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Integral-type contractions and weakly compatible mappings in 2-metric spaces

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Abstract

This research explores f.p theorems by utilizing weakly compatible mappings in 2-m.s, focusing on contractive conditions of the integral type. By broadening the notion of compatibility, the study demonstrates the unique common fixed point for four self-mappings that meet certain rational inequality constraints.

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Keywords: *Fixed point (f. p), 2-metric space (2-m.s), Property (E.A).*

1. Introduction

The Banach Contraction Principle, established by Stefan Banach in 1922 [4], is a basic conclusion in f.p theory. It provides a strong framework for showing the existence and uniqueness of fixed points under specific conditions. This principle has been instrumental in solving various mathematical and engineering problems. Over the years, researchers have extended and generalized this theorem through various approaches, leading to significant advancements in the field [2], [3], [5], [7], [11], [12].

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In 1976, Jungck [8] proposed the concept of commute mappings, which served as a basis for studying common fixed points in m.s. [9] Theorems related to common fixed points typically require conditions such as commutativity, continuity, contractiveness, the completeness of the space, and specific containment properties of the mappings. These results are crucial for analyzing the behavior and interactions of complex mappings.

Gähler [6] initiated the concept of 2-m.s as a generalization of traditional m.s. 2-m.s have gained attention because of their utility in advanced mathematical studies.

The property (E.A), initially proposed by Aamri and Moutawakil in 2002 [1], has further enriched the f.p theory. Many researchers have used this property to establish new f.p results, extending its applicability to various mappings [13], [14], [10]. This research expands upon these fundamental concepts by exploring weakly compatible mappings in 2-m.s under integral-type contractive conditions, to offer new perspectives and significant contributions to the field.

Definition 1.1: Let Ξ be a non-empty set and $\rho: \Xi \times \Xi \times \Xi \rightarrow \mathbb{R}$ is a function fulfilling the following properties:

(i) Non-negativity: For any two distinct element $\alpha, \pi \in \Xi$, there exists some element τ of Ξ such that $\rho(\alpha, \pi, \tau) \neq 0$.

(ii) The value of $\rho(\alpha, y, \tau)$ is zero whenever at least two of α, π and τ are the same.

(iii) Symmetry: The function ρ remains unchanged under any permutation of its three arguments,

i.e.,$\rho(\alpha, \pi, \tau) = \rho(\alpha, \tau, \pi) = \rho(\pi, \tau, \alpha)$ for all α, π, τ in Ξ.

(iv) Rectangular Inequality: For any four elements α, π, τ, w in Ξ, the inequality

$$\rho(\alpha, \pi, \tau) \leq \rho(\alpha, \pi, w) + \rho(\alpha, w, \tau) + \rho(w, y, \tau)$$

hold.

If these conditions are satisfied, then the function ρ is a 2-metric on Ξ and the pair (Ξ, ρ) is mentioned as a 2-m.s.

Definition 1.2: Let (Ξ, ρ) be a 2-m.s. A subset $Y \subseteq \Xi$ is defined as closed if, whenever a sequence $\{\alpha_n\}$ in Y converges to some element α, it follows that α belongs to Y.

Definition 1.3: Let $\mathfrak{K}$ and $\mathfrak{L}$ be self-mappings on a set Ξ. The pair $(\mathfrak{K}, \mathfrak{L})$ is said to exhibit property (E.A) if there exists a sequence $\{\alpha_n\}$ such that

$$\lim_{n\to\infty} \mathfrak{L}(\alpha_n) = \lim_{n\to\infty} \mathfrak{K}(\alpha_n) = t,$$

for some point $t \in \Xi$.

Definition 1.4: Let $\mathfrak{K}$ and $\mathfrak{L}$ be self-mappings in the set Ξ. The pair $(\mathfrak{K},\mathfrak{L})$ is termed weakly compatible if they commute at their points of coincidence, that is, $\mathfrak{K}\mathfrak{L}\alpha = \mathfrak{L}\mathfrak{K}\alpha$ whenever $\mathfrak{K}\alpha = \mathfrak{L}\alpha$.

In this work, $\xi: \mathbb{R}^+ \to \mathbb{R}^+$ represents a non-negative Lebesgue integrable and summable function, which satisfies $\int_0^{\varepsilon} \xi(v)dv > 0$ for every $\varepsilon > 0$. Additionally, $\eta: [0,\infty) \to [0,\infty)$ is a right-continuous function such that $\eta(0) = 0$ and $\eta(t) < t$ for all $t > 0$.

Theorem 1.5: *Let Ξ be a 2-m.s and let $\mathfrak{K},\mathfrak{L},\mathfrak{S}$ and $\mathfrak{T}$ be four self-mappings defined on Ξ that satisfy the following condition.*

1. *Integral Contractive Condition:*
$$\int_0^{\rho(\mathfrak{K}\alpha,\mathfrak{L}\pi,\tau)} \varsigma(v)dv = \eta\left(\int_0^{\mu(\alpha,\pi,\tau)} \varsigma(v)dv\right) \tag{1.1}$$
for all $\alpha,\pi,\tau \in \Xi$, where
$$\mu(\alpha,\pi,\tau) = max\{\rho(\mathfrak{T}\alpha,\mathfrak{S}\pi,\tau), \rho(\mathfrak{T}\alpha,\mathfrak{K}\alpha,\tau), \rho(\mathfrak{S}\pi,\mathfrak{L}\pi,\tau),$$
$$\frac{1}{2}\rho(\mathfrak{T}\alpha,\mathfrak{L}\pi,\tau) + \rho(\mathfrak{S}\pi,\mathfrak{K}\alpha,\tau)$$
2. *Mapping Containment and Closedness:*
$$\mathfrak{K}(\Xi) \subseteq \mathfrak{S}(\Xi), \mathfrak{L}\Xi) \subseteq \mathfrak{T}(\Xi)$$
and either $\mathfrak{T}(\Xi)$ or $\mathfrak{S}(\Xi)$ is closed.
3. *Property (E.A): Either pair $(\mathfrak{K},\mathfrak{T})$ or $(\mathfrak{L},\mathfrak{S})$ satisfy the property (E.A).*
4. *Weakly compatibility: The pairings $(\mathfrak{K},\mathfrak{T})$ and $(\mathfrak{L},\mathfrak{S})$ are weakly compatible.*

Under these conditions, $\mathfrak{K},\mathfrak{L},\mathfrak{S},\mathfrak{T}$ possess a unique common f.p in Ξ.

Proof: Suppose the pair $(\mathfrak{K},\mathfrak{T})$ satisfies property (E.A). Then there exists a sequence $\{\alpha_n\}$ in Ξ such that
$$\lim_{n\to\infty} \mathfrak{K}(\alpha_n) = \lim_{n\to\infty} \mathfrak{T}(\alpha_n) = \beta,$$
where $\beta \in \Xi$.

This implies that both mappings $\mathfrak{K}$ and $\mathfrak{T}$ converge to the same limit point β as n approaches infinity. Suppose that $\mathfrak{K}(\Xi) \subseteq \mathfrak{S}(\Xi)$

Therefore $\mathfrak{K}\alpha_n = \mathfrak{S}\pi_n$ for each n, for some sequence $\{\pi_n\}$ in Ξ. Hence
$$\lim_{n\to\infty} \mathfrak{S}\pi_n = \beta$$

Next we prove that $\lim_{n\to\infty}\mathfrak{L}\pi_n = \beta$

We put $\alpha = \alpha_n, \pi = \pi_n$ in (1.1)
$$\int_0^{\rho(\mathfrak{K}\alpha_n,\mathfrak{L}\pi_n,\tau)} \varsigma(v)dv \le \eta\left(\int_0^{\mu(\alpha_n,\pi_n,\tau)} \varsigma(v)dv\right) \tag{1.2}$$
for all $n \in \mathfrak{L}$. where
$$\mu(\alpha_n,\pi_n,\tau) = max\left\{\begin{matrix} \varrho(\mathfrak{T}\alpha_n,\mathfrak{S}\pi_n,\tau), \varrho(\mathfrak{T}\alpha_n,\mathfrak{K}\alpha_n,\tau), \\ \varrho(\mathfrak{S}\pi_\mathrm{n},\mathfrak{L}\pi_n,\tau), \\ \frac{1}{2}\varrho(\mathfrak{T}\alpha_n,\mathfrak{L}\pi_n,\tau) + \varrho(\mathfrak{S}\pi_n,\alpha_n,\tau) \end{matrix}\right\}$$

On letting $n \to \infty$

$$\lim_{n\to\infty} \mu(\alpha_n, \pi_n, \tau) = \lim_{n\to\infty} max \begin{Bmatrix} 0,0, \varrho(\beta, \mathfrak{L}\pi_n, \tau), \\ \frac{1}{2}\varrho(\beta, \mathfrak{L}\pi_n, \tau) \end{Bmatrix}$$

$$= \lim_{n\to\infty} \rho(\beta, \mathfrak{L}\pi_n, \tau)$$

$$\lim_{n\to\infty} \int_0^{\mu(\beta,\mathfrak{L}\pi_n,\tau)} \varsigma(v)dv \le \lim_{n\to\infty} \eta\left(\int_0^{\varrho(\beta,\mathfrak{L}\pi_n,\tau)} \varsigma(v)dv\right)$$

$$< \lim_{n\to\infty} \left(\int_0^{\varrho(\beta,\mathfrak{L}\pi_n,\tau)} \varsigma(v)dv\right)$$

If $\lim_{n\to\infty} \varrho(\beta, \mathfrak{L}\pi_n, \tau) \neq 0$. which is a contradiction.

Therefore $\lim_{n\to\infty} \varrho(\beta, \mathfrak{L}\pi_n, \tau) = 0$

$$\lim_{n\to\infty} \mathfrak{L}\pi_n = \beta$$

Suppose $\mathfrak{T}(\Xi)$ is closed. We can find a point $\pi \in \Xi$ such that

$$\beta = \mathfrak{T}\pi$$

To prove $\mathfrak{K}\pi = \beta$, we put $\alpha = \pi, \pi = \pi_n$ in (1.1)

$$\int_0^{\varrho(\mathfrak{K}\pi,\mathfrak{L}\pi_n,\tau)} \varsigma(v)dv \le \eta\left(\int_0^{\mu(\pi,\pi_n,\tau)} \varsigma(v)dv\right) \tag{1.3}$$

for all $n \in \mathfrak{L}$.

$$\mu(\pi, \pi_n, \tau) = max\{\varrho(\mathfrak{T}\pi, \mathfrak{S}\pi_n, \tau), \varrho(\mathfrak{T}\pi, \mathfrak{K}\pi, \tau), \varrho(\mathfrak{S}\pi_n, \mathfrak{L}\pi_n, \tau), \frac{1}{2}[\varrho(\mathfrak{T}\pi, \mathfrak{L}\pi_n, \tau) + \varrho(\mathfrak{S}\pi_n, \mathfrak{K}\pi, \tau)]\}$$

On letting $n \to \infty$

$$\lim_{n\to\infty} \mu(\pi, \pi_n, \tau) = \max\{\varrho(\beta, \beta, \tau), \varrho(r, \mathfrak{K}\pi, \tau), \varrho(r, r, \tau) + \frac{1}{2}[\varrho(\beta, \beta, \tau) + \varrho(\beta, \mathfrak{K}\pi, \tau)]\}$$

$$= \varrho(\beta, \mathfrak{K}\pi, \tau)$$

$$\int_0^{\varrho(\mathfrak{K}\pi,\mathfrak{L}\pi_n,\tau)} \varsigma(v)dv = \lim_{n\to\infty} \int_0^{\varrho(\mathfrak{K}\pi,\mathfrak{L}\pi_n,\tau)}$$

$$\le \eta\left(\int_0^{\varrho(\mathfrak{K}\pi,\beta,\tau)} \varsigma(v)dv\right)$$

$$< \int_0^{\varrho\varrho(\beta,\mathfrak{K}\pi,\tau)} \varsigma(v)dv \text{ if } \rho\rho(\beta, \mathfrak{K}\pi, \tau) \neq 0$$

This results in a contradiction.

Hence, $\varrho(\beta, \mathfrak{K}\pi, \tau) = 0$ which implies that $\mathfrak{K}\pi = \beta$.

Since $\beta \in \mathfrak{K}(\Xi) \subseteq \mathfrak{S}(\Xi)$, we have $\beta = \mathfrak{S}\kappa$ for some element κ in the set Ξ.

To prove $\mathfrak{L}\kappa = \beta$, Put $\alpha = \pi, \pi = \kappa$ in (1.1)

$$\int_0^{\varrho(\mathfrak{K}\pi,\mathfrak{L}\kappa,\tau)} \varsigma(v)dv \le \eta\left(\int_0^{\mu(\pi,\kappa,\tau)} \varsigma(v)dv\right) \tag{1.4}$$

where,

$$\mu(\pi, \kappa, \tau) = \max\{\varrho(\mathfrak{T}\pi, \mathfrak{S}\kappa, \tau), \varrho(\mathfrak{T}\pi, \mathfrak{K}\pi, \tau), \varrho(\mathfrak{S}\kappa, \mathfrak{L}\kappa, \tau), \frac{1}{2}\varrho(\mathfrak{T}\pi, \mathfrak{L}\kappa, \tau) + \varrho(\mathfrak{S}\kappa, \mathfrak{K}\pi, \tau)\}$$

$$\mu(\pi,\kappa,\tau) = \varrho(\beta,\mathfrak{L}\kappa,\tau)$$
$$\int_0^{\varrho(\beta,\mathfrak{L}\kappa,\tau)} \varsigma(v)dv \le \eta\left(\int_0^{\varrho(\beta,\mathfrak{L}\kappa,\tau)} \varsigma(v)dv\right)$$
$$< \int_0^{\varrho(\beta,\mathfrak{L}\kappa,\tau)} \varsigma(v)dv \text{ if } \varrho(\beta,\mathfrak{L}\kappa,\tau) \ne 0$$

Which would lead to a contradiction.

Hence $\varrho(\beta,\mathfrak{L}\kappa,\tau) = 0$ and $\mathfrak{L}\kappa = \beta$.

Given the weak compatibility of $(\mathfrak{K},\mathfrak{T})$ and $(\mathfrak{L},\mathfrak{S})$, it follows that $\mathfrak{T}\mathfrak{K}\pi = \mathfrak{K}\mathfrak{T}\pi$ and $\mathfrak{S}\mathfrak{L}\kappa = \mathfrak{L}\mathfrak{S}\kappa$.

This leads to $\mathfrak{T}\beta = \mathfrak{K}\beta, \mathfrak{S}\beta = \mathfrak{L}\beta$.

Next, We establish $\mathfrak{L}\beta = \beta$, put $\alpha = \pi, \pi = \beta$ in (1.1)

$$\int_0^{\varrho(\mathfrak{K}\pi,\mathfrak{L}\beta,\tau)} \varsigma(v)dv \le \eta\left(\int_0^{\mu(\pi,\beta,\tau)} \varsigma(v)dv\right) \tag{1.5}$$

where,

$$\mu(\pi,\beta,\tau) = \{max\{\varrho(\mathfrak{T}\pi,\mathfrak{S}\beta,\tau), \varrho(\mathfrak{T}\pi,\mathfrak{K}\pi,\tau), \varrho(\mathfrak{S}\beta,\mathfrak{L}\beta,\tau),$$
$$\frac{1}{2}[\varrho(\mathfrak{T}\pi,\mathfrak{L}\beta,\tau) + \varrho(\mathfrak{S}\beta,\mathfrak{K}\pi,\tau)]\}$$
$$\mu(u,\beta,\tau) = \varrho(\beta,\mathfrak{L}\beta,\tau)$$
$$\int_0^{\varrho(\pi,\mathfrak{L}\beta,\tau)} \varsigma(v)dv \le \eta\left(\int_0^{\varrho(\beta,\mathfrak{L}\beta,\tau)} \varsigma(v)dv\right)$$
$$< \int_0^{\varrho(\beta,\mathfrak{L}\beta,\tau)} \varsigma(v)dv \text{ if } \varrho(\beta,\mathfrak{L}\beta,\tau) \ne 0$$

Which is a contradiction.

Therefore $\varrho(\beta,\mathfrak{L}\beta,\tau) = 0$ which implies $\mathfrak{L}\beta = \beta$.

Similarly we prove $\mathfrak{K}\beta = \mathfrak{T}\beta = \beta$.

Therefore $\mathfrak{K}\beta = \mathfrak{T}\beta = \mathfrak{S}\beta = \mathfrak{L}\beta = \beta$.

Hence β is common f.p of $\mathfrak{T}, \mathfrak{K}, \mathfrak{S}$ and $\mathfrak{L}$.

Uniqueness: Let's assume that $\beta^* \ne \beta$ is a common f.p of $\mathfrak{T}, \mathfrak{K}, \mathfrak{S}$ and $\mathfrak{L}$. Then $\mathfrak{K}\beta^* = \mathfrak{T}\beta^* = \mathfrak{S}\beta^* = \mathfrak{L}\beta^* = \beta^*$

$$\int_0^{\varrho(\beta,\beta^*,\tau)} \varsigma(v)dv = \int_0^{\varrho(\mathfrak{K}\beta,\mathfrak{K}\beta^*,\tau)} \varsigma(v)dv \le \eta\left(\int_0^{\mu(\beta,\beta^*,\tau)} \varsigma(v)dv\right)$$
$$\mu(\beta,\beta^*,\tau) = max\begin{Bmatrix} \varrho(\mathfrak{T}\beta,\mathfrak{S}\beta^*,\tau), d(\mathfrak{T}\beta,\mathfrak{K}\beta,\tau), \\ \varrho(\mathfrak{S}\beta^*,\mathfrak{L}\beta^*,\tau), \\ \frac{1}{2}[\varrho(\mathfrak{T}\beta,\mathfrak{L}\beta^*,\tau) + \varrho(\mathfrak{S}\beta^*,\mathfrak{K}\beta,\tau)] \end{Bmatrix}$$
$$\mu(\beta,\beta^*,\tau) = \varrho(\beta,\beta^*,\tau)$$
$$\int_0^{\varrho(\beta,\beta^*,\tau)} \varsigma(v)dv \le \eta\left(\int_0^{\varrho(\beta,\beta^*,\tau)} \varsigma(v)dv\right) < \int_0^{\varrho(\beta,\beta^*,\tau)} \varsigma(v)dv$$

Which is a contradiction.

Therefore, $\beta = \beta^*$.

Likewise, the proof stems from the (E.A) property of $(\mathfrak{L},\mathfrak{S})$.

Corollary 1: *Let Ξ be a 2-metric space, and let $\mathfrak{K}, \mathfrak{L}, \mathfrak{S}, \mathfrak{T}$ be four self-mappings defined on Ξ, satisfying the following condition:*

$$\int_0^{\varrho(\mathfrak{K}\alpha,\mathfrak{L}\pi,\tau)} \varsigma(v)dv = \eta\left(\int_0^{\mu(\alpha,\pi,\tau)} \varsigma(v)dv\right)$$

for all $\alpha, \pi, \tau \in \Xi$, *where*

$$\mu(\alpha, \pi, \tau) = \max\{\varrho(\mathfrak{T}\alpha, \mathfrak{S}\pi, \tau), \varrho(\mathfrak{T}\alpha, \mathfrak{K}\alpha, \tau), \varrho(\mathfrak{S}\pi, \mathfrak{L}\pi, \tau)\}.$$

The following conditions are also satisfied:

1. $\mathfrak{K}(\Xi) \subseteq \mathfrak{S}(\Xi)$, $\mathfrak{L}(\Xi) \subseteq \mathfrak{T}(\Xi)$, and $\mathfrak{T}(\Xi)$ or $\mathfrak{S}(\Xi)$ are closed.
2. Either the pair $(\mathfrak{K}, \mathfrak{T})$ or $(\mathfrak{L}, \mathfrak{S})$ fulfills Property (E.A).
3. The pairs $(\mathfrak{K}, \mathfrak{T})$ and $(\mathfrak{L}, \mathfrak{S})$ exhibit weak compatibility.

As a result, the mappings $\mathfrak{K}, \mathfrak{L}, \mathfrak{S}, \mathfrak{T}$ have f.p in Ξ.

2. Conclusion

In this work, a new f.p result is established with the help of four self-mappings in the spaces 2-metric by applying integral-type contractive conditions. The study highlights the significance of the (E.A.) property and weak compatibility in ensuring the existence and uniqueness of a common f.p. These results not only generalize existing theorems in f.p theory but also extend their applicability to broader mathematical models, including systems of integral equations.

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