

Fractional derivative and the inverse function theorem

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Abstract

Our objective here is to present some novel applications of the conformable fractional derivative. Especially, the inverse function theorem is generalized.

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1. Introduction

The inverse function theorem is fundamental concept in mathematical analysis, optimization, and various applied disciplines. However, these classical concepts have traditionally relied on the framework of real and

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complex analysis, often assuming strict differentiability and Lipschitz continuity of functions.

In recent years, the emergence of the conformable fractional derivative has ushered in a new era of mathematical tools that allow us to explore the Inverse function theorem in more general and flexible settings. The conformable fractional derivative is a mathematical operator that extends the classical notions of differentiation by incorporating non-local and singular behavior. [4, 5, 6, 7, 8, 12].

The conformable fractional derivative is a specific type of fractional derivative that has gained attention in recent years. It's a derivative operator that preserves certain properties of standard calculus while still capturing the essence of fractional differentiation. The conformable fractional derivative is defined in such a way that it maintains the Leibniz rule and the chain rule, making it a convenient tool for various applications, as well as it offers a more consistent transition to fractional calculus. This makes it easier to integrate fractional calculus into existing mathematical frameworks and apply it to real-world problems. The conformable fractional derivative has shown promise in modeling complex processes that exhibit fractal-like behavior, such as anomalous diffusion, viscoelasticity, and non-local phenomena [3, 9, 10].

In summary, fractional calculus introduces the concept of derivatives and integrals with non-integer orders, allowing for the analysis of systems and phenomena that exhibit fractional behavior. This makes it a powerful tool for understanding and modeling various complex processes in different scientific and engineering domains.

One of the motivations behind the conformable fractional derivative is to provide a more intuitive and straightforward way to handle fractional differentiation. It can be particularly useful in modeling real-world phenomena involving irregularities or anomalous behavior, such as in the fields of physics, engineering, and finance. Additionally, the conformable fractional derivative helps bridge the gap between traditional integer-order differentiation and more complex fractional-order differentiation.

The conformable fractional derivative was first proposed by [8] in 2014 as an alternative approach to fractional calculus. The conformable fractional derivative was introduced as an attempt to address certain limitations and ambiguities associated with traditional fractional derivatives, such as the Riemann-Liouville and Caputo derivatives [8]. These traditional derivatives are defined using integral operators and involve non-local properties, making their interpretation and application complex in some cases.

The most natural definition of fractional differentiation is the conformable one defined in [8] for each $f : [0, +\infty[\rightarrow \mathbb{R}$ by:

$$\forall \alpha \in]0.1[, f^{(\alpha)}(t) := \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad \forall t > 0.$$

The conformable fractional derivative involves naturally a low computational cost, because it does not require the evaluation of other special functions. The motivation behind this study lies in harnessing the power of the conformable fractional derivative. Indeed, we generalize some theorems replacing the classical derivative f' by the conformable fractional one $f^{(\alpha)}$.

2. Preliminaries

The derivative of $f^{-1}(y) = x$, the inverse of $f(x) = y$, can be found using the following steps:

- (1) Write down the inverse function;
- (2) Differentiate, respect to y , the equation's both sides;
- (3) Solve for $(f^{-1})'(y)$ to get the derivative of the inverse function.

We summarize this result in this theorem:

Theorem 2.1: (Inverse Function Theorem)

If $f : I \rightarrow \mathbb{R}$ is continuous and strictly monotonic on I , then :

- f is bijective in I onto $f(I)$;
- f^{-1} is also continuous and strictly monotonic on $f(I)$;
- If $f'(x_0) \neq 0$ exists then, $(f^{-1})'(f(x_0))$ exists and is given by

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

So, the function $f : I \rightarrow f(I)$ is bijective such that :

$$\begin{cases} x \in I \\ f(x) = y \end{cases} \Leftrightarrow \begin{cases} y \in f(I) \\ f^{-1}(y) = x \end{cases}$$

Our objective is to give a new inverse function theorem in the context of conformable fractional derivatives who generalizes the classical theorem.

3. Main results

We generalize here the inverse function theorem. We also establish conditions ensuring the irrationality of a function. In the next tow theorems we take $\alpha \in]0, 1]$, $a > 0$ and $f : [a, b] \rightarrow \mathbb{R}$.

Theorem 3.1: *If f is continuous strictly monotonic and α -differentiable at x_0 with $f'(x_0) \neq 0$, then f^{-1} is α -differentiable at $f(x_0) = y_0 \in f([a, b])$ and*

$$(f^{-1})^{(\alpha)}(f(x_0)) = \frac{(f(x_0))^{1-\alpha}}{f'(x_0)} = \frac{(f(f^{-1}(y_0)))^{1-\alpha}}{f'(f^{-1}(y_0))}.$$

Proof: The previous theorem 2.1 implies that f is bijective, f^{-1} is continious and strictly monotonic, and so the result is true for $\alpha = 1$. Now, let $\alpha \in]0, 1[$ and pose $x = f^{-1}(y_0 + (y - y_0) y_0^{1-\alpha})$. Then $f(x) = y_0 + \varepsilon y_0^{1-\alpha}$, $y = (f(x) - y_0) y_0^{\alpha-1} + y_0$, $\varepsilon = y - y_0$. By the continuity of f^{-1} and the definition of the α -conformable derivative we get :

$$\begin{aligned} (f^{-1})^{(\alpha)}(y_0) &= \lim_{\varepsilon \rightarrow 0} \frac{f^{-1}(y_0 + \varepsilon y_0^{1-\alpha}) - f^{-1}(y_0)}{\varepsilon} \\ &= \lim_{y \rightarrow y_0} \frac{f^{-1}(y_0 + (y - y_0) y_0^{1-\alpha}) - f^{-1}(y_0)}{y - y_0} \\ &= \lim_{x \rightarrow x_0} \frac{x - x_0}{(f(x) - y_0) y_0^{\alpha-1}} \\ &= \frac{(f(f^{-1}(y_0)))^{1-\alpha}}{f'(f^{-1}(y_0))}. \end{aligned}$$

Example 1: Let $f(x) = xe^x$, $\forall x \in [1, 2]$. It's clear that f is continious and strictly increasing from $[1, 2]$ to $[e, 2e^2]$. For $\alpha = \frac{1}{2}$ and by [8] one has :

$$f^{(\frac{1}{2})}(x) = \sqrt{x}(x+1)e^x.$$

f is then $\frac{1}{2}$ -differentiable at each $x_0 \in [1, 2]$ and $f'(x_0) \neq 0$.

Then, for $f(x_0) = y_0 \in [e, 2e^2]$, $(f^{-1})^{(\frac{1}{2})}(y_0)$ exists and is given by

$$\begin{aligned} (f^{-1})^{(\frac{1}{2})}(y_0) &= \frac{(f(f^{-1}(y_0)))^{1-\frac{1}{2}}}{f'(f^{-1}(y_0))} \\ &= \frac{(f(x_0))^{1-\frac{1}{2}}}{f'(x_0)} \\ &= \frac{x_0^{\frac{1}{2}} e^{\frac{x_0}{2}}}{(x_0 + 1) e^{x_0}}. \end{aligned}$$

In particular, for $x_0 = 1$ and $f(x_0) = e$ we have $(f^{-1})^{(1)}(e) = \frac{\sqrt{e}}{2e}$.

Theorem 3.2: *If f is α -differentiable, $f^{(\alpha)}$ is continuous and*

$$\alpha \frac{f(b)-f(a)}{b^\alpha - a^\alpha} \geq \sup\{f^{(\alpha)}(x), x \in [a, b]\},$$

then f is an irrational function.

Proof: Since $f^{(\alpha)}$ is continuous on the compact $[a, b]$, then $f^{(\alpha)}$ is bounded and reaches its limits, and by [8, Theorem 2.2] we have the equality $\alpha \frac{f(b)-f(a)}{b^\alpha - a^\alpha} = \sup\{f^{(\alpha)}(x), x \in [a, b]\}$.

Let us define the irrational function g by

$$g(x) = f(a) + \frac{f(b)-f(a)}{b^\alpha - a^\alpha} (x^\alpha - a^\alpha).$$

It is clear that $g(a) = f(a)$ and $g(b) = f(b)$.

By [8, Theorem 2.2], $h(x) = f(x) - g(x)$ is an α -differentiable function and for each x one has :

$$\begin{aligned} h^{(\alpha)}(x) &= f^{(\alpha)}(x) - g^{(\alpha)}(x) \\ &= x^{1-\alpha} (f'(x) - g'(x)) \\ &= x^{1-\alpha} \left(f'(x) - \alpha \left(\frac{f(b)-f(a)}{b^\alpha - a^\alpha} \right) x^{\alpha-1} \right) \\ &= f^{(\alpha)}(x) - \sup\{f^{(\alpha)}(y), y \in [a, b]\} \leq 0. \end{aligned}$$

By [11, Theorem 6], h is then decreasing on the considered interval. Thus ,

$$0 = h(b) \leq h(x) \leq h(a) = 0, \forall x \in [a, b].$$

Finally, $f(x) = g(x)$ for each x and the result is required.

We close this paper by noticing that the previous theorem 3.1 can be extended in $\mathbb{R}''$.

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