, An Introduction to Numerical Methods – A MATLAB Approach, CRC Press, Boca Raton, FL (2001).

[8] E. Kreyszig, *Advanced Engineering Mathematics*, 9th ed., Wiley (2006), ISBN-13: 978-0-471-72897-9.

[9] D. Levy, Introduction to Numerical Analysis, [Online]. Available: https://www.scribd.com/document/331408400/Introduction-to-Numerical-Analysis (2010).

[10] W. Malfliet, "The Tanh method: A tool for solving certain classes of nonlinear evaluations and wave equations," *Computational and Applied Mathematics*, vol. 164-165, pp. 529–541 (2004).

[11] J. H. Mathews and K. F. Kurtis, Numerical Methods Using MATLAB, 4th ed., New Jersey, USA (2004), ISBN: 0-13-065248-2.

[12] S. Mechee and J. K. Mshachal, "Derivation of embedded explicit RK type methods for directly solving class of seventh order ordinary differential equations," *Journal of Interdisciplinary Mathematics,* vol. 22, no. 8, pp. 1451–1456 (2019), DOI: 10.1080/09720502.2019.1700936.

[13] N. E. Tatar, "(L) type activation functions in a system of ordinary differential equations," *Journal of Information and Optimization Sciences*, vol. 39, no. 8, pp. 1637–1645 (2018), DOI: 10.1080/02522667.2017.1383661.

[14] Q. I. Rahman and G. Schmeisser, "Characterization of the speed of convergence of the trapezoidal rule," *Numerische Mathematik,* vol. 57, no. 1, pp. 123–138 (1990).

[15] V. Dukkipati Rao, Numerical Methods, New Age International Publisher (2010), ISBN 10: 8122428134, ISBN 13: 9788122428131.

[16] S. S. Sastry, Introductory Methods of Numerical Analysis, 5th ed., PHI (2012), ISBN 10: 8120345924, ISBN 13: 9788120345928.

[17] A. M. Wazwaz, "New compact and noncompact solutions for two variants of a modified Camassa-Holm equation," *Applied Mathematics and Computation*, vol. 163, pp. 1165–1179 (2005).

[18] A. M. Wazwaz, "A class of nonlinear fourth order variant of a generalized Camassa-Holm equation with compact and noncompact solutions," *Applied Mathematics and Computation*, vol. 165, no. 2, pp. 485–501 (2005).

[19] J. A. C. Weidman, "Numerical integration of periodic functions: A few examples," *The American Mathematical Monthly,* vol. 109, no. 1, pp. 21–36 (2002).

[20] Y. L. Chou, "Generalization and new proof for almost everywhere convergence to imply local convergence in measure," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 8, pp. 2195–2199 (2021).

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