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Geometric progression of length four on the circle

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Abstract

A sequence of rational points (P_1, P_2, P_3,...) on a circle is said to constitute an r-geometric progression if either the sequence of their x-coordinates {a_1, a_2, a_3,...} or their y-coordinates {b_1, b_2, b_3,...} forms a geometric progression with common ratio r. This article examines a specific construction of such a progression, focusing on a sequence of four points located on a circle whose center lies along the x-axis. As a consequence, it is deduced that a certain family of unit circles whose centers lie on the x-axis cannot contain a rational sequence of length four in geometric progression of the form {1, r, r^2, r^3}.

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1. Introduction

A sequence of rational points $(P_1, P_2, P_3, ...)$, where $P_i = (a_i, b_i)$, lying on an algebraic curve in the plane, is said to be an r-arithmetic progression or an r-geometric progression if either the sequence of x-coordinates $\{a_1, a_2, a_3, ...\}$ or the sequence of y-coordinates $\{b_1, b_2, b_3, ...\}$ forms an arithmetic progression or a geometric progression with common difference or common ratio r, respectively. The exploration of arithmetic progressions and other special sequences arise in number theory through various approaches. For instance, Shukur et al. [18] investigated arithmetic progressions in connection with sums of powers. The study of arithmetic progressions on different models of elliptic curves has attracted the attention of numerous mathematicians [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Choudhry and Juyal [15] explored sequences of three

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rational points in arithmetic progression on the unit circle. Subsequently, Ciss and Moody [16] extended this line of inquiry to longer arithmetic progressions on general conic sections. More recently, Çelik, Sadek, and Soydan [17] established the existence of sequences of rational points in geometric progression on the unit circle, with a minimum length of three. Specifically, they demonstrated that the sequence of x-coordinates $\left(a_1 = \frac{5}{13}, a_2 = \frac{3}{5}, a_3 = \frac{117}{125}\right)$ forms a geometric progression on the unit circle defined by the equation $x^2 + y^2 = 1$. The problem of finding a long sequence of rational points on various curves in the literature often leads to the study of high rank elliptic curves, or more generally, Diophantine equations.

In this paper, we will construct a circle of irrational radius containing a rational sequence of length four of the form $1, r, r^2, r^3$. In particular, the non-existence of a rational sequence of length four on a family of unit circles of the form $y^2 = -x^2 - gx - f$ with centers on the x-axis is deduced.

2. Main Result

Theorem 1: *There exists a circle with center on the x-axis of the form $y^2 = -x^2 - gx - f$ containing the r-geometric progression $x = 1, r, r^2, r^3$ of length four, where r is a given positive rational number.*

The coefficients g and f are given by

$$g = \frac{1}{8}(-5 - 3r - 2r^2 - 6r^3 - 2r^6 + 2r^7 + r^8 - r^9)$$
$$f = r^6 - \frac{1}{64}(-5 + 2r + r^2 + 4r^3 + r^4 + 2r^5 - r^6)^2$$

Proof: If we write $p(x) = (x-1)(x-r)(x-r^2)(x-r^3)$ and $q(x) = x^2 + vx + w$, respectively, we obtain

$$\begin{aligned} p(x) - q(x)^2 = {} & (-1 - r - r^2 - r^3 - 2v)\, x^3 \\ & + (r + r^2 + 2r^3 + r^4 + r^5 - v^2 - 2w)x^2 \end{aligned} \tag{2.1}$$

In the equation (2.1), equating the coefficient of x^3 and x^2 to 0 and 1 respectively, we get

$$-1 - r - r^2 - r^3 - 2v = 0$$
$$r + r^2 + 2r^3 + r^4 + r^5 - v^2 - 2w = 1$$

Solving for v and w, we get

$$v = \frac{1}{2}(-1 - r - r^2 - r^3) \tag{2.2}$$

$$w = \frac{1}{8}(-5 + 2r + r^2 + 4r^3 + r^4 + 2r^5 - r^6) \tag{2.3}$$

Thus we may write equation (2.1) as

$$p(x) - q(x)^2 = x^2 + gx + f$$

where

$$g = -r^3 - r^4 - r^5 - r^6 - 2vw \tag{2.4}$$

$$f = r^6 - w^2 \tag{2.5}$$

Using the values of v and w in equations (2.4) and (2.5), we get

$$g = \frac{1}{8}(-5 - 3r - 2r^2 - 6r^3 - 2r^6 + 2r^7 + r^8 - r^9) \tag{2.6}$$

$$f = r^6 - \frac{1}{64}(-5 + 2r + r^2 + 4r^3 + r^4 + 2r^5 - r^6)^2 \tag{2.7}$$

We thus get, $p(x) - q(x)^2 = x^2 + gx + f = -(-x^2 - gx - f) = -h(x)$ where $h(x) = -x^2 - gx - f$. Since $h(x)$ is a square at every zero $x = 1, r, r^2, r^3$ of $p(x)$, we see that $y^2 = -x^2 - gx - f$ is a circle with center $\left(-\frac{g}{2}, 0\right)$ and radius $R = \sqrt{\frac{g^2}{4} - f}$. For such a circle to be real, we must have $\frac{g^2}{4} - f > 0$ or $g^2 - 4f > 0$. Using the values of g and f, this inequality implies

$$\frac{1}{64}(5 + 3r + 2r^2 + 6r^3 + 2r^6 - 2r^7 - r^8 + r^9)^2$$
$$+4\left(r^6 - \frac{1}{64}(-5 + 2r + r^2 + 4r^3 + r^4 + 2r^5 - r^6)^2\right) > 0 \tag{2.8}$$

It can be checked that the inequality (2.8) is valid for all the positive values of r. This proves the result.

Example 2: Setting $r = 2$ in equation (2.6) and (2.7), we have $g = \frac{-195}{8}$ and $f = \frac{1495}{64}$ so that we have a circle $y^2 = -x^2 - gx - f = -x^2 + 195/8x - (1495/64)$ containing a rational sequence $\left(1, \frac{1}{8}\right), \left(2, \frac{37}{8}\right), \left(2^2, \frac{61}{8}\right), \left(2^3, \frac{83}{8}\right)$. The radius and center of this circle are $R = \sqrt{\frac{g^2}{4} - f} = \frac{\sqrt{32045}}{16}$ and (195/16, 0), respectively.

Next we prove the non-existence of such a rational sequence on a family of unit circles with center on the x-axis.

Corollary 3: *There is no rational sequence of the form* $\{1, r, r^2, r^3\}$ *on a family of unit circles with center on the* x*-axis given by Theorem 1.*

Proof: In Theorem 1, we note that $y^2 = -x^2 - gx - f$ is a family of circles with center on the x -axis. In particular, setting radius $R = \sqrt{\frac{g^2}{4} - f} = 1$ we have

$$g^2 - 4f = 4 \tag{2.9}$$

Substituting values of g and f from equations (2.6) and (2.7) in the equation (2.9), after simplification, we obtain

$$r^{18} - 2r^{17} - 3r^{16} + 8r^{15} - 8r^{13} + 20r^{12} - 24r^{11} - 14r^{10} + 4r^9 + 46r^8$$
$$+24r^7 - 56r^6 - 8r^5 + 68r^4 - 72r^3 + 5r^2 - 50r - 131 = 0 \tag{2.10}$$

By rational root theorem, for the existence of any possible positive rational solution $r = \frac{m}{n}, m, n \in \mathbf{Z}$ of equation (2.10), we must have that m divides 131 and n divides 1. Thus the only possible positive rational solutions of the equation (2.10) are 1 and 131, both of which do not satisfy it. In fact, a computation shows that $r \approx -0.887295$ and $r \approx 1.43321$ are the only two real roots of equation (2.10). Hence we conclude that the family of unit circles mentioned in Corollary 3 cannot contain a rational sequence of the form $\{1, r, r^2, r^3\}$.

3. Conclusion

In the Example 2, we observe that the radius of the circle is an irrational number. It would be interesting to look for the circles with rational radii containing such a rational sequence of the form $\{1, r, r^2, r^3\}$. Thus we have the following open problem:

Question 4: Does there exist a circle with rational radius containing a sequence of rational points in geometric progression of the form $\{1, r, r^2, r^3\}$?

The elementary approach to Question 4 does not lead to anything interesting. Consider a general circle : $H(x, y) = x^2 + y^2 + fx + gy + h$. We want the points $(1, a), (r, b), (r^2, c), (r^3, d)$ on $H(x, y)$ where ($1, r, r^2, r^3$) form a geometric progression with common ratio r (a rational number), and a, b, c, d are arbitrary rational numbers. The solution to the following system of equations is sought: $H(1, a) = 0, H(r, b) = 0, H(r^2, c) = 0, H(r^3, d) = 0$, that is,

$$1 + a^2 + f + ga + h = 0 \tag{3.1}$$

$$r^2 + b^2 + fr + gb + h = 0 \tag{3.2}$$

$$(r^2)^2 + c^2 + fr^2 + gc + h = 0 \tag{3.3}$$

$$(r^3)^2 + d^2 + fr^3 + gd + h = 0 \tag{3.4}$$

We solve the first three linear equations (3.1), (3.2) and (3.3) for f, g, h, we obtain,

$$f = \frac{(b-a)(b^2-c) + (a^2-c^2)(b-c) + a(b-c)(r^4-r)}{(1-r)(b-c-ar+br)}$$

$$g = \frac{b^2-c^2-a^2r+b^2r+r^2-r^4}{b-c-ar+br}$$

$$h = -\frac{(b^2+r-(1+a^2)r)(c-ar^2) + (b-ar)(c^2-(1+a^2)r^2+r^4)}{(b-c-ar+cr+ar^2-br^2)}$$

Using the values of f, g and h in equation (3.4), we obtain

$$\frac{(a-b)r^7 + (-a+c)r^6 + (-a+b)r^5 + (b-d)r^4 + (a-c)r^3 + \left[ac(a-c) + b^2(a-c) - b(a^2-c^2+1) + d\right]r^2}{ar-b(r+1)+c}$$
$$+ \frac{(a-b)(a+b-c-d)(c-d)r - (b-c)(b-d)(c-d)}{ar-b(r+1)+c} = 0$$

This is a seventh degree equation in rational number r, which we are required to solve for rational numbers r, a, b, c and d, which is quite complicated.

It is important to note that Theorem 2 produces geometric progressions of rational points solely for circles centered along the x-axis A linear transformation can always make the center of any general circle lie on the x-axis. But a linear transformation, while maintaining arithmetic progressions in x-coordinates, will not preserve geometric progressions in x-coordinates. Thus it would be interesting to construct a circle with center not necessarily on the x-axis, containing a geometric progression of length four or more.

Finally, Corollary 3 suggests the following conjecture:

Conjecture 5: The unit circle does not admit a geometric progression of rational points of length five or greater.

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