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Fixed point results using self mapping in multiplicative metric space

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Abstract

In this article, we proved fixed point theorems using contraction mappings in the metric space. Specifically, we deal among the space of multiplicative metric in which we obtain fixed point using the self mappings. At the beginning of this paper, we discuss few essential definitions and lemmas that contribute to our main theorems. Our main theorem includes few fixed points results, which were derived by proving the existence together with the uniqueness of fixed point in the multiplicative metric space. Eventually, we obtain the results involving fixed points using double self mappings in the complete multiplicative metric space.

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1. Introduction

The Theory of Fixed Point is significant in various fields such as topology, mathematical economics, physics, biology and engineering. Fixed point theory is a mathematical discipline focused on characterizing all self-maps within a given context in which at least one element remains unchanged. Specifically, fixed point techniques have proven helpful in research related to the theories of integral equations, differential equations, and optimization theory. They have also proven valuable across many other fields including biology, chemistry, and economics. A usual non-empty abstract space which is equipped with a distance function is defined as a metric space. In 1922, Stefan Banach initiated the “Banach Contraction Principle”, a fundamental result, which is extensively acknowledged as the cornerstone of ‘Metric Fixed Point Theory’. This foundation led to the proof of various fixed point theorems within different classes of metric spaces.

Following the definition of metric space, numerous authors have developed various generalizations by relaxing or eliminating certain conditions, modifying the metric function, or abstracting the notion. Consequently, plenty of extensions of metric spaces have been constructed, including b-metric space, G-metric space, multiplicative metric space, and quasi-metric space. A Non-Newtonian calculus called “Multiplicative Calculus” (otherwise called exponential calculus) formulated by Grossman and Katz [4], is a family of non-linear calculi, which doesn't have additive operators. Multiplicative calculus is defined by two operators known as “multiplicative derivative” and “multiplicative integral”, which exhibit an inverse relationship equivalent to the classical calculus's derivative and integral relationship formulated by Newton and Leibniz. Bashirov et al. [3] proceeded that multiplicative calculus is more effective than newtonian calculus for deriving assorted problems from distinct fields.

In 2008, by defining multiplicative distance, they supplied foundation for “Multiplicative Metric Space” (MMS) [3]. Shrivastava, Sharma and Bhardwaj have proved fixed point theorems in complete metric space [14]. We tend to obtain multiplicative contractions in complete MMS. Many authors have proved fixed point results in MMS [1],[12] and [13]. Fixed point results for various multiplicative contractions in MMS were obtained in [15]. One can refer [6],[8] and [16] for recent multiplicative contractions obtained in MMS

2. Preliminaries

Definition 2.1: [3] A mapping $\rho_m : \Xi \times \Xi \to [1, \infty)$ on non-empty set Ξ is known as multiplicative metric, if the below criteria holds:

1) $\rho_m(\mu, \xi) \geq 1$, for all $\mu, \xi \in \Xi$.
2) $\rho_m(\mu, \xi) = \rho_m(\xi, \mu)$, for all $\mu, \xi \in \Xi$.
3) $\rho_m(\mu, \xi) \leq \rho_m(\mu, \eta).\rho_m(\eta, \xi)$, for all $\mu, \xi, \eta \in \Xi$.

Hence the ordered pair (Ξ, ρ_m) is called MMS.

Example 2.2: [12] The metric function ρ_m satisfies multiplicative metric conditions, if $\rho_m : \mathbb{R} \times \mathbb{R} \to [1,\infty)$ defined by $\rho_m(\mu,\xi) = c^{|\mu-\xi|}$, where $\mu, \xi \in \mathbb{R}$ with $c > 1$.

Thus, the pair $(\mathbb{R}, \rho_m)$ is known as MMS.

Definition 2.3: [11] Suppose (Ξ, ρ_m) is MMS with the mapping $Q : \Xi \to \Xi$ is called a multiplicative contraction mapping if $\rho_m(Q\kappa_1, Q\kappa_2) \leq [\rho_m(\kappa_1,\kappa_2)]^{\lambda}$, where $\lambda \in [0,1), \forall \kappa_1, \kappa_2 \in \Xi$.

Lemma 2.4: [11] *A sequence $\{\kappa_n\}$ in MMS (Ξ,ρ_m), is said to be a Multiplicative convergent to an element $\xi \in \Xi$, iff $lim_{n\to\infty} \rho_m(\kappa_n,\xi) = 1$.*

Lemma 2.5: [11] *A sequence $\{\kappa_n\}$ in MMS (Ξ,ρ_m), is called Multiplicative Cauchy iff $lim_{n,m\to\infty} \rho_m(\kappa_m,\kappa_n) = 1$.*

Lemma 2.6: [11] *We say that (Ξ,ρ_m) is complete with respect to Multiplicative metric, whenever all multiplicative Cauchy sequence in Ξ is multiplicative convergent to an element $\kappa \in \Xi$.*

We demonstrate results that establish a unique fixed point (u.f.p) and a unique common fixed point (u.c.f.p) in the complete MMS. Note that our obtained main results involving multiplicative contractions for obtaining unique fixed points, will neglect few rational type contractions from the obtained result of metric space in the existing literature.

3. Main Results

Theorem 3.1: *Suppose (Ξ,ρ_m) is the MMS which is complete in which Γ is a self mapping on Ξ that satisfies the below condition:*

$$\begin{aligned}\rho_m(\Gamma^{p+1}\nu, \Gamma^{p+2}\zeta) \geq & \left[\frac{\rho_m^2(\nu,\Gamma^{p+1}\nu).\rho_m(\zeta,\Gamma^{p+2}\zeta)}{\rho_m(\nu,\Gamma^{p+2}\zeta)}\right]^{\alpha} \\ & .[\rho_m(\nu,\Gamma^{p+1}\nu).\rho_m(\zeta,\Gamma^{p+1}\nu)]^{\beta} \\ & .[\rho_m(\nu,\Gamma^{p+1}\nu).\rho_m(\zeta,\Gamma^{p+1}\nu)]^{\frac{\gamma}{2}} \\ & .[\rho_m(\nu,\Gamma^{p+2}\zeta).\rho_m(\zeta,\Gamma^{p+2}\zeta)]^{\frac{\delta}{2}}\end{aligned} \tag{3.1}$$

for all $\nu,\zeta \in \Xi$, $\alpha,\beta,\gamma,\delta \geq 0$, $2\beta - 2\alpha + \gamma + \delta > 2$, $2\alpha + 2\beta + \gamma > 1$ and 'p' is any non-negative integer, then Γ has a u.f.p in the given complete MMS (Ξ,ρ_m).

Proof: We prove this theorem for $p = 0$.

Now putting $p = 0$ in Equation (3.1), we have

$$\begin{aligned}\rho_m(\Gamma\nu, \Gamma^2\zeta) \geq & \left[\frac{\rho_m^2(\nu,\Gamma\nu).\rho_m(\zeta,\Gamma^2\zeta)}{\rho_m(\nu,\Gamma^2\zeta)}\right]^{\alpha}.[\rho_m(\nu,\Gamma\nu).\rho_m(\zeta,\Gamma\nu)]^{\beta} \\ & .[\rho_m(\nu,\Gamma\nu).\rho_m(\zeta,\Gamma\nu)]^{\frac{\gamma}{2}}\end{aligned}$$

$$.\,[\rho_m(\nu,\Gamma^2\zeta).\,\rho_m(\zeta,\Gamma^2\zeta\,)]^{\frac{\delta}{2}}$$

Consider an arbitrary sequence $\{\nu_n\} \in \Xi$ such that

$$\nu_n = \Gamma\nu_{n+1},\ n = 0,1,2,\ldots$$

Now $\rho_m(\nu_0,\nu_1) = \rho_m(\Gamma\nu_2,\Gamma^2\nu_2)$

$$\geq \left[\frac{\rho_m^2(\nu_2,\Gamma\nu_2).\rho_m(\nu_2,\Gamma^2\nu_2)}{\rho_m(\nu_2,\Gamma^2\nu_2)}\right]^{\alpha}.\,[\rho_m^2(\nu_2,\Gamma\nu_2)]^{\beta}$$

$$.[\rho_m(\nu_2,\Gamma\nu_2).\,\rho_m(\nu_2,\Gamma\nu_2)]^{\frac{\gamma}{2}}$$

$$.[\rho_m(\nu_2,\Gamma^2\nu_2).\,\rho_m(\nu_2,\Gamma^2\nu_2\,)]^{\frac{\delta}{2}}$$

$$\geq \left[\frac{\rho_m^2(\nu_2,\nu_1).\rho_m(\nu_2,\nu_0)}{\rho_m(\nu_2,\nu_0)}\right]^{\alpha}.\,[\rho_m^2(\nu_2,\nu_1)]^{\beta}$$

$$.[\rho_m(\nu_2,\nu_1).\,\rho_m(\nu_2,\nu_1)]^{\frac{\gamma}{2}}$$

$$.\,[\rho_m(\nu_2,\nu_0).\,\rho_m(\nu_2,\nu_0)]^{\frac{\delta}{2}}$$

$$\rho_m(\nu_0,\nu_1) \geq [\rho_m(\nu_2,\nu_1)]^{2\alpha}.\,[\rho_m(\nu_2,\nu_1)]^{2\beta}$$

$$.\,[\rho_m(\nu_2,\nu_1)]^{\gamma}.\,[\rho_m(\nu_2,\nu_0)]^{\delta}.$$

By applying multiplicative triangle inequality, we have

$$\rho_m(\nu_0,\nu_1) \geq [\rho_m(\nu_2,\nu_1)]^{2\alpha}.\,[\rho_m(\nu_2,\nu_1)]^{2\beta}$$

$$.\,[\rho_m(\nu_2,\nu_1)]^{\gamma}.\left[\frac{\rho_m(\nu_2,\nu_1)}{\rho_m(\nu_0,\nu_1)}\right]^{\delta}.$$

$$[\rho_m(\nu_1,\nu_2)]^{(2\alpha+2\beta+\gamma+\delta)} \leq [\rho_m(\nu_0,\nu_1)]^{(1+\delta)}.$$

$$\rho_m(\nu_1,\nu_2) \leq [\rho_m(\nu_0,\nu_1)]^{\left(\frac{1+\delta}{2\alpha+2\beta+\gamma+\delta}\right)}.$$

In the same manner, we have

$$\rho_m(\nu_2,\nu_3) \leq [\rho_m(\nu_1,\nu_2)]^{\left(\frac{1+\delta}{2\alpha+2\beta+\gamma+\delta}\right)}.$$

$$\rho_m(\nu_2,\nu_3) \leq [\rho_m(\nu_0,\nu_1)]^{\left(\left(\frac{1+\delta}{2\alpha+2\beta+\gamma+\delta}\right)^2\right)}.$$

In general, we write

$$\rho_m(\nu_n,\nu_{n+1}) \leq [\rho_m(\nu_0,\nu_1)]^{\left(\left(\frac{1+\delta}{2\alpha+2\beta+\gamma+\delta}\right)^n\right)}.$$

$$\rho_m(\nu_n,\nu_{n+1}) \leq [\rho_m(\nu_0,\nu_1)]^{k^n},$$

where $k = \frac{1+\delta}{2\alpha+2\beta+\gamma+\delta} < 1$.

Since $0 \leq k < 1$, so for $n \to \infty$, $k^n \to 0$.

Hence $\rho_m(\nu_n,\nu_{n+1}) \to 1$.

Therefore $\{\nu_n\}$ is Multiplicative Cauchy sequence.

As Ξ is a complete MMS, therefore $\exists$ an element $\xi \in \Xi$ such that

$$\{\nu_n\} \to \xi.$$

Now $\rho_m(\xi,\Gamma\xi) = \rho_m(\Gamma\xi,\Gamma^2\nu_{n+2})$

$$\geq \left[\frac{\rho_m^2(\xi,\Gamma\xi).\rho_m(\nu_{n+2},\Gamma^2\nu_{n+2})}{\rho_m(\xi,\Gamma^2\nu_{n+2})}\right]^{\alpha}$$
$$.[\rho_m(\xi,\Gamma\xi).\rho_m(\nu_{n+2},\Gamma\xi)]^{\beta}$$
$$.[\rho_m(\xi,\Gamma\xi).\rho_m(\nu_{n+2},\Gamma\xi)]^{\frac{\gamma}{2}}$$
$$.[\rho_m(\xi,\Gamma^2\nu_{n+2}).\rho_m(\nu_{n+2},\Gamma^2\nu_{n+2})]^{\frac{\delta}{2}}$$
$$\geq \left[\frac{\rho_m^2(\xi,\Gamma\xi).\rho_m(\nu_{n+2},\nu_n)}{\rho_m(\xi,\nu_n)}\right]^{\alpha}$$
$$.[\rho_m(\xi,\Gamma\xi).\rho_m(\nu_{n+2},\Gamma\xi)]^{\beta}$$
$$.[\rho_m(\xi,\Gamma\xi).\rho_m(\nu_{n+2},\Gamma\xi)]^{\frac{\gamma}{2}}$$
$$.[\rho_m(\xi,\nu_n).\rho_m(\nu_{n+2},\nu_n)]^{\frac{\delta}{2}}.$$

Applying limit

$$\rho_m(\xi,\Gamma\xi) \geq \left[\frac{\rho_m^2(\xi,\Gamma\xi).\rho_m(\xi,\xi)}{\rho_m(\xi,\xi)}\right]^{\alpha}.[\rho_m^2(\xi,\Gamma\xi)]^{\beta}$$
$$.[\rho_m(\xi,\Gamma\xi).\rho_m(\xi,\Gamma\xi)]^{\frac{\gamma}{2}}$$
$$.[\rho_m(\xi,\xi).\rho_m(\xi,\xi)]^{\frac{\delta}{2}}$$
$$\rho_m(\xi,\Gamma\xi) \geq [\rho_m(\xi,\Gamma\xi)]^{(2\alpha+2\beta+\gamma)}.$$
$$[\rho_m(\xi,\Gamma\xi)]^{((2\alpha+2\beta+\gamma)-1)} \leq 1.$$

This gives $\rho_m(\xi,\Gamma\xi)=1 \Rightarrow \Gamma\xi=\xi$.

Therefore the mapping Γ holds the fixed point ξ.

Assume that the mapping Γ holds fixed point η distinct from ξ, then by Equation (3.1), we get

$$\rho_m(\xi,\eta) = \rho_m(\Gamma\xi,\Gamma^2\eta)$$
$$\geq \left[\frac{\rho_m^2(\xi,\Gamma\xi).\rho_m(\eta,\Gamma^2\eta)}{\rho_m(\xi,\Gamma^2\eta)}\right]^{\alpha}.[\rho_m(\xi,\Gamma\xi).\rho_m(\eta,\Gamma\xi)]^{\beta}$$
$$.[\rho_m(\xi,\Gamma\xi).\rho_m(\eta,\Gamma\xi)]^{\frac{\gamma}{2}}$$
$$.[\rho_m(\xi,\Gamma^2\eta).\rho_m(\eta,\Gamma^2\eta)]^{\frac{\delta}{2}}$$
$$\geq \left[\frac{\rho_m^2(\xi,\xi).\rho_m(\eta,\eta)}{\rho_m(\xi,\eta)}\right]^{\alpha}.[\rho_m(\xi,\xi).\rho_m(\eta,\xi)]^{\beta}$$
$$.[\rho_m(\xi,\xi).\rho_m(\eta,\xi)]^{\frac{\gamma}{2}}.[\rho_m(\xi,\eta).\rho_m(\eta,\eta)]^{\frac{\delta}{2}}$$
$$\rho_m(\xi,\eta) \geq [\rho_m(\xi,\eta)]^{\left((\beta-\alpha)+\left(\frac{\gamma+\delta}{2}\right)\right)} \quad \Rightarrow [\rho_m(\xi,\eta)]^{\left(\left((\beta-\alpha)+\left(\frac{\gamma+\delta}{2}\right)\right)-1\right)} \leq 1.$$
$$\Rightarrow [\rho_m(\xi,\eta)]^{\left(\left(\frac{2\beta-2\alpha+\gamma+\delta}{2}\right)-1\right)} \leq 1.$$

Since $2\beta-2\alpha+\gamma+\delta>2$, thus $\rho_m(\xi,\eta)=1 \Rightarrow \xi=\eta$.

Thus Γ has a u.f.p in the complete MMS (Ξ,ρ_m).

Theorem 3.2: *Given (Ξ, ρ_m) be a MMS which is complete. If Γ is a self-mapping on Ξ such that the below condition is satisfied:*

$$\rho_m(\Gamma^{p+1}\nu, \Gamma^{p+2}\zeta) \geq [\min\{\rho_m(\nu, \Gamma^{p+2}\zeta), \rho_m(\zeta, \Gamma^{p+1}\nu)\}]^{\alpha} \cdot [\rho_m(\nu, \Gamma^{p+1}\nu). \rho_m(\zeta, \Gamma^{p+2}\zeta)]^{\frac{\beta}{2}} \tag{3.2}$$

for all $\nu, \zeta \in \Xi$, $\alpha > 1$, $\beta > 2$ and 'p' is any non-negative integer. Then Γ has a u.f.p in the complete MMS (Ξ, ρ_m).

Proof: We prove this theorem for $p = 0$.

Now putting $p = 0$ in Equation (3.2), we have

$$\rho_m(\Gamma\nu, \Gamma^2\zeta) \geq [\min\{\rho_m(\nu, \Gamma^2\zeta), \rho_m(\zeta, \Gamma\nu)\}]^{\alpha} \cdot [\rho_m(\nu, \Gamma\nu). \rho_m(\zeta, \Gamma^2\zeta)]^{\frac{\beta}{2}}.$$

Let us take a sequence $\{\nu_n\} \in \Xi$ which is defined by

$$\nu_n = \Gamma\nu_{n+1},\ n = 0,1,2, \dots \text{ and } \nu_0 \in \Xi.$$

Now consider,

$$\begin{aligned}
\rho_m(\nu_n, \nu_{n-1}) &= \rho_m(\Gamma\nu_{n+1}, \Gamma^2\nu_{n+1}) \\
&\geq [\min\{\rho_m(\nu_{n+1}, \Gamma^2\nu_{n+1}), \rho_m(\nu_{n+1}, \Gamma\nu_{n+1})\}]^{\alpha} \\
&\quad \cdot [\rho_m(\nu_{n+1}, \Gamma\nu_{n+1}). \rho_m(\nu_{n+1}, \Gamma^2\nu_{n+1})]^{\frac{\beta}{2}} \\
&\geq [\min\{\rho_m(\nu_{n+1}, \nu_{n-1}), \rho_m(\nu_{n+1}, \nu_n)\}]^{\alpha} \\
&\quad \cdot [\rho_m(\nu_{n+1}, \nu_n). \rho_m(\nu_{n+1}, \nu_{n-1})]^{\frac{\beta}{2}} \\
&\geq \min\left\{\begin{array}{l} \left([\rho_m(\nu_{n+1}, \nu_{n-1})]^{\left(\alpha+\frac{\beta}{2}\right)}. [\rho_m(\nu_{n+1}, \nu_n)]^{\left(\frac{\beta}{2}\right)}\right), \\ \left([\rho_m(\nu_{n+1}, \nu_n)]^{\left(\alpha+\frac{\beta}{2}\right)}. [\rho_m(\nu_{n+1}, \nu_{n-1})]^{\left(\frac{\beta}{2}\right)}\right) \end{array}\right\}.
\end{aligned}$$

Applying multiplicative triangle inequality, we have

$$\geq \min\left\{\begin{array}{l} \left(\left[\dfrac{\rho_m(\nu_n, \nu_{n+1})}{\rho_m(\nu_{n-1}, \nu_n)}\right]^{\left(\alpha+\frac{\beta}{2}\right)} . [\rho_m(\nu_{n+1}, \nu_n)]^{\left(\frac{\beta}{2}\right)}\right), \\ \left([\rho_m(\nu_{n+1}, \nu_n)]^{\left(\alpha+\frac{\beta}{2}\right)}. \left[\dfrac{\rho_m(\nu_n, \nu_{n+1})}{\rho_m(\nu_{n-1}, \nu_n)}\right]^{\left(\frac{\beta}{2}\right)}\right) \end{array}\right\}$$

$$\geq \min\left\{\begin{array}{l} \left([\rho_m(\nu_n, \nu_{n+1})]^{\left(\alpha+\frac{\beta}{2}+\frac{\beta}{2}\right)}. [\rho_m(\nu_{n-1}, \nu_n)]^{\left(-\left(\alpha+\frac{\beta}{2}\right)\right)}\right), \\ \left([\rho_m(\nu_n, \nu_{n+1})]^{\left(\alpha+\frac{\beta}{2}+\frac{\beta}{2}\right)}. [\rho_m(\nu_{n-1}, \nu_n)]^{\left(-\left(\frac{\beta}{2}\right)\right)}\right) \end{array}\right\}$$

$$\rho_m(\nu_{n-1},\nu_n) \geq [\rho_m(\nu_n,\nu_{n+1})]^{\left(\min\left\{\frac{\alpha+\beta}{1+\left(\alpha+\frac{\beta}{2}\right)},\frac{\alpha+\beta}{1+\frac{\beta}{2}}\right\}\right)}$$

$$\rho_m(\nu_{n-1},\nu_n) \geq [\rho_m(\nu_n,\nu_{n+1})]^{\left(\frac{\alpha+\beta}{1+\left(\alpha+\frac{\beta}{2}\right)}\right)}$$

$$\rho_m(\nu_n,\nu_{n+1}) \leq [\rho_m(\nu_{n-1},\nu_n)]^{\left(\frac{2+2\alpha+\beta}{2(\alpha+\beta)}\right)}$$

$$\rho_m(\nu_n,\nu_{n+1}) \leq [\rho_m(\nu_{n-1},\nu_n)]^l,$$

where $l = \frac{2+2\alpha+\beta}{2(\alpha+\beta)} < 1$.

Similarly we can show that

$$\rho_m(\nu_n,\nu_{n-1}) \leq [\rho_m(\nu_{n-1},\nu_{n-2})]^l.$$

$$\rho_m(\nu_{n+1},\nu_n) \leq [\rho_m(\nu_{n-1},\nu_{n-2})]^{l^2}.$$

Thus $\rho_m(\nu_{n+1},\nu_n) \leq [\rho_m(\nu_1,\nu_0)]^{l^n}$.

Since $0 \leq l < 1$, so for $n \to \infty$, $l^n \to 0$.

So, we obtain $\rho_m(\nu_{n+1},\nu_n) \to 1$.

Hence, $\{\nu_n\}$ is Multiplicative Cauchy sequence.

We know that Ξ is complete, therefore $\exists\, z \in \Xi$ such that $\{v_n\} \to z$.

$$\begin{aligned}\text{Now } \rho_m(z,\Gamma z) &= \rho_m(\Gamma z,\Gamma^2\nu_{n+2})\\ &\geq [\min\{\rho_m(z,\Gamma^2\nu_{n+2}),\rho_m(\nu_{n+2},\Gamma z)\}]^\alpha\\ &\quad .[\rho_m(z,\Gamma z).\,\rho_m(\nu_{n+2},\Gamma^2\nu_{n+2})]^{\frac{\beta}{2}}\\ &\geq [\min\{\rho_m(z,\nu_n),\rho_m(\nu_{n+2},\Gamma z)\}]^\alpha\\ &\quad .[\rho_m(z,\Gamma z).\,\rho_m(\nu_{n+2},\nu_n)]^{\frac{\beta}{2}}\end{aligned}$$

Employing limit

$$\rho_m(z,\Gamma z) \geq [\min\{\rho_m(z,z),\rho_m(z,\Gamma z)\}]^\alpha\ .\ [\rho_m(z,\Gamma z).\,\rho_m(z,z)]^{\frac{\beta}{2}}$$

$$\rho_m(z,\Gamma z) \geq [\rho_m(z,\Gamma z)]^{\frac{\beta}{2}}$$

$$[\rho_m(z,\Gamma z)]^{\left(\frac{\beta}{2}-1\right)} \leq 1.$$

$$\rho_m(z,\Gamma z) = 1 \Rightarrow \Gamma z = z.$$

As a result, the mapping Γ has the fixed point z . Let w represent different fixed point other than z for the mapping Γ, then by Equation (3.2), we have

$$\begin{aligned}\rho_m(z,w) &= \rho_m(\Gamma z,\Gamma^2 w)\\ &\geq [\min\{\rho_m(z,\Gamma^2 w),\rho_m(w,\Gamma z)\}]^\alpha\\ &\quad .[\rho_m(z,\Gamma z).\,\rho_m(w,\Gamma^2 w)]^{\frac{\beta}{2}}\end{aligned}$$

$$\rho_m(z,w) \geq [\rho_m(z,w)]^\alpha$$

$$[\rho_m(z,w)]^{(\alpha-1)} \leq 1.$$
$$\rho_m(z,w) = 1, \text{ since } \alpha > 1.$$
$$\Rightarrow z = w.$$

Thus Γ has u.f.p in complete MMS.

Theorem 3.3: *Suppose a pair of mappings $S, T : \Xi \to \Xi$ in a complete MMS (Ξ, ρ_m) such that the below condition is satisfied:*

$$\rho_m(S^{p+1}\nu, T^{p+2}\zeta) \geq [\min\{\rho_m(\nu, T^{p+2}\zeta), \rho_m(\zeta, S^{p+1}\nu)\}]^\alpha \quad (3.3)$$
$$. [\rho_m(\nu, S^{p+1}\nu). \rho_m(\zeta, T^{p+2}\zeta)]^{\frac{\beta}{2}}$$
$$. [\rho_m(\nu, T^{p+2}\zeta). \rho_m(\zeta, S^{p+1}\nu)]^{\frac{\gamma}{2}}$$

for all $\nu, \zeta \in \Xi$, $\alpha, \beta, \gamma > 1$ and 'p' is any non-negative integer. Then the mappings S, T has a u.c.f.p in the complete MMS (Ξ, ρ_m).

Proof: We prove this theorem for $p = 0$.

Now putting $p = 0$ in Equation (3.3), we have

$$\rho_m(S\nu, T^2\zeta) \geq [\min\{\rho_m(\nu, T^2\zeta), \rho_m(\zeta, S\nu)\}]^\alpha$$
$$. [\rho_m(\nu, S\nu). \rho_m(\zeta, T^2\zeta)]^{\frac{\beta}{2}}$$
$$. [\rho_m(\nu, T^2\zeta). \rho_m(\zeta, S\nu)]^{\frac{\gamma}{2}}$$

Let $\nu_0 \in \Xi$. A sequence $\{\nu_n\} \in \Xi$ is defined as follows

$$\nu_{2n} = T\nu_{2n+1}, \nu_{2n-1} = S\nu_{2n}, \ldots$$

Consider

$$\rho_m(\nu_{2n+1}, \nu_{2n}) = \rho_m(S\nu_{2n+2}, T^2\nu_{2n+2})$$
$$\geq \left[\min\left\{\begin{matrix}\rho_m(\nu_{2n+2}, T^2\nu_{2n+2}), \\ \rho_m(\nu_{2n+2}, S\nu_{2n+2})\end{matrix}\right\}\right]^\alpha$$
$$. [\rho_m(\nu_{2n+2}, S\nu_{2n+2}). \rho_m(\nu_{2n+2}, T^2\nu_{2n+2})]^{\frac{\beta}{2}}$$
$$. [\rho_m(\nu_{2n+2}, T^2\nu_{2n+2}). \rho_m(\nu_{2n+2}, S\nu_{2n+2})]^{\frac{\gamma}{2}}$$
$$\geq \left[\min\left\{\begin{matrix}\rho_m(\nu_{2n+2}, \nu_{2n}), \\ \rho_m(\nu_{2n+2}, \nu_{2n+1})\end{matrix}\right\}\right]^\alpha$$
$$. [\rho_m(\nu_{2n+2}, \nu_{2n+1}). \rho_m(\nu_{2n+2}, \nu_{2n})]^{\frac{\beta}{2}}$$
$$. [\rho_m(\nu_{2n+2}, \nu_{2n}). \rho_m(\nu_{2n+2}, \nu_{2n+1})]^{\frac{\gamma}{2}}$$
$$\geq \min\left\{\begin{matrix}\left([\rho_m(\nu_{2n+2}, \nu_{2n})]^{\left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}\right)}. [\rho_m(\nu_{2n+2}, \nu_{2n+1})]^{\left(\frac{\beta+\gamma}{2}\right)}\right), \\ \left([\rho_m(\nu_{2n+2}, \nu_{2n+1})]^{\left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}\right)}. [\rho_m(\nu_{2n+2}, \nu_{2n})]^{\left(\frac{\beta+\gamma}{2}\right)}\right)\end{matrix}\right\}$$

After applying multiplicative triangle inequality, we get

$$\rho_m(\nu_{2n+1},\nu_{2n}) \geq \min\left\{\begin{array}{l}\left(\left[\frac{\rho_m(\nu_{2n+2},\nu_{2n+1})}{\rho_m(\nu_{2n+1},\nu_{2n})}\right]^{\left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}\right)}.[\rho_m(\nu_{2n+2},\nu_{2n+1})]^{\left(\frac{\beta+\gamma}{2}\right)}\right),\\ \left([\rho_m(\nu_{2n+2},\nu_{2n+1})]^{\left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}\right)}.\left[\frac{\rho_m(\nu_{2n+2},\nu_{2n+1})}{\rho_m(\nu_{2n+1},\nu_{2n})}\right]^{\left(\frac{\beta+\gamma}{2}\right)}\right)\end{array}\right\}$$

$$\geq \min\left\{\begin{array}{l}\left([\rho_m(\nu_{2n+2},\nu_{2n+1})]^{(\alpha+\beta+\gamma)}.[\rho_m(\nu_{2n+1},\nu_{2n})]^{\left(-\left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}\right)\right)}\right),\\ \left([\rho_m(\nu_{2n+2},\nu_{2n+1})]^{(\alpha+\beta+\gamma)}.[\rho_m(\nu_{2n+1},\nu_{2n})]^{\left(-\left(\frac{\beta+\gamma}{2}\right)\right)}\right)\end{array}\right\}$$

$$\rho_m(\nu_{2n+1},\nu_{2n}) \geq [\rho_m(\nu_{2n+2},\nu_{2n+1})]^{\left(\frac{2(\alpha+\beta+\gamma)}{(2+2\alpha+\beta+\gamma)}\right)}$$

$$\rho_m(\nu_{2n+2},\nu_{2n+1}) \leq [\rho_m(\nu_{2n+1},\nu_{2n})]^{\left(\frac{(2+2\alpha+\beta+\gamma)}{2(\alpha+\beta+\gamma)}\right)}$$

$$\rho_m(\nu_{2n+2},\nu_{2n+1}) \leq [\rho_m(\nu_{2n+1},\nu_{2n})]^{r},$$

where $r = \frac{(2+2\alpha+\beta+\gamma)}{2(\alpha+\beta+\gamma)} < 1.$

Similarly, $\rho_m(\nu_{2n+1},\nu_{2n}) \leq [\rho_m(\nu_{2n},\nu_{2n-1})]^r$.

$\rho_m(\nu_{2n+2},\nu_{2n+1}) \leq [\rho_m(\nu_{2n},\nu_{2n-1})]^{r^2}$.

Continuing in this way, we have

$\rho_m(\nu_{2n+2},\nu_{2n+1}) \leq [\rho_m(\nu_1,\nu_0)]^{r^n}$

Since $0 \leq r < 1$, so for $n \to \infty, r^n \to 0$.

So, we have $\rho_m(\nu_{2n+2},\nu_{2n+1}) \to 1$.

Hence $\{\nu_n\}$ is Multiplicative Cauchy sequence.

Since Ξ is complete, $\exists\, z \in \Xi$ such that $\{\nu_n\} \to z$.

Next we illustrate that the mappings S, T possesses a common fixed point z.

$$\begin{aligned}\rho_m(z,Sz) = \rho_m(\nu_n,Sz) &= \rho_m(Sz,T^2\nu_{n+2})\\ &\geq [\min\{\rho_m(z,T^2\nu_{n+2}),\rho_m(\nu_{n+2},Sz)\}]^{\alpha}\\ &\quad.[\rho_m(z,Sz).\rho_m(\nu_{n+2},\mathrm{T}^2\nu_{n+2})]^{\frac{\beta}{2}}\\ &\quad.[\rho_m(z,T^2\nu_{n+2}).\rho_m(\nu_{n+2},Sz)]^{\frac{\gamma}{2}}\\ &\geq [\min\{\rho_m(z,\nu_n),\rho_m(\nu_{n+2},Sz)\}]^{\alpha}\\ &\quad.[\rho_m(z,Sz).\rho_m(\nu_{n+2},\nu_n)]^{\frac{\beta}{2}}\\ &\quad.[\rho_m(z,\nu_n).\rho_m(\nu_{n+2},Sz)]^{\frac{\gamma}{2}}\\ &\geq [\min\{\rho_m(z,z),\rho_m(z,Sz)\}]^{\alpha}\\ &\quad.[\rho_m(z,Sz).\rho_m(z,z)]^{\frac{\beta}{2}}\\ &\quad.[\rho_m(z,z).\rho_m(z,Sz)]^{\frac{\gamma}{2}}\end{aligned}$$

$$\rho_m(z, Sz) \geq (1)^{\alpha}.\rho_m(z, Sz)^{\left(\frac{\beta+\gamma}{2}\right)}$$

$$[\rho_m(z, Sz)]^{\left(\left(\frac{\beta+\gamma}{2}\right)-1\right)} \leq 1.$$

$$\rho_m(z, Sz) = 1 \Rightarrow z = Sz.$$

Hence the mapping S possess the fixed point z.

In an equivalent way, It is possible for us to demonstrate that T possesses fixed point z.

Consequently, the mappings S, T holds the common fixed point z .

Inorder to show their uniqueness, take two fixed points c, d which is common for mappings S and T.

Then
$$\begin{aligned}\rho_m(c, d) = \rho_m(Sc, Td) &= \rho_m(Sc, T^2d)\\ &\geq [\min\{\rho_m(c, T^2d), \rho_m(d, Sc)\}]^{\alpha}\\ &\quad .[\rho_m(c, Sc).\rho_m(d, T^2d)]^{\frac{\beta}{2}}\\ &\quad .[\rho_m(c, T^2d).\rho_m(d, Sc)]^{\frac{\gamma}{2}}\\ &\geq [\min\{\rho_m(c, d), \rho_m(d, c)\}]^{\alpha}\\ &\quad .[\rho_m(c, c).\rho_m(d, d)]^{\frac{\beta}{2}}\\ &\quad .[\rho_m(c, d).\rho_m(d, c)]^{\frac{\gamma}{2}}\end{aligned}$$

$$\rho_m(c, d) \geq [\rho_m(c, d)]^{(\alpha+\gamma)}$$

$$[\rho_m(c, d)]^{(\alpha+\gamma-1)} \leq 1.$$

$$\rho_m(c, d) = 1 \Rightarrow c = d.$$

Hence the mappings S and T possess u.c.f.p in complete MMS (Ξ, ρ_m).

4. Conclusion

In this article, we obtained few results of fixed point in MMS. Furthermore, an u.c.f.p holds for the pair of self mappings in the space of MMS, which is complete. Similarly, various fixed point results can be acquired in the space of MMS, where any contraction mapping consistently possess the existence together with uniqueness of fixed point, particularly in the case of a complete MMS. One can refer [2], [7] and [9] for few results concerning fixed points, which has been proved recently in MMS. Also one can refer [5] and [10] for current findings of new fixed point results.

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